

Global Behavior of Some Rational Second Order Difference Equations

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Abstract

We investigate the global behavior of five special cases of the difference equation of the form

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, \dots$$

with nonnegative parameters and initial conditions such that $A + B + C > 0$ and with several equilibrium points. We will prove that for four of these five equations the unique equilibrium is globally asymptotically stable and for the fifth equation that the unique equilibrium is stable but not asymptotically stable. We will extend some of the obtained results to difference equations of any order.

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1 Introduction and Preliminaries

Consider the following difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, \dots \quad (1.1)$$

with nonnegative parameters and initial conditions such that $A + B + C > 0$. Equation (1.1) was considered in great details in [1,2] and all 49 special cases, when one or more parameters were equal to zero, were considered. In number of cases the authors gave the complete or partial description of the global dynamics and in the other cases they left number of conjectures and open problems to be answered. In particular, they left several open problems and conjectures for the following five equations:

$$x_{n+1} = \frac{\alpha}{1 + x_n x_{n-1}}, \quad n = 0, 1, \dots, \quad (1.2)$$

$$x_{n+1} = \frac{\alpha + x_n x_{n-1}}{A + x_n x_{n-1}}, \quad n = 0, 1, \dots, \quad (1.3)$$

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_n x_{n-1}}, \quad n = 0, 1, \dots, \quad (1.4)$$

$$x_{n+1} = \frac{\alpha + x_{n-1}}{A + x_n x_{n-1}}, \quad n = 0, 1, \dots, \quad (1.5)$$

and

$$x_{n+1} = \frac{\alpha}{(1 + x_n)x_{n-1}}, \quad n = 0, 1, \dots \quad (1.6)$$

In this paper we solve all posed conjectures for Equations (1.2)–(1.6). Our main results for Equations (1.2)–(1.6) are:

Theorem 1.1. *The unique positive equilibrium of Eq. (1.2) is globally asymptotically stable.*

Theorem 1.2. *The unique positive equilibrium of Eq. (1.3) is globally asymptotically stable.*

Theorem 1.3. *The unique positive equilibrium of Eq. (1.4) is globally asymptotically stable.*

Theorem 1.4. *The unique positive equilibrium of Eq. (1.5) is globally asymptotically stable.*

Theorem 1.5. *The unique positive equilibrium of Eq. (1.6) is stable but not asymptotically stable. All nonequilibrium solutions of Eq. (1.6) belong to exactly one of the invariants of the form*

$$H(x_n, x_{n-1}) = x_n + x_{n+1} + x_n x_{n+1} + \alpha \left(\frac{1}{x_n} + \frac{1}{x_{n-1}} \right) = H(x_0, x_{-1}), \quad n = 0, 1, \dots \quad (1.7)$$

which are simple, closed curves. Consequently, no solution converges to a limit.

Let I be some interval of real numbers and let $f \in C[I \times I, I]$. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots, \quad (1.8)$$

where f is a real function in all variables. There are several global attractivity results for Eq. (1.8) which give the sufficient conditions for all solutions to approach a equilibrium. See [1–3, 6–13]. The following global attractivity result was established in [7]:

Theorem 1.6. *Let $l \in \{1, 2, \dots\}$. Suppose that on some interval $I \subseteq \mathbb{R}$ Eq. (1.8) has the linearization*

$$x_{n+l} = \sum_{i=1-l}^m g_i x_{n-i},$$

where the nonnegative functions $g_i : I^{k+l} \rightarrow \mathbb{R}$ are such that $\sum_{i=1-l}^m g_i = 1$ is satisfied.

Assume that there exists $A > 0$ such that

$$g_{1-l} \geq A, \quad n = 0, 1, \dots .$$

Then if $x_{l-1}, \dots, x_{-k} \in I$,

$$\lim_{n \rightarrow \infty} x_n = L \in I.$$

The following result is useful in determining Lyapunov functions if difference equation has an invariant.

Consider the difference equation

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n), \quad n = 0, 1, \dots, \quad (1.9)$$

where \mathbf{x}_n is in \mathbb{R}^k and $\mathbf{f} : D \rightarrow D$ is continuous, where $D \subset \mathbb{R}^k$.

Theorem 1.7 (Discrete Dirichlet Theorem). *Consider (1.9), where $\mathbf{x}_n \in \mathbb{R}^k$ and $\mathbf{f} : D \rightarrow D$ is continuous with $D \subset \mathbb{R}^k$. Suppose that $I : \mathbb{R}^k \rightarrow \mathbb{R}$ is a continuous invariant of (1.9). If I attains an isolated local minimum or maximum value at the equilibrium point $\bar{\mathbf{x}}$ of this system, then there exists a Lyapunov function equal to*

$$\pm(I(\mathbf{x}) - I(\bar{\mathbf{x}}))$$

and so the equilibrium $\bar{\mathbf{x}}$ is stable.

Theorem 1.7 is a discrete analogue of the Dirichlet theorem, see [10].

2 Proofs of Main Results

In this section we give the proofs of Theorems 1.1–1.5.

Proof of Theorem 1.1. The proof of the theorem was given in [1] in the case where $\alpha \leq 2$. Here we give a unified proof for all values of α . Every solution of Eq. (1.2) satisfies

$$\frac{\alpha}{1 + \alpha^2} \leq x_n \leq \alpha, \quad n = 1, 2, \dots \quad (2.1)$$

Every solution of Eq. (1.2) satisfies $\alpha = x_{n+1}(1 + x_n x_{n-1})$ and so by using it in the first iterate of Eq. (1.2) we obtain

$$x_{n+2} = \frac{x_{n+1}(1 + x_n x_{n-1})}{1 + x_n x_{n+1}}$$

which means that a solution of Eq. (1.2) satisfies the following equation on the interval $I = [0, \infty)$

$$x_{n+2} = \frac{1}{1 + x_n x_{n+1}} x_{n+1} + \frac{x_n x_{n+1}}{1 + x_n x_{n+1}} x_{n-1} = g_{-1} x_{n+1} + g_1 x_{n-1}, \quad n = 0, 1, \dots, \quad (2.2)$$

where

$$g_1 = \frac{x_n x_{n+1}}{1 + x_n x_{n+1}}, \quad g_{-1} = \frac{1}{1 + x_n x_{n+1}}.$$

Clearly $g_{-1} + g_1 = 1$. By using the estimate (2.1) we obtain

$$g_{-1} = \frac{1}{1 + x_n x_{n+1}} \geq \frac{1}{1 + \alpha^2} > 0,$$

and so all conditions of Theorem 1.6 are satisfied on the interval $I = [0, \infty)$, which implies that the unique positive equilibrium \bar{x} is global attractor and because it is locally asymptotically stable, see [1], it is also globally asymptotically stable. \square

Remark 2.1. Theorem 1.1 solves [1, Conjectures 2.2 and 8.1]. By using different method, this result was first obtained in [12].

Proof of Theorem 1.2. The proof of the theorem was given in [1] in the case where $\alpha \leq A$. Here we give a proof for the case $\alpha > A$. As it was shown in [1], every solution of Eq. (1.3) satisfies

$$\frac{\min\{\alpha, 1\}}{\max\{A, 1\}} \leq x_n \leq \frac{\max\{\alpha, 1\}}{\min\{A, 1\}}, \quad n = 1, 2, \dots \quad (2.3)$$

Every solution of Eq. (1.2) satisfies $\alpha = x_{n+1}(A + x_n x_{n-1}) - x_n x_{n-1}$ and so by using it in the first iterate of Eq. (1.3) we obtain

$$x_{n+2} = \frac{A x_{n+1} + x_{n+1} x_n x_{n-1} - x_{n-1} x_n + x_{n+1} x_n}{A + x_n x_{n+1}},$$

which means that a solution of Eq. (1.3) satisfies the following equation on the interval $I = [1, \infty)$

$$x_{n+2} = \frac{A + x_n}{A + x_n x_{n+1}} x_{n+1} + \frac{x_n(x_{n+1} - 1)}{A + x_n x_{n+1}} x_{n-1} = g_{-1} x_{n+1} + g_1 x_{n-1}, \quad n = 0, 1, \dots, \tag{2.4}$$

where

$$g_{-1} = \frac{A + x_n}{A + x_n x_{n+1}}, \quad g_1 = \frac{x_n(x_{n+1} - 1)}{A + x_n x_{n+1}}.$$

Clearly $g_{-1} + g_1 = 1$. By using the estimate (2.3) we obtain

$$g_{-1} = \frac{A + x_n}{A + x_n x_{n+1}} \geq \frac{A}{A + \left(\frac{\max\{\alpha, 1\}}{\min\{A, 1\}}\right)^2} > 0.$$

Furthermore, $x_{n+1} > 1$ for $n = 1, 2, \dots$ if and only if $\alpha > A$. Thus, when $\alpha > A$ all conditions of Theorem 1.6 are satisfied on the interval $I = [1, \infty)$, which implies that the unique positive equilibrium \bar{x} is global attractor and because it is locally asymptotically stable, see [1], it is also globally asymptotically stable. \square

Remark 2.2. Thorem 1.2 solves [1, Conjectures 12.1]. This result was obtained by a different method by E. Drymonis from University of Rhode Island.

Proof of Theorem 1.3. The proof of the theorem was given in [2] in the case where $\alpha \leq \beta$. Here we give a unified proof for all cases. As it was shown in [2], every solution of Eq. (1.3) satisfies

$$\beta \leq x_n \leq \beta + \frac{\alpha}{\beta^2} + \frac{1}{\beta}, \quad n = 1, 2, \dots \tag{2.5}$$

Every solution of Eq. (1.4) satisfies $\alpha = x_{n-1}x_nx_{n+1} - \beta x_nx_{n-1} - x_{n-1}$ and so by plugging it in Eq. (1.4), after one iteration, we obtain

$$x_{n+2} = \frac{x_{n-1}x_nx_{n+1} - \beta x_nx_{n-1} - x_{n-1} + \beta x_{n+1}x_n}{x_nx_{n+1}}$$

which means that a solution of Eq. (1.4) satisfies the following embedded third order difference equation on the interval $I = (0, \infty)$ for $n = 0, 1, \dots$

$$x_{n+2} = \frac{\beta x_n}{x_n x_{n+1}} x_{n+1} + \frac{1}{x_n x_{n+1}} x_n + \frac{x_n x_{n+1} - \beta x_n - 1}{x_n x_{n+1}} x_{n-1} = g_{-1} x_{n+1} + g_0 x_n + g_1 x_{n-1}, \tag{2.6}$$

where

$$g_{-1} = \frac{\beta x_n}{x_n x_{n+1}}, \quad g_0 = \frac{1}{x_n x_{n+1}}, \quad g_1 = \frac{x_n x_{n+1} - \beta x_n - 1}{x_n x_{n+1}}.$$

Clearly $g_{-1} + g_0 + g_1 = 1$. By using the estimate (2.5) we obtain

$$g_{-1} = \frac{\beta x_n}{x_n x_{n+1}} \geq \frac{\beta}{\beta + \frac{\alpha}{\beta^2} + \frac{1}{\beta}} = \frac{\beta^3}{\alpha + \beta^3 + \beta} > 0.$$

Furthermore, $g_0 > 0$ and $g_1 \geq 0$ if and only if $x_n x_{n+1} - \beta x_n - 1 \geq 0$, which immediately follows from Eq. (1.4). Thus, all conditions of Theorem 1.6 are satisfied on the interval $I = (0, \infty)$ for Eq. (2.6), which implies that every solution of that equation converges to a finite limit. This implies that the unique positive equilibrium \bar{x} of Eq. (1.4) is global attractor and because it is locally asymptotically stable, see [2], it is also globally asymptotically stable. \square

Remark 2.3. Theorem 1.3 solves [2, Conjectures 11.1].

Proof of Theorem 1.4. The proof of the theorem was given in [2] in the case where $\alpha \leq A$. Here we give a proof for the remaining case $\alpha > A$. As it was shown in [2, Lemma 3.5], every solution of Eq. (1.3) eventually satisfies

$$\frac{A}{\alpha} \leq x_n \leq \frac{\alpha}{A}, \quad n = N, N + 1, \dots \tag{2.7}$$

Without loss of generality we can assume that $N = 1$ in (2.7). Every solution of Eq. (1.5) satisfies $\alpha = (A + x_{n-1}x_n)x_{n+1} - x_{n-1}$ and so by plugging it in Eq. (1.5), after one iteration, we obtain

$$x_{n+2} = \frac{(A + x_{n-1}x_n)x_{n+1} - x_{n-1} + x_n}{A + x_n x_{n+1}}$$

which means that a solution of Eq. (1.4) satisfies the following embedded third order difference equation on the interval $I = [0, \infty)$ for $n = 0, 1, \dots$

$$x_{n+2} = g_{-1}x_{n+1} + g_0x_n + g_1x_{n-1}, \tag{2.8}$$

where

$$g_{-1} = \frac{A}{A + x_n x_{n+1}}, \quad g_0 = \frac{1}{A + x_n x_{n+1}}, \quad g_1 = \frac{x_n x_{n+1} - 1}{A + x_n x_{n+1}}.$$

By using the estimate (2.7) we obtain

$$g_{-1} = \frac{A}{A + x_n x_{n+1}} \geq \frac{A}{A + \left(\frac{\alpha}{A}\right)^2} > 0.$$

Furthermore, $g_0 > 0$ and $g_1 \geq 0$ if and only if $x_n x_{n+1} - 1 \geq 0$, which, by using Eq. (1.5) and estimate (2.7) reduces to $x_n \geq \frac{A}{\alpha}$, which is satisfied. Finally, $g_{-1} + g_0 + g_1 = 1$.

Thus, all conditions of Theorem 1.6 are satisfied on the interval $I = [0, \infty)$, which implies that the unique positive equilibrium \bar{x} is a global attractor and because it is locally asymptotically stable, see [2], it is also globally asymptotically stable. \square

Remark 2.4. Theorem 1.4 solves [2, Conjectures 3.7].

Proof of Theorem 1.5. Note that the invariant (1.7) was obtained in [1]. To complete this proof we will use Theorem 1.7 and Morse’s lemma, see [14]. The straight-forward computation shows that the invariant $H(u, v)$ given with Eq. (1.7) attains the unique minimum, in the first quadrant, at the equilibrium point $E(\bar{x}, \bar{x})$. Furthermore, the invariant $H(u, v)$ satisfies all conditions of Morse’s lemma [14] and so the level sets $H(u, v) = H(x_{-1}, x_0)$ are diffeomorphic to circles and so are simple closed curves. Indeed, the determinant of the Hessian matrix of $H(u, v)$ evaluated at $E(\bar{x}, \bar{x})$ has the form:

$$\det \text{Hessian}(E) = \frac{4\alpha^2 - \bar{x}^6}{\bar{x}^6} = \frac{(\alpha + \bar{x}^2)(2\alpha + \bar{x}^3)}{\bar{x}^6} > 0.$$

Thus, all solutions stays on the family of closed curves and so are bounded and can not approach the equilibrium point.

In view of Theorem 1.7 the corresponding Lyapunov function $V(u, v)$ has the form

$$V(u, v) = H(u, v) - H(\bar{x}, \bar{x}) = H(u, v) - \frac{\bar{x}^2 + 3\alpha}{\bar{x}}$$

which proves the stability of the equilibrium.

It should be mentioned that the unique positive equilibrium point is elliptic fixed point, that is, the characteristic roots of the linearized equation around the equilibrium are complex conjugate numbers on the unit circle. This indicates that the dynamics of Eq. (1.6) is very complicated and similar to the one of Lyness’ equation, see [4, 5]. \square

Remark 2.5. Theorem 1.5 solves [1, Conjectures 3.1 and 3.2].

3 Some Extensions to Higher Order Difference Equations

By using Theorem 1.6 we will extend Theorems 1.1 and 1.2 to the following difference equations:

$$x_{n+1} = \frac{\alpha}{1 + \prod_{i=0}^k x_{n-i}}, \quad n = 0, 1, \dots, \tag{3.1}$$

and

$$x_{n+1} = \frac{\alpha + \prod_{i=0}^k x_{n-i}}{A + \prod_{i=0}^k x_{n-i}}, \quad n = 0, 1, \dots, \tag{3.2}$$

where all parameters are positive and the initial conditions are nonnegative.

Theorem 3.1. *The unique positive equilibrium of Eq. (3.1) is the global attractor.*

Proof. The equilibrium solutions \bar{x} of Eq. (3.1) satisfy

$$x^{k+1} + x - \alpha = 0$$

and so by Descartes rule of signs there is the unique positive equilibrium.

Every solution of Eq. (3.1) satisfies

$$\frac{\alpha}{1 + \alpha^{k+1}} \leq x_n \leq \alpha, \quad n = 1, 2, \dots \quad (3.3)$$

Every solution of Eq. (3.1) satisfies $\alpha = x_{n+1}(1 + \prod_{i=0}^k x_{n-i})$ and so by using it in the first iterate of Eq. (3.1) we obtain

$$x_{n+2} = \frac{x_{n+1}(1 + \prod_{i=0}^k x_{n-i})}{1 + \prod_{i=0}^k x_{n+1-i}}.$$

After some regrouping we obtain that a solution of Eq. (3.1) satisfies the following equation on the interval $I = [0, \infty)$ for $n = 0, 1, \dots$

$$x_{n+2} = \frac{1}{1 + \prod_{i=0}^k x_{n+1-i}} x_{n+1} + \frac{\prod_{i=0}^k x_{n+1-i}}{1 + \prod_{i=0}^k x_{n+1-i}} x_{n-k} = g_{-1} x_{n+1} + g_k x_{n-k}, \quad (3.4)$$

where

$$g_{-1} = \frac{1}{1 + \prod_{i=0}^k x_{n+1-i}}, \quad g_k = \frac{\prod_{i=0}^k x_{n+1-i}}{1 + \prod_{i=0}^k x_{n+1-i}}.$$

Clearly $g_{-1} + g_k = 1$. By using the estimate (3.3) we obtain

$$g_{-1} = \frac{1}{1 + \prod_{i=0}^k x_{n+1-i}} \geq \frac{1}{1 + \alpha^{k+1}} > 0,$$

and so all conditions of Theorem 1.6 are satisfied, which implies that the unique positive equilibrium \bar{x} is global attractor. \square

Theorem 3.2. *Assume that $\alpha \geq A$. Then every solution of Eq. (3.2) with nonnegative initial conditions converges to a positive equilibrium.*

Proof. The equilibrium solutions \bar{x} of Eq. (3.2) satisfy

$$x^{k+2} - x^{k+1} + Ax - \alpha = 0$$

and so by Descartes rule of signs there is either one or three positive equilibrium points.

Every solution of Eq. (3.2) satisfies the estimate (2.3).

Every solution of Eq. (3.2) satisfies $\alpha = x_{n+1}(A + \prod_{i=0}^k x_{n-i}) - \prod_{i=0}^k x_{n-i}$ and so by using it in the first iterate of Eq. (3.2) we obtain

$$x_{n+2} = \frac{x_{n+1}(A + \prod_{i=0}^k x_{n-i}) - \prod_{i=0}^k x_{n-i} + \prod_{i=0}^k x_{n+1-i}}{A + \prod_{i=0}^k x_{n+1-i}}.$$

After some regrouping we obtain that a solution of Eq. (3.2) satisfies the following equation on the interval $I = [0, \infty)$ for $n = 0, 1, \dots$

$$x_{n+2} = \frac{A + \prod_{i=1}^k x_{n+1-i}}{A + \prod_{i=0}^k x_{n+1-i}} x_{n+1} + \frac{(x_{n+1} - 1) \prod_{i=0}^{k-1} x_{n-i}}{A + \prod_{i=0}^k x_{n+1-i}} x_{n-k} = g_{-1} x_{n+1} + g_k x_{n-k}, \quad (3.5)$$

where

$$g_{-1} = \frac{A + \prod_{i=1}^k x_{n+1-i}}{A + \prod_{i=0}^k x_{n+1-i}}, \quad g_k = \frac{(x_{n+1} - 1) \prod_{i=0}^{k-1} x_{n-i}}{A + \prod_{i=0}^k x_{n+1-i}}.$$

Clearly $g_{-1} + g_k = 1$. By using the estimate (2.3) we obtain

$$g_{-1} = \frac{A + \prod_{i=1}^k x_{n+1-i}}{A + \prod_{i=0}^k x_{n+1-i}} \geq \frac{A}{A + \left(\frac{\max\{\alpha, 1\}}{\min\{A, 1\}}\right)^{k+1}} > 0.$$

Furthermore, $g_k \geq 0$ if and only if $x_{n+1} \geq 1$, which by Eq. (3.2) is equivalent to $\alpha \geq A$. Thus all conditions of Theorem 1.6 are satisfied and so every solution converges to a finite limit, that is to an equilibrium. \square

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