International Journal of Difference Equations
ISSN 0973-6069, Volume 7, Number 2, pp. 139–152 (2012)
http://campus.mst.edu/ijde

On the Character of q-quadratic Transformations

Thomas Ernst
Uppsala University
Department of Mathematics
P.O. Box 480, SE-751 06 Uppsala, Sweden
thomas@math.uu.se

Abstract

We show that q-quadratic transformations can be divided into two classes. The first one, which uses the q-binomial theorem in the proof, gives a formula which holds numerically for certain values of the parameters. The second class gives formal equalities. We prove two new q-quadratic transformations corresponding to examples of Rainville.

AMS Subject Classifications: 33D15, 33F05, 33D05.
Keywords: q-hypergeometric series, quadratic transformation, numerical values.

1 Introduction

The quadratic q-hypergeometric transformations, q-analogues of the quadratic hypergeometric transformations are not well known. And even if you see such an equation, it is not easy to understand what the equation means. The aim of this paper is to explain the character of these transformations on the basis of six examples. We will see that there are two categories of quadratic q-hypergeometric transformations. The first category is illustrated by cases one to four, the q-binomial theorem is used in the proof. The first category has good possibilities for numerical confirmation. The second category is illustrated with cases five and six, higher summation formulas are used in the proof. The second category are formal equalities.

In general, the q-case gives a larger convergence region. We also illustrate with an example of Whipple [18] where the proof almost succeeds.

This article is organized as follows. In this section we give the first definitions and introduce the q-hypergeometric functions. In section two and three follow two well-known transformations, for the last proof we refer to [6]. In section three follows an extensive investigation of the numerical character of this transformation.

Received September 2, 2012; Accepted October 30, 2012
Communicated by Martin Bohner

In the fourth section, we give a q-analogue of an example by Rainville [16], which has a similar numerical character. Then follows a similar example in the fifth section of Srivastava and Jain. In the final section follows a formal q-analogue of an example by Rainville [16], which explains a transformation formula due to Whipple [18].

We first briefly describe the q-umbral calculus [3]– [5]. This method is a combination of ideas from Heine 1846 [12] and Gasper–Rahman [10]. The advantages of this method have been summarized in [3, p. 495].

Definition 1.1. Let $\delta > 0$ be an arbitrary small number. We will use the following branch of the logarithm: $-\pi + \delta < \mathrm{Im}(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane. The power function is defined by $q^a \equiv e^{a\log(q)}$. The variables $a, b, c, \ldots \in \mathbb{C}$ denote certain parameters. The variables i, j, k, l, m, n, p, r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit. The q-analogues of a complex number a, and the factorial are defined as follows:

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{0,1\}, \tag{1.1}$$

$$\{n\}_q! \equiv \prod_{k=1}^{n}\{k\}_q,\ \{0\}_q!,\ q \in \mathbb{C}\backslash\{0,1\}. \tag{1.2}$$

Definition 1.2. The q-shifted factorial is defined by

$$\langle a;q\rangle_n \equiv \prod_{m=0}^{n-1}(1-q^{a+m}). \tag{1.3}$$

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notation

$$\langle a,b;q\rangle_n \equiv \langle a;q\rangle_n\langle b;q\rangle_n. \tag{1.4}$$

Furthermore

$$(a;q)_\infty \equiv \prod_{m=0}^{\infty}(1-aq^m),\ 0<|q|<1. \tag{1.5}$$

$$(a;q)_\alpha \equiv \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty},\ a \neq q^{-m-\alpha}, m = 0,1,\ldots. \tag{1.6}$$

The following notation is often used when we have long exponents.

$$\mathrm{QE}(x) \equiv q^x. \tag{1.7}$$

Definition 1.3. The operator

$$\widetilde{\ }: \frac{\mathbb{C}}{\mathbb{Z}} \to \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{\pi i}{\log q}. \tag{1.8}$$

A consequence of (1.8) is

$$\widetilde{\langle a; q \rangle_n} = \prod_{m=0}^{n-1} (1 + q^{a+m}). \tag{1.9}$$

To be able to treat a general root of unity, we will need another generalization of the tilde operator. In the equations (1.10) to (1.21) we assume that $(m, l) = 1$.

Definition 1.4. The generalized tilde operator

$$\widetilde{\tfrac{m}{l}} : \frac{\mathbb{C}}{\mathbb{Z}} \to \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{2\pi i m}{l \log q}. \tag{1.10}$$

This means

$$\langle \widetilde{\tfrac{m}{l}} a; q \rangle_n = \prod_{m=0}^{n-1} (1 - e^{2\pi i \frac{m}{l}} q^{a+m}). \tag{1.11}$$

Furthermore we define

$$\widetilde{{}^{\frac{m}{l}}\langle a; q \rangle_n} \equiv \langle \widetilde{\tfrac{m}{l}} a; q \rangle_n. \tag{1.12}$$

We will also need another generalization of the tilde operator

$$\widetilde{{}_k\langle a; q \rangle_n} \equiv \prod_{m=0}^{n-1} (\sum_{i=0}^{k-1} q^{i(a+m)}). \tag{1.13}$$

This leads to

$$\widetilde{{}_2\langle a; q \rangle_n} \equiv \widetilde{\langle a; q \rangle_n}. \tag{1.14}$$

$$\widetilde{{}_1\langle a; q \rangle_n} \equiv 1, \tag{1.15}$$

and

$$\langle \widetilde{{}_k a}; q \rangle_n \equiv \widetilde{{}_k\langle a; q \rangle_n}. \tag{1.16}$$

The $\triangle(q; l; \lambda)$ operator is defined by

$$\langle \triangle(q; l; \lambda); q \rangle_n \equiv \prod_{m=0}^{l-1} \langle \frac{\lambda + m}{l}; q \rangle_n \times_l \langle \widetilde{\frac{\lambda + m}{l}}; q \rangle_n. \tag{1.17}$$

The following, simple rules follow from (1.10). Some of them were previously known in other representation in [10].

Theorem 1.5.

$$\widetilde{{}_{\frac{m}{l}}a} \pm b \equiv \widetilde{{}_{\frac{m}{l}}(a \pm b)} \text{ mod } \frac{2\pi i}{\log q}, \tag{1.18}$$

$$\sum_{k=1}^{n} \widetilde{{}_{\frac{1}{n}} \pm a_k} \equiv \sum_{k=1}^{n} \pm a_k \text{ mod } \frac{2\pi i}{\log q}, \tag{1.19}$$

$$\frac{m}{l} \times \widetilde{a} \equiv_{\frac{m}{2l}} \widetilde{\frac{am}{l}} \text{ mod } \frac{2\pi i}{\log q}, \tag{1.20}$$

$$\text{QE}(\widetilde{{}_{\frac{m}{l}}a}) = \text{QE}(a)e^{\frac{2\pi i m}{l}}, \tag{1.21}$$

where the second equation is a consequence of the fact that we work mod$\frac{2\pi i}{\log q}$.

Theorem 1.6.

$$\langle \widetilde{a}; q^2 \rangle_n = \langle \widetilde{{}_{\frac{1}{4}}a}, \widetilde{{}_{\frac{3}{4}}a}; q \rangle_n. \tag{1.22}$$

$$\langle a; q^p \rangle_n = \prod_{k=0}^{p-1} \langle \widetilde{{}_{\frac{k}{p}}a}; q \rangle_n, \text{ where } p \text{ is an uneven prime number.} \tag{1.23}$$

$$\langle a; q^k \rangle_n = \langle a; q \rangle_n \times_k \widetilde{\langle a; q \rangle}_n. \tag{1.24}$$

This leads to the following q-analogue of Rainville [16, p. 22, (2)].

Theorem 1.7.

$$\langle a; q \rangle_{kn} = \prod_{m=0}^{k-1} \langle \frac{a+m}{k}; q \rangle_n \times_k \langle \widetilde{\frac{a+m}{k}}; q \rangle_n. \tag{1.25}$$

Definition 1.8. We define a q-hypergeometric series by

$$\begin{aligned} &{}_{p+p'}\phi_{r+r'}(\hat{a}_1, \ldots, \hat{a}_p; \hat{b}_1, \ldots, \hat{b}_r | q, z || s_1, \ldots, s_{p'}; t_1, \ldots, t_{r'}) \\ &\equiv {}_{p+p'}\phi_{r+r'} \left[\begin{matrix} \hat{a}_1, \ldots, \hat{a}_p \\ \hat{b}_1, \ldots, \hat{b}_r \end{matrix} | q, z || \begin{matrix} s_1, \ldots, s_{p'} \\ t_1, \ldots, t_{r'} \end{matrix} \right] \\ &= \sum_{n=0}^{\infty} \frac{\langle \hat{a}_1; q \rangle_n \ldots \langle \hat{a}_p; q \rangle_n}{\langle 1; q \rangle_n \langle \hat{b}_1; q \rangle_n \ldots \langle \hat{b}_r; q \rangle_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+r+r'-p-p'} \times \\ &\quad z^n \prod_{k=1}^{p'} (s_k; q)_n \prod_{k=1}^{r'} (t_k; q)_n^{-1}, \end{aligned} \tag{1.26}$$

where

$$\widehat{a} \equiv a \vee \widetilde{a} \vee \widetilde{{}_{\frac{m}{l}}a} \vee_k \widetilde{a} \vee \triangle(q; l; \lambda). \tag{1.27}$$

Remark 1.9. In a few cases, when $0 < |q| < 1$, the parameter $\hat{a}$ in (1.26) will be the real plus infinity. If we want to be formal, we could introduce a symbol ∞_{H} with property

$$\langle \infty_{\mathrm{H}}; q \rangle_n = \langle \infty_{\mathrm{H}} + \alpha; q \rangle_n = 1, \ \alpha \in \mathbb{C}, \ 0 < |q| < 1. \tag{1.28}$$

The symbol ∞_{H} corresponds to the parameter 0 in [10, p. 4]. We will denote ∞_{H} by ∞.

The factors $\langle \widehat{a}; q \rangle_n$ then correspond to multiplication by 1.

Remark 1.10. The parameters $\hat{a}_i$ and $\hat{b}_i$ on the left-hand side of | in (1.26) are exponents; they are periodic mod$\dfrac{2\pi i}{\log q}$.

We use the notation $\cong$ for formal equality.

2 First Case

Jackson's [14] q-analogue of the Euler–Pfaff transformation:

$$_2\phi_1(a, b; c|q; z) \cong \frac{1}{(z; q)_a} \, _2\phi_2(a, c - b; c|q; zq^b||-; (zq^a; q)_k), \tag{2.1}$$

$|z| < 1$ and $|\dfrac{z}{z-1}| < 1$. The convergence area in (2.1) for general q is a little larger than indicated. The right-hand side also converges faster than the left-hand side.

3 Second Case

We first write down the two equalities.

Theorem 3.1. *Kummer [1, p. 125 (3.1.4)], [13, p. 169], [2, p. 9 (2)], [15, p. 78 (53)], [16, p. 67 (3)], [10, p. 59 (3.1.4)]:*

$$_2\mathrm{F}_1(a, b; 1 + a - b; x) = (1 - x)^{-a} \, _2\mathrm{F}_1 \left(\tfrac{a}{2}, \tfrac{a+1}{2} - b; 1 + a - b; \frac{-4x}{(1-x)^2} \right). \tag{3.1}$$

Theorem 3.2. *[6] A q-analogue of* (3.1)*:*

$$\begin{aligned} _2\phi_1(a, b; 1 + a - b|q; xq^{\frac{a+1}{2} - b}) &\cong \sum_{m=0}^{\infty} \frac{\langle \frac{a}{2}, \frac{a+1}{2} - b, \widetilde{\frac{a}{2}}, \widetilde{\frac{a+1}{2}}; q \rangle_m (-x)^m q^{-\binom{m}{2}}}{\langle 1, 1 + a - b; q \rangle_m (xq^{-m}; q)_{a+2m}} \\ &\equiv \, _5\phi_3 \left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2} - b, \widetilde{\frac{a}{2}}, \widetilde{\frac{a+1}{2}}, \infty \\ 1 + a - b \end{matrix} |q; x|| \begin{matrix} - \\ (xq^{-k}; q)_{a+2k} \end{matrix} \right]. \end{aligned} \tag{3.2}$$

A more practical form is the finite version

$$
{}_2\phi_1(-n,b;1-n-b|q;xq^{\frac{-n+1}{2}-b}) \cong \sum_{m=0}^{\frac{n}{2}} \frac{\langle \frac{-n}{2}, \frac{-n+1}{2}-b, \widetilde{\frac{-n}{2}}, \widetilde{\frac{-n+1}{2}}; q\rangle_m (-x)^m q^{-\binom{m}{2}}}{\langle 1, 1-n-b; q\rangle_m (xq^{-m};q)_{-n+2m}}
$$

$$
\equiv {}_5\phi_3\left[\begin{matrix} \frac{-n}{2}, \frac{-n+1}{2}-b, \widetilde{\frac{-n}{2}}, \widetilde{\frac{-n+1}{2}}, \infty \\ 1-n-b \end{matrix} |q;x|| \begin{matrix} - \\ (xq^{-k};q)_{-n+2k} \end{matrix}\right], \; n \text{ even.} \tag{3.3}
$$

We always treat the two cases $x < 0$ and $x > 0$ separately. The value $x = 1$ is of course not allowed. The right-hand side of (3.1) has the function value $\frac{-4x}{(1-x)^2}$. If we solve the inequality $|4x| < (1-x)^2$ with respect to $x < 1$, we arrive at the solutions $x < -1$ and $-1 < x < 3 - 2\sqrt{2} \approx .17$. It turns out that this is very important for the q-case. The hypergeometric formula (3.1) has three variables. Now let $a = 4.3$ and $b = 2.7$. For $x > 0$, the above formula applies until $x = .96$ and for $x < 0$ to $x = -.9895$. The right-hand side of (3.1) $= F(x)$ is a meromorphic continuation of the left-hand side for $x < -1$ with the following values:

x	$F(x)$
-1.1	0.0361114
$-3.$	0.00244793
$-6.$	0.000226233

Now let $q = .9$. For $x < 0$ the left-hand side of (3.2) converges until $x = -.994$ and has the values

x	$G_{.9}(x)$
$-.99$	0.0650276
$-.994.$	0.0644883

The right-hand side of (3.2) $= F_q(x)$ is a meromorphic continuation of the left side for $x < 0$ with the following values

x	$F_{.9}(x)$
$-5.$	0.000797
$-7.$	0.000241997
$-9.$	0.0000952031

For $x > 0$ the right-hand side of (3.2) has singularities for $x = q^m, m \geq 0,\ 0 < q < 1$. Put $q = .9$ again. We show a table for the left side of (3.2) $= G_q(x)$ and $F_q(x)$. For $F_q(x)$ we have also indicated the partial sum of k, see below. The meromorphic continuation

$F_q(x)$ is very sensitive.

x	$G_{.9}(x)$	$F_{.9}(x)$	k
.005	1.01914	1.01914	2
.01	1.03873	1.03873	2
.015	1.05879	1.05879	2
.02	1.07932	1.07932	2
.025	1.10034	1.10034	3
.03	1.12187	1.12187	3
.035	1.14391	1.14391	3
.04	1.16648	1.16648	4
.045	1.1896	1.1896	4
.05	1.21329	1.21329	4
.055	1.23755	1.23755	4
.06	1.26242	1.26242	5
.065	1.28789	1.28789	4
.07	1.314	1.314	6
.075	1.34076	1.34076	9
.08	1.3682	1.3682	7
.085	1.39632	1.39632	6
.09	1.42515	1.42515	6
.095	1.45472	1.45472	7

The rest of the table follows below. As explained above, we seemingly cannot go further than $x = .17$. The point $x = .15$ is singular, since $.9^{18} \approx .15$.

x	$G_{.9}(x)$	$F_{.9}(x)$	k
.1	1.48504	1.48504	8
.105	1.51614	1.51614	9
.11	1.54804	1.54804	10
.115	1.58077	1.58077	11
.12	1.61435	1.61436	11
.125	1.64881	1.64883	11
.13	1.68417	1.68412	10
.135	1.72047	1.72055	9
.14	1.75773	1.75787	9
.145	1.79598	1.79574	8
.15	1.83527	∞	–
.155	1.87561	1.87504	8
.16	1.91704	1.91801	9
.165	1.9596	1.95833	8
.17	2.00333	2.00154	6

4 Third Case

Theorem 4.1. *A q-analogue of [16, Ex. 9, p. 69]*

$$\begin{aligned}&{}_3\phi_1(-n,a,\infty;1+a+n|q;zq^n)\cong\sum_{m=0}^{\infty}\frac{\langle\frac{a}{2},\frac{a+1}{2},\widetilde{\frac{a}{2}},\widetilde{\frac{a+1}{2}};q\rangle_m(-z)^m q^{-\binom{m}{2}+mn}}{\langle 1,1+a+n;q\rangle_m(zq^{-m};q)_{a+2m}}\\&\equiv\ {}_5\phi_3\left[\begin{matrix}\frac{a}{2},\frac{a+1}{2},\widetilde{\frac{a}{2}},\widetilde{\frac{a+1}{2}},\infty\\1+a+n\end{matrix}|q;zq^n||\begin{matrix}-\\(zq^{-k};q)_{a+2k}\end{matrix}\right].\end{aligned}\tag{4.1}$$

Proof. By the q-binomial theorem, the right side can be written as

$$\sum_{m,k=0}^{\infty}\frac{\langle\frac{a}{2},\frac{a+1}{2},\widetilde{\frac{a}{2}},\widetilde{\frac{a+1}{2}};q\rangle_m\langle a+2m;q\rangle_k(-z)^m z^k q^{-mk-\binom{m}{2}+mn}}{\langle 1,1+a+n;q\rangle_m\langle 1;q\rangle_k}.\tag{4.2}$$

The coefficient of z^l is

$$\begin{aligned}&\sum_{m=0}^{l}\frac{\langle\frac{a}{2},\frac{a+1}{2},\widetilde{\frac{a}{2}},\widetilde{\frac{a+1}{2}};q\rangle_m\langle a+2m;q\rangle_{l-m}(-1)^m q^{\frac{m^2+m}{2}+m(n-l)}}{\langle 1,1+a+n;q\rangle_m\langle 1;q\rangle_{l-m}}.=\\&\frac{\langle a;q\rangle_l}{\langle 1;q\rangle_l}\sum_{m=0}^{l}\frac{\langle a+l,-l;q\rangle_m}{\langle 1+a+n,1;q\rangle_m}q^{m(1+n)}=\frac{\langle 1+n-l,a;q\rangle_l}{\langle 1+a+n,1;q\rangle_l}=\\&\frac{\langle -n,a;q\rangle_l}{\langle 1+a+n,1;q\rangle_l}q^{nl-\binom{l}{2}}(-1)^l.\end{aligned}\tag{4.3}$$

This is equal to the coefficient of z^l on the left hand side. □

Now put $a=4.3, n=20, q=.9$.
For $x<0$ we have the following table for the left-hand side of (4.1) $=G_q(x)$

x	$G_{.9}(x)$	x	$G_{.9}(x)$
-1	0.0764614	-1.05	0.0694025
-1.1	0.0631296	-1.15	0.0575488
-1.2	0.0525919	-1.25	0.0482326
-1.3	0.0445264	-1.35	0.0417049
-1.4	0.0403837	-1.45	0.0420015
-1.5	0.0497192	-1.55	0.0702004
-1.6	0.117047	-1.65	0.217254
-1.7	0.423068	-1.75	0.833271

Then the function value increases fast.

The difference $G_q(x)-F_q(x)$ is still relatively small (<0.001) until $x=-1.4$.

For $x > 0$ the right-hand side of (4.1) has singularities for $x = q^m, m \geq 0,\ 0 < q < 1$. Put $n = 5,\ q = .9$. We show a table for the left side of (4.1) $= G_q(x)$ and $F_q(x)$. For $F_q(x)$ we have also indicated the number of terms k.

x	$G_{.9}(x)$	$F_{.9}(x)$	k
.005	1.01133	1.01133	2
.01	1.02278	1.02278	3
.015	1.03437	1.03437	2
.02	1.04608	1.04608	3
.025	1.05792	1.05792	3
.03	1.06989	1.06989	3
.035	1.082	1.082	4
.04	1.09424	1.09424	4
.045	1.10662	1.0662	4
.05	1.11913	1.11913	5
.055	1.13179	1.13179	4
.06	1.14458	1.14458	5
.065	1.15751	1.15751	5
.07	1.17059	1.17059	5
.075	1.18381	1.18381	6
.08	1.19717	1.19717	5
.085	1.21069	1.21069	6
.09	1.22435	1.22435	6
.095	1.23816	1.23816	6

5 Fourth Case

The following transformation follows from a simple substitution in a known quadratic transformation in [16, p. 65, (1)].

$$ {}_2\mathrm{F}_1(b,a;2a;2\frac{x}{1+x}) = (1+x)^b\ {}_2\mathrm{F}_1\left(\tfrac{b}{2},\tfrac{b+1}{2};a+\tfrac{1}{2};x^2\right). \tag{5.1} $$

[17, p. 222, 4.6]. A q-analogue of (5.1).

$$ {}_3\phi_2\left[\begin{matrix} b,a,\widetilde{a} \\ 2a \end{matrix}\ |q;x||\ \begin{matrix} - \\ (-xq^b;q)_k \end{matrix}\right] = (-x;q)_b\ {}_2\phi_1\left[\begin{matrix} \frac{b}{2},\frac{b+1}{2} \\ a+\frac{1}{2} \end{matrix}\ |q^2;x^2\right]. \tag{5.2} $$

The value $x = -1$ is of course not allowed. The left-hand side of (5.1) has the function value $\dfrac{2x}{(1+x)}$. If we solve the inequality $|2x| < |1+x|$ with respect to $|x| < 1$, we arrive at the solution $-\dfrac{1}{3} < x < 1$. It again turns out that this is very important for the q-case.

Now put $a = 4.3, b = 2.8, q = .9$. We show a table for the difference left side minus right-hand side ($D = G_{.9}(x) - F_{.9}(x)$) in (5.2):

x	$D(10^{-15})$	x	$D(10^{-15})$
.1	1.9984	−.1	1.11022
.2	−3.10862	−.2	−3.33067
.3	2.22045	−.3	.94369
.4	5.32907	−.4	.555112
.5	5.32907	−.5	.138778
.6	−1.59872	−.6	−19.0403
.7	1.77636	−.7	334.857
.8	−1.77636	−.8	5600.26
.9	−65.7252	−.9	−11154.

For $x < 0$ the function $G_{.9}(x)$ converges until -1 with the value 3.3412 $\times 10^6$. For $x > 0$ $G_{.9}(x)$ converges until .95 with the value 23.

For $x < 0$ the function $F_{.9}(x)$ converges until -.95 with the value 0.008764. For $x > 0$ $F_{.9}(x)$ converges until .925 with the value 36.5454.

6 Fifth Case

First follows the hypergeometric transformation.

Theorem 6.1. *Gauss [11, p. 225 (101)], [15, p. 78 (51)], [10, p. 68 (3.5.6)], [16, p. 65 (12)]:*

$$ {}_2\mathrm{F}_1(a,b;1+a-b;x^2) = (1-x)^{-2a}\,{}_2\mathrm{F}_1\left(a, a+\tfrac{1}{2}-b; 1+2a-2b; \frac{-4x}{(1-x)^2}\right). \quad (6.1)$$

Theorem 6.2. *[6] A q-analogue of* (6.1)*:*

$$ {}_2\phi_1\left[\begin{matrix} a,-l \\ a+1+l, \end{matrix} |q^2, y^2\right] \cong $$
$$ {}_5\phi_3\left[\begin{matrix} a+l+\frac{1}{2}, \widetilde{a+l+\frac{1}{2}}, a, \widetilde{a}, \infty \\ 2a+1+2l \end{matrix} |q; yq^{-a-l-\frac{3}{2}}|| \begin{matrix} - \\ (yq^{-k-a-\frac{1}{2}-l};q)_{2a+2k} \end{matrix}\right], \; l \in \mathbb{N}. \quad (6.2)$$

This is a formal equation that illustrates this character of q-calculus.

7 Sixth Case

A q-analogue of [16, (1), p.90] influenced by Whipple [18].

Theorem 7.1. *If neither $a-b$ nor $a-c$ nor a is a negative integer,*

$$
{}_3\phi_2(a,b,c;1+a-b,1+a-c|q;xq^{1+a-b-c}) \cong
{}_6\phi_4\left[\begin{array}{c}\triangle(q;2;a),1+a-b-c,\infty \\ 1+a-b,1+a-c\end{array}|q;x||\begin{array}{c}- \\ (xq^{-k};q)_{a+2k}\end{array}\right]. \tag{7.1}
$$

Proof. We put $\Gamma_q\,[] \equiv \Gamma_q\left[\begin{array}{c}1+a-b,1+a-c \\ 1+a,1+a-b-c\end{array}\right]$. Then

$$
\begin{aligned}
&\sum_k^{\infty}\frac{\langle a,b,c;q\rangle_k x^k q^{k(1+a-b-c)}}{\langle 1,1+a-b,1+a-c;q\rangle_k}\\
&=\Gamma_q\,[]\sum_k^{\infty}\frac{\langle a,b,c;q\rangle_k x^k q^{k(1+a-b-c)}}{\langle 1;q\rangle_k\langle 1+a;q\rangle_{2k}}\Gamma_q\left[\begin{array}{c}1+a+2k,1+a-b-c \\ 1+a-b+k,1+a-c+k\end{array}\right]\\
&=\Gamma_q\,[]\sum_k^{\infty}{}_2\phi_1\left[\begin{array}{c}b+k,c+k \\ 1+a+2k\end{array}|q;q^{1+a-b-c}\right]\frac{\langle a,b,c;q\rangle_k x^k q^{k(1+a-b-c)}}{\langle 1;q\rangle_k\langle 1+a;q\rangle_{2k}}\\
&=\Gamma_q\,[]\sum_{n,k=0}^{\infty}\frac{\langle a;q\rangle_k\langle b,c;q\rangle_{n+k}x^k q^{(n+k)(1+a-b-c)}}{\langle 1;q\rangle_k\langle 1;q\rangle_n\langle 1+a;q\rangle_{n+2k}}\\
&=\Gamma_q\,[]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\langle a;q\rangle_k\langle b,c;q\rangle_n x^k q^{n(1+a-b-c)}}{\langle 1;q\rangle_k\langle 1;q\rangle_{n-k}\langle 1+a;q\rangle_{n+k}}\\
&=\Gamma_q\,[]\sum_{n=0}^{\infty}{}_3\phi_1(-n,a,\infty;1+a+n|q;xq^n)\frac{\langle b,c;q\rangle_n q^{n(1+a-b-c)}}{\langle 1,1+a;q\rangle_n}\\
&=\Gamma_q\,[]\sum_{n=0}^{\infty}{}_5\phi_3\left[\begin{array}{c}\frac{a}{2},\frac{a+1}{2},\widetilde{\frac{a}{2}},\widetilde{\frac{a+1}{2}},\infty \\ 1+a+n\end{array}|q;zq^n||\begin{array}{c}- \\ (xq^{-k};q)_{a+2k}\end{array}\right]\frac{\langle b,c;q\rangle_n q^{n(1+a-b-c)}}{\langle 1,1+a;q\rangle_n}\\
&=\Gamma_q\,[]\sum_{n,k=0}^{\infty}\frac{(-1)^k\langle a;q\rangle_{2k}\langle b,c;q\rangle_n q^{-\binom{k}{2}+n(1+a+k-b-c)}x^k}{\langle 1;q\rangle_k\langle 1;q\rangle_n\langle 1+a;q\rangle_{n+k}(xq^{-k};q)_{a+2k}}\\
&=\Gamma_q\,[]\sum_{k=0}^{\infty}{}_2\phi_1\left[\begin{array}{c}b,c \\ 1+a+k\end{array}|q;q^{1+a+k-b-c}\right]\frac{(-1)^k\langle a;q\rangle_{2k}q^{-\binom{k}{2}}x^k}{\langle 1;q\rangle_k\langle 1+a;q\rangle_k(xq^{-k};q)_{a+2k}}\\
&=\Gamma_q\,[]\sum_{k=0}^{\infty}\Gamma_q\left[\begin{array}{c}1+a+k,1+a-b-c+k \\ 1+a-b+k,1+a-c+k\end{array}\right]\frac{(-1)^k\langle a;q\rangle_{2k}q^{-\binom{k}{2}}x^k}{\langle 1;q\rangle_k\langle 1+a;q\rangle_k(xq^{-k};q)_{a+2k}}\\
&=\sum_{k=0}^{\infty}\frac{\langle\triangle(q;2;a),1+a-b-c;q\rangle_k(-x)^k q^{-\binom{k}{2}}}{\langle 1,1+a-b,1+a-c;q\rangle_k(xq^{-k};q)_{a+2k}}=\text{RHS}.
\end{aligned}
\tag{7.2}
$$

□

We next try to find a q-analogue of the Whipple [18] quadratic transformation

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} -n, b, c \\ 1-b-n, 1-c-n \end{matrix} \; x\right] \\
&= (1-x)^n \, {}_3\phi_2\left[\begin{matrix} -\frac{n}{2}, \frac{-n+1}{2}, 1-b-c-n \\ 1-b-n, 1-c-n \end{matrix} \; \frac{-4x}{(1-x)^2}\right].
\end{aligned}
\tag{7.3}
$$

We follow the proof from Rainville [16, (1), p. 89]. We expand the function

$$
A \equiv \, {}_2\phi_1\left[\begin{matrix} b, c \\ b+c \end{matrix} \; |q, t(1 \oplus_q xt \boxminus_q x)\right]
$$

in two different ways.

$$
\begin{aligned}
A &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\langle b, c; q\rangle_n t^n (1 \boxminus_q x)^{n-k}(xt)^k}{\langle b+c; q\rangle_n \langle 1; q\rangle_k \langle 1; q\rangle_{n-k}} = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]} \frac{\langle b, c; q\rangle_{n-k} t^n (1 \boxminus_q x)^{n-2k} x^k}{\langle b+c; q\rangle_{n-k} \langle 1; q\rangle_k \langle 1; q\rangle_{n-2k}} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]} \frac{\langle b, c; q\rangle_n \langle -n; q\rangle_{2k} \langle 1-b-c-n; q\rangle_k (-1)^k t^n (1 \boxminus_q x)^{n-2k} x^k}{\langle 1, b+c; q\rangle_n \langle 1, 1-b-n, 1-c-n; q\rangle_k} q^{kn-3\binom{k}{2}} \\
&= \sum_{n,k=0}^{\infty} \frac{\langle b, c; q\rangle_n \langle -n; q\rangle_{2k} \langle 1-b-c-n; q\rangle_k (-1)^k t^n (1 \boxminus_q x)^n x^{-k}}{\langle 1, b+c; q\rangle_n \langle 1, 1-b-n, 1-c-n; q\rangle_k (1 \boxminus_q x^{-1} q^{1-n})^{2k}} \times \\
&\text{QE}\left(-kn - 3\binom{k}{2} + 2k + \binom{2k}{2}\right) \\
&= \sum_{n=0}^{\infty} \frac{\langle b, c; q\rangle_n t^n (1 \boxminus_q x)^n}{\langle 1, b+c; q\rangle_n} \sum_{k=0}^{\infty} \frac{\langle -n; q\rangle_{2k} \langle 1-b-c-n; q\rangle_k (-1)^k x^k}{\langle 1, 1-b-n, 1-c-n; q\rangle_k (x \boxminus_q q^{1-n})^{2k}} \times \\
&\text{QE}\left(-kn + 3k + \binom{k}{2}\right) \\
&= \sum_{n=0}^{\infty} \frac{\langle b, c; q\rangle_n t^n (1 \boxminus_q x)^n}{\langle 1, b+c; q\rangle_n} \times \\
&{}_5\phi_5\left[\begin{matrix} \triangle(q; 2; -n), 1-b-c-n \\ 1-b-n, 1-c-n, \infty \end{matrix} \; |q; x^{-1}q^{3-n} || \begin{matrix} - \\ (x^{-1}q^{1-n}; q)_{2k} \end{matrix}\right].
\end{aligned}
\tag{7.4}
$$

By a slightly different calculation we have

$$A=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\langle b,c;q\rangle_n t^n(-t\boxplus_q 1)^k(-x)^k}{\langle b+c;q\rangle_n\langle 1;q\rangle_k\langle 1;q\rangle_{n-k}}=\sum_{n,k=0}^{\infty}\frac{\langle b,c;q\rangle_{n+k} t^{n+k}(-t\boxplus_q 1)^k(-x)^k}{\langle b+c;q\rangle_{n+k}\langle 1;q\rangle_k\langle 1;q\rangle_n}$$
$$=\sum_{n,k=0}^{\infty}\frac{\langle b+k,c+k;q\rangle_n t^{n+2k}\langle b,c;q\rangle_k(t^{-1};q)_k x^k}{\langle 1,b+c;q\rangle_k\langle 1,b+c+k;q\rangle_n}$$
$$=\sum_{k=0}^{\infty}{}_2\phi_1\left(b+k,c+k;b+c+k|q,t\right)\frac{t^{2k}\langle b,c;q\rangle_k(t^{-1};q)_k x^k}{\langle 1,b+c;q\rangle_k}$$
$$=\sum_{k=0}^{\infty}\frac{1}{(t;q)_k}\,{}_2\phi_1\left(b,c;b+c+k|q,tq^k\right)\frac{t^{2k}\langle b,c;q\rangle_k(t^{-1};q)_k x^k}{\langle 1,b+c;q\rangle_k}$$
$$=\sum_{n,k=0}^{\infty}\frac{t^{n+2k}\langle b,c;q\rangle_n\langle b,c;q\rangle_k(t^{-1};q)_k x^k q^{kn}}{(t;q)_k\langle b+c;q\rangle_{n+k}\langle 1;q\rangle_n\langle 1;q\rangle_k}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{t^{n+k}\langle b,c;q\rangle_{n-k}\langle b,c;q\rangle_k(t^{-1};q)_k x^k q^{k(n-k)}}{(t;q)_k\langle b+c;q\rangle_n\langle 1;q\rangle_{n-k}\langle 1;q\rangle_k}. \tag{7.5}$$

We cannot equate the coefficients of t^n in the two expressions for A, and the proof fails.

8 Conclusion

There are a lot of q-quadratic transformations, and the future will show how many they are. For each hypergeometric formula duplicates can occur with quite different properties. The logarithmic method of [7] and [9] will clearly facilitate the comparison to the hypergeometric case.

References

[1] G. E. Andrews, R. Askey and R.Roy, *Special functions.* Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, Cambridge, 1999. xvi+664 pp.

[2] W. N. Bailey, *Generalized hypergeometric series.* Cambridge 1935, reprinted by Stechert-Hafner, New York, 1964.

[3] T. Ernst, A method for q-calculus. *J. nonlinear Math. Physics* **10** No.4 (2003), 487–525.

[4] T. Ernst, Some new formulas involving Γ_q functions. *Rendiconti di Padova* **118**, 159–188 (2007)

[5] T. Ernst, A renaissance for a q-umbral calculus. Proceedings of the International Conference Munich, Germany 25–30 July 2005. World Scientific, (2007)

[6] T. Ernst, Examples of a q-umbral calculus. *Advan. Stud. Contemp. Math.* **16**, No. 1 (2008), 1–22.

[7] T. Ernst, Zur Theorie der Γ_q-Funktion, *Proceedings Jangjeon Math. Soc.* **14** (2011), 91–113.

[8] T. Ernst. q-analogues of general reduction formulas by Buschman and Srivastava, together with the important $\triangle(l;\lambda)$ operator reminding of MacRobert, *Demonstratio Mathematica* **XLIV** (2011), 285–296.

[9] T. Ernst, *A comprehensive treatment of* q*-calculus*, Birkhäuser 2012.

[10] G.Gasper and M.Rahman, *Basic hypergeometric series,* Cambridge, 1990.

[11] C.F. Gauss, *Werke 3*. 1876.

[12] E. Heine, Über die Reihe $1+[(1-q^{\alpha})(1-q^{\beta})]/[(1-q)(1-q^{\gamma})]x+[(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})]/[(1-q)(1-q^2)(1-q^{\gamma}(1-q^{\gamma+1})]x^2+\ldots$ *J. Reine Angew. Math.* **32** (1846), 210–212.

[13] P. Henrici, *Applied and computational complex analysis*. Vol. 2. Special functions—integral transforms—asymptotics—continued fractions. Wiley Interscience [John Wiley & Sons], New York-London-Sydney, 1977.

[14] F. H. Jackson, Transformations of q-series. *Mess. Math.* **39** (1910), 145–153.

[15] E. E. Kummer, Über die hypergeometrische Reihe ... *J. für Math.* **15** (1836), 39–83 and 127–172.

[16] E. D. Rainville, *Special functions.* Reprint of 1960 first edition. Chelsea Publishing Co., Bronx, N.Y., 1971.

[17] H. M. Srivastava and V.K. Jain, q-series identities and reducibility of basic double hypergeometric functions. *Canad. J. Math.* **38**, no. 1, (1986), 215–231.

[18] F. J. W.Whipple, Some transformations of generalized hypergeometric series. Proceedings L. M. S. (2) **26** (1927), 257–272.