International Journal of Difference Equations
ISSN 0973-6069, Volume 7, Number 1, pp. 79–97 (2012)
http://campus.mst.edu/ijde

Oscillatory Behaviour of a Class of Nonlinear Fourth Order Functional Difference Equations

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Abstract

In this work, the solution space of a class of nonlinear fourth order functional difference equations of the form

$$(E)\Delta^2(r(n)\Delta^2((y(n)+p(n)y(\tau(n))))) + q(n)G(y(\alpha(n))) - h(n)H(y(\beta(n))) = 0$$

is studied under the assumption

$$\sum_{n=0}^{\infty} \frac{n}{r(n)} = \infty$$

where $\tau(n) \le n-1$, $\alpha(n) \le n-1$ and $\beta(n) \le n-1$ such that $\lim_{n\to\infty} \tau(n) = \infty = \lim_{n\to\infty} \alpha(n) = \infty = \lim_{n\to\infty} \beta(n)$. Also, the forced equation of (E) is studied for various ranges of $p(n)$.

AMS Subject Classifications: 39A10, 39A12.
Keywords: Oscillation, nonoscillation, neutral, delay difference equations, asymptotic behaviour.

1 Introduction

Consider the fourth order nonlinear functional difference equations of the form

$$\Delta^2(r(n)\Delta^2((y(n)+p(n)y(\tau(n))))) + q(n)G(y(\alpha(n))) - h(n)H(y(\beta(n))) = 0 \quad (1.1)$$

Received April 18, 2011; Accepted June 28, 2012
Communicated by Allan Peterson

and its associated forced equations

$$\Delta^2(r(n)\Delta^2((y(n)+p(n)y(\tau(n))))) + q(n)G(y(\alpha(n))) \\ - h(n)H(y(\beta(n))) = f(n), \tag{1.2}$$

where r, p, q, h and f are real valued functions defined on $N(n_0) = \{n_0, n_0+1, \ldots\}$, $n_0 \geq 0$ such that $r(n) > 0, q(n) > 0, h(n) > 0$ for $n \geq n_0, G$ and $H \in C(R, R)$ are nondecreasing with $uG(u) > 0, vH(v) > 0$ for $u, v \neq 0$, and τ, α, β are increasing functions such that $\tau(n) \leq n-1$, $\alpha(n) \leq n-1$ and $\beta(n) \leq n-1$ and $\lim_{n\to\infty} \tau(n) = \infty = \lim_{n\to\infty} \alpha(n) = \infty = \lim_{n\to\infty} \beta(n)$.

The objective of this work is to study the solution space of (1.1) and (1.2) under the assumption

(H$_0$) $\sum_{n=0}^{\infty} \frac{n}{r(n)} = \infty.$

Because (1.1)–(1.2) are highly nonlinear, it is interesting to study both equations under (H$_0$). If $h(n) \equiv 0$, then (1.1) and (1.2) reduce to

$$\Delta^2(r(n)\Delta^2((y(n)+p(n)y(\tau(n))))) + q(n)G(y(\alpha(n))) = 0 \tag{1.3}$$

and

$$\Delta^2(r(n)\Delta^2((y(n)+p(n)y(\tau(n))))) + q(n)G(y(\alpha(n))) = f(n) \tag{1.4}$$

respectively. In [7], the author has studied (1.3) and (1.4) under the assumption (H$_0$) and $\tau(n) = n-\tau, \alpha(n) = n-\alpha$. It shows that if $q(n) < 0$, then also we can predict analogous results for oscillation and asymptotic behaviour of solutions of (1.3) and (1.4). But the problem is still left if $q(n)$ changes sign. In particular, if $q(n) = q^+(n) - q^-(n)$, where $q^+(n) = \max\{0, q(n)\}$ and $q^-(n) = \max\{-q(n), 0\}$, then (1.3) and (1.4) can be viewed as

$$\Delta^2(r(n)\Delta^2((y(n)+p(n)y(\tau(n))))) + q^+(n)G(y(\alpha(n))) \\ - q^-(n)G(y(\alpha(n))) = 0 \tag{1.5}$$

and

$$\Delta^2(r(n)\Delta^2((y(n)+p(n)y(\tau(n))))) + q^+(n)G(y(\alpha(n))) \\ - q^-(n)G(y(\alpha(n))) = f(n) \tag{1.6}$$

respectively. Clearly, (1.5)–(1.6) is a particular case of (1.1)–(1.2) and the present work is devoted to study the more general functional difference equations of the type (1.1)–(1.2) rather than (1.5)–(1.6). On the other hand, (1.3)–(1.4) is a special case of (1.1)–(1.2) and hence the study of (1.1)–(1.2) is more illustrative in view of (H$_0$).

Keeping in view the above fact, the motivation of the present work has come from the work of [7]. We may note that, there is almost no work in this direction as long as the functional equations (1.1)–(1.2) are concerned.

For the last decade, the study of the behaviour of the solutions of functional differential/difference equations with positive and negative coefficients of first, second and higher order is a major area of research. Most of the work dealt with the existence of positive solutions of the functional equations. However, much attention has not been given to oscillation results. We refer the reader to some of the works [5, 6, 8–10].

In the present paper, the author has made an attempt to study the solution space of the functional difference equations (1.1) and (1.2) under the assumption (H_0) with different ranges of $p(n)$. It is noticed that the solution when it is bounded, either oscillates or converges to zero. But, when the solution is bounded, it oscillates.

Definition 1.1 (See [3]). Define $\rho = -\min_{n\geq 0}\{\tau(n), \alpha(n), \beta(n)\}$. By a solution of (1.1), we mean a sequence of real numbers $(y(n))_{n\geq -\rho}$ which satisfies (1.1) for all $n \geq 0$. It is clear that, for each choice of real numbers $C_{-\rho}, C_{-\rho+1}, \ldots, C_{-1}, C_0$,there exists a unique solution $(y(n))_{n\geq -\rho}$ of (1.1) which satisfies the initial conditions $y(-\rho) = C_{-\rho}, y(-\rho{+}1) = C_{-\rho+1}, \ldots, y(-1) = C_{-1}, y(0) = C_0$. As usual, a solution $(y(n))_{n\geq -\rho}$ of (1.1) is called oscillatory if the terms $y(n)$ of the sequence are neither eventually positive nor eventually negative, and otherwise the solution is said to be nonoscillatory.

2 Some Lemmas

This section deals with some established results which are useful throughout our discussion.

Lemma 2.1 (See [1]). *Let $\{f_n\}$,$\{q_n\}$ and $\{p_n\}$ be the sequences of reals defined for $n \geq N_0 > 0$ such that*

$$f_n = q_n - p_n q_{\tau(n)}, \quad n \geq N_1 > N_0,$$

where $\{\tau(n)\}$ is an increasing unbounded sequence such that $\tau(n) \leq n - 1$. Suppose that p_n satisfies one of the following three conditions:

$$-1 < -b_1 \leq p_n \leq 0, \quad -b_2 \leq p_n \leq -b_3 < -1, \quad \text{and } 0 \leq p_n \leq b_4 < \infty,$$

for all $n \in N$, where b_1, b_2, b_3 and b_4 are constants. If $q_n > 0$ for $n \geq N_0$, $\liminf_{n\to\infty} q_n = 0$ and $\lim_{n\to\infty} f_n = L$ exists, then $L = 0$.

Lemma 2.2 (See [2, 3]). *If $p(n) > 0$ for all $n \geq n_0 \geq 0$ and*

$$\liminf_{n\to\infty} \sum_{j=\delta(n)}^{n-1} p(j) > \frac{1}{e},$$

then $\Delta x(n) + p(n)x(\delta(n)) \leq 0, n \geq n_0 \geq 0$ *can not have an eventually positive solution.*

Lemma 2.3 (See [7]). *Let* (H_0) *hold. Let* u *be a real valued function on* $[0,\infty)$ *such that* $\Delta^2(r(n)\Delta^2 u(n)) \leq 0$ *for large* n. *If* $u(n) > 0$ *ultimately, then one of cases (a) and (b) holds for large* n, *and if* $u(n) < 0$ *ultimately, then one of cases (b),(c),(d) and (e) holds for large* n, *where*
(a) $\Delta u(n) > 0,\ \Delta^2 u(n) > 0$ *and* $\Delta(r(n)\Delta^2 u(n) > 0$,
(b) $\Delta u(n) > 0,\ \Delta^2 u(n) < 0$ *and* $\Delta(r(n)\Delta^2 u(n) > 0$,
(c) $\Delta u(n) < 0,\ \Delta^2 u(n) < 0$ *and* $\Delta(r(n)\Delta^2 u(n) > 0$,
(d) $\Delta u(n) < 0,\ \Delta^2 u(n) < 0$ *and* $\Delta(r(n)\Delta^2 u(n) < 0$,
(e) $\Delta u(n) < 0,\ \Delta^2 u(n) > 0$ *and* $\Delta(r(n)\Delta^2 u(n) > 0$.

Lemma 2.4 (See [4]). *Let the conditions of Lemma 2.3 hold. If* $u(n) > 0$ *ultimately, then* $u(n) > R_N(n-1)\Delta(r(n)\Delta^2 u(n))$, *where*

$$R_N(n) = \sum_{t=N}^{n-1}\sum_{s=N}^{t-1}\frac{(s-N)}{r(s)}.$$

3 Oscillation Criteria for (1.1)

In this section, sufficient conditions are obtained for the oscillation and asymptotic behaviour of solutions of the functional difference equations (1.1) under the assumption (H_0). In the sequel, we use the following hypotheses:

(H_1) there exists $\lambda > 0$ such that for every $u, v > 0,\ u, v \in \mathcal{R}$, $G(u) + G(v) \geq \lambda G(u+v)$;

(H_2) $G(uv) = G(u)G(v)$, $H(uv) = H(u)H(v)$;

(H_3) $Q(n) = \min\{q(n),\ q(\tau(n))\}$, $n \in N(n_0)$;

(H_4) $\tau(\alpha(n)) = \alpha(\tau(n))$, $n \in N(n_0)$;

(H_5) G is sublinear and $\displaystyle\int_0^{\pm c}\frac{dx}{G(x)} < \infty$, for $c > 0$;

(H_6) $\displaystyle\sum_{s=0}^{\infty}\frac{s}{r(s)}\sum_{n=s}^{\infty} nh(n) < \infty$;

(H_7) $\displaystyle\sum_{n=n_0}^{\infty} Q(n) = \infty$;

(H_8) $\displaystyle\liminf_{n\to\infty}\frac{G(x)}{x} \geq \gamma > 0$;

(H$_9$) $\sum_{n=n_0}^{\infty} q(n) = \infty$;

(H$_{10}$) $\liminf_{n\to\infty} \sum_{j=\alpha(n)}^{n-1} G(R_N(\alpha(j)-1))q(j) > (\gamma e G(1-a))^{-1}, 0 < a < 1$;

(H$_{11}$) $\sum_{n=n_0+N}^{\infty} G(R_N(\alpha(n)-1))Q(n) = \infty$;

(H$_{12}$) $\frac{G(x_1)}{x_1^\sigma} \geq \frac{G(x_2)}{x_2^\sigma}$ for $x_1 \geq x_2 > 0$ and $\sigma \geq 1$;

(H$_{13}$) $\sum_{n=n_0+N}^{\infty} R_N^\sigma(\alpha(n)-1)Q(n) = \infty$.

Remark 3.1. (H$_{10}$) implies that $\sum_{j=n+\rho}^{\infty} G(R_N(\alpha(j)-1))q(j) = \infty$. Indeed, if $\sum_{j=n+\rho}^{\infty} G(R_N(\alpha(j)-1))q(j) = b < \infty$, then for $n \geq n_0 > N+\rho$,

$$\liminf_{n\to\infty} \sum_{j=\alpha(n)}^{n-1} G(R_N(\alpha(j)-1))q(j) = \liminf_{n\to\infty}\Big(\sum_{j=n_1}^{n-1} - \sum_{j=n_1}^{\alpha(n)}\Big)G(R_N(\alpha(j)-1))q(j)$$
$$\leq b - b = 0,$$

a contradiction.

Theorem 3.2. *Let* $0 \leq p(n) \leq a < 1$. *Suppose that* (*H*$_0$), (*H*$_2$), (*H*$_6$), *and* (*H*$_8$)–(*H*$_{10}$) *hold. Then every solution of* (1.1) *either oscillates or tends to zero as* $n \to \infty$.

Proof. On the contrary, $y(n)$ be a non-oscillatory solution of (1.1) on $[n_0, \infty)$. Then $y(n) > 0$ or < 0, for $n \geq n_0$. Therefore, there exists $n_1 > n_0$ such that $y(n) > 0, y(\tau(n)) > 0, y(\alpha(n)) > 0$ and $y(\beta(n)) > 0$, for $n \geq n_1$. Setting

$$z(n) = y(n) + p(n)y(\tau(n)), \tag{3.1}$$

$$K(n) = \sum_{s=n}^{\infty} \frac{(s-n+1)}{r(s)} \sum_{t=s}^{\infty}(t-s+1)h(t)H(y(\beta(t))) \tag{3.2}$$

and

$$w(n) = z(n) - K(n) = y(n) + p(n)y(\tau(n)) - K(n) \tag{3.3}$$

we obtain

$$\Delta^2(r(n)\Delta^2 w(n)) = -q(n)G(y(\alpha(n))) < 0 \tag{3.4}$$

for $n \geq n_1$. Consequently, $w(n), \Delta w(n), \Delta(r(n)\Delta^2 w(n))$ are monotonic real valued functions on $[n_1, \infty)$. In what follows, either $w(n) > 0$ or $w(n) < 0$, for $n \geq n_1$. Suppose the former holds. By Lemma 2.3, any one of cases (a) and (b) holds. In each case, $w(n)$ is nondecreasing. We note that $K(n) > 0, \Delta K(n) < 0$ and $\Delta^2 K(n) > 0$ implies that $\lim_{n\to\infty} K(n) = 0$ due to (H_6). Hence for $0 < \Delta w(n) = \Delta z(n) - \Delta K(n)$, it is immediate to say that either$\Delta z(n) > 0$ or $\Delta z(n) > 0$. Let $n_2 > n_1$ be such that $\Delta z(n) > 0$, for $n \geq n_2$. Then

$$\begin{aligned}(1-p(n))z(n) \leq & z(n) - p(n)z(\tau(n)) \\ =& y(n) + p(n)y(\tau(n)) - p(n)y(\tau(n)) - p(n)p(\tau(n))y(\tau(\tau(n))) \\ =& y(n) - p(n)p(\tau(n))y(\tau(\tau(n))) < y(n),\end{aligned}$$

that is,

$$y(n) > (1-a)z(n) > (1-a)w(n), \quad n \geq n_2.$$

Consequently, (3.4) yields

$$G((1-a)w(\alpha(n)))q(n) \leq -\Delta^2(r(n)\Delta^2 w(n)).$$

Upon using Lemma 2.4 and (H_2), the last inequality becomes

$$\begin{aligned}G(1-a)q(n)G(R_N(\alpha(n)-1))G(\Delta r(\alpha(n))\Delta^2 w(\alpha(n))) \\ \leq -\Delta^2(r(n)\Delta^2 w(n))\end{aligned} \tag{3.5}$$

for $n \geq N > n_2$. Let $\lim_{n\to\infty} \Delta(r(n)\Delta^2 w(n)) = C$, $C \in [0, \infty)$. If $0 < C < \infty$, then there exist $C_1 > 0$ and $n_3 > N$ such that $\Delta(r(n)\Delta^2 w(n)) > C_1$, for $n \geq n_3$ and hence (3.5) yields

$$G(1-a)G(C_1)G(R_N(\alpha(n)-1))q(n) \leq -\Delta^2(r(n)\Delta^2 w(n)).$$

Therefore, for $n \geq n_3$

$$\sum_{n=n_3}^{\infty} G(1-a)G(C_1)G(R_N(\alpha(n)-1))q(n) < \infty,$$

a contradiction due to Remark 3.1. Thus $C = 0$. Using (H_8), it follows that $G(\Delta(r(n)\Delta^2 w(n))) \geq \gamma\Delta(r(n)\Delta^2 w(n))$, for $n \geq n_3$ and hence (3.5) becomes

$$\gamma G(1-a)q(n)G(R_N(\alpha(n)-1))\Delta(r(\alpha(n))\Delta^2 w(\alpha(n))) \leq -\Delta^2(r(n)\Delta^2 w(n)),$$

that is,

$$\Delta u(n) + \gamma G(1-a)G(R_N(\alpha(n)-1))q(n)u(\alpha(n)) \leq 0, \; n \geq n_3. \tag{3.6}$$

From Lemma 2.2, it follows that (3.6) has no positive solution due to (H_{10}), a contradiction to the fact that $\Delta(r(n)\Delta^2 w(n)) > 0$, for $n \geq n_3$. Ultimately, $\Delta z(n) < 0$, for $n \geq n_2$. Because $\lim_{n\to\infty} K(n)$ exists and $w(n)$ is monotonic, then $\lim_{n\to\infty} z(n)$ exists. Further, $\lim_{n\to\infty} \Delta(r(n)\Delta^2 w(n))$ exists implies that (since (H_9) holds)

$$\sum_{n=n_0}^{\infty} q(n)G(y(\alpha(n)) < \infty$$

and hence it is easy to verify that $\liminf_{n\to\infty} y(n) = 0$. Consequently, $\lim_{n\to\infty} z(n) = 0$ due to Lemma 2.1. Because $z(n) \geq y(n)$, then $\lim_{n\to\infty} y(n) = 0$.

Assume that the later holds. Then

$$y(n) \leq z(n) = y(n) + p(n)y(\tau(n)) < K(n),$$

that is, $y(n)$ is bounded (since $K(n)$ is bounded). By Lemma 2.3, any one of cases (b)-(e) holds.

Case (b) Proceeding as above, it is easy to show that $\lim_{n\to\infty} y(n) = 0$.

Cases (c)and (d) These two cases are not possible due to the fact that $w(n) < 0$, $y(n)$ is bounded, $\lim_{n\to\infty} K(n)$ exists and hence $\lim_{n\to\infty} w(n)$ exists. This fact is contradictory when we sum successively $\Delta^2 w(n) \leq 0$ from n_1 to $n-1$, where $\lim_{n\to\infty} w(n) = -\infty$.

Case (e) $(r(n)\Delta^2 w(n))$ is nondecreasing on $[n_1, \infty)$. Thus for $n \geq n_1$, $(r(n)\Delta^2 w(n)) \geq (r(n_1)\Delta^2 w(n_1))$, that is,

$$n\Delta^2 w(n) \geq \frac{n}{r(n)}(r(n_1)\Delta^2 w(n_1)). \tag{3.7}$$

Summing (3.7) from n_1 to $(n-1)$, we obtain

$$\sum_{s=n_1}^{n-1} s\Delta^2 w(s) \geq (r(n_1)\Delta^2 w(n_1)) \sum_{s=n_1}^{n-1} \frac{s}{r(s)},$$

that is,

$$n\Delta w(n) \geq n_1 \Delta w(n_1) + w(n+1) - w(n_1+1)(r(n_1)\Delta^2 w(n_1)) \sum_{s=n_1}^{n-1} \frac{s}{r(s)} > 0$$

as $n \to \infty$, a contradiction.

Finally, we suppose that $y(n) < 0$, for $n \geq n_0$. From (H_2), we may note that $G(-u) = -G(u)$ and $H(-u) = -H(u)$, for $u \in \mathcal{R}$.Indeed, $G(1)G(1) = G(1)$ and $G(-1)G(-1) = G(1)$ implies that $G(-1) = -1$ and $G(1) = 1$. Hence putting $x(n) = -y(n)$ in (1.1), for $n \geq n_0$, we obtain $x(n) > 0$ and

$$\Delta^2(r(n)\Delta^2((x(n) + p(n)x(\tau(n))))) + q(n)G(x(\alpha(n))) - h(n)H(x(\beta(n))) = 0.$$

Proceeding as above, we can show that every solution of (1.1) either oscillates or converges to zero as $n \to \infty$. This completes the proof of the theorem. □

Remark 3.3. By Theorem 3.2, we have that $y(n)$ is bounded ultimately when $w(n) < 0$, for $n \geq n_1$. Hence the case $w(n) < 0$ doesn't arise, if $y(n)$ is unbounded. Therefore, we have proved the following theorem.

Theorem 3.4. *Let* $0 \leq p(n) \leq a < 1$. *Suppose that* (H_0), (H_2), (H_6), (H_8) *and* (H_{10}) *hold. Then every unbounded solution of* (1.1) *oscillates.*

Theorem 3.5. *Let* $0 \leq p(n) \leq a < \infty$. *Assume that* $\tau(n) \geq \alpha(n)$ *for all* $n \in N(n_0)$. *If* (H_0)–(H_7) *and* (H_{11}) *hold, then every solution of* (1.1) *either oscillates or tends to zero as* $n \to \infty$.

Proof. Let $y(n)$ be a nonoscillatory solution of (1.1) such that $y(n) > 0$, for $n \geq n_0$. The proof for the case $y(n) < 0$, for $n \geq n_0$ is similar. Setting $z(n)$ and $K(n)$ as in (3.1) and (3.2), we obtain (3.3) and (3.4) for $n \geq n_1 > n_0$. From Lemma 2.3, it follows that any one of cases (a) and (b) holds when $w(n) > 0$, for $n \geq n_1$. Upon using (1.1), we obtain

$$\begin{aligned}
0 &= \Delta^2(r(n)\Delta^2 w(n)) + q(n)G(y(\alpha(n))) + G(a)\Delta^2(r(\tau(n))\Delta^2 w(\tau(n)) \\
&\quad + G(a)q(\tau(n))G(y(\alpha(\tau(n)))) \\
&\geq \Delta^2(r(n)\Delta^2 w(n)) + G(a)\Delta^2(r(\tau(n))\Delta^2 w(\tau(n)) + Q(n)[G(y(\alpha(n))) \\
&\quad + G(a)G(y(\alpha(\tau(n))))] \\
&= \Delta^2(r(n)\Delta^2 w(n)) + G(a)\Delta^2(r(\tau(n))\Delta^2 w(\tau(n)) + Q(n)[G(y(\alpha(n))) \\
&\quad + G(ay(\tau(\alpha(n))))] \\
&\geq \Delta^2(r(n)\Delta^2 w(n)) + G(a)\Delta^2(r(\tau(n))\Delta^2 w(\tau(n)) + \lambda Q(n)G(z(\alpha(n)))
\end{aligned}$$

due to (H_1), (H_2), (H_3) and (H_4). Since $z(n) \leq w(n)$, then the last inequality becomes

$$\Delta^2(r(n)\Delta^2 w(n)) + G(a)\Delta^2(r(\tau(n))\Delta^2 w(\tau(n)) + \lambda Q(n)G(w(\alpha(n))) \leq 0, \quad (3.8)$$

that is,

$$\lambda Q(n) \leq -\frac{\Delta^2(r(n)\Delta^2 w(n)) + G(a)\Delta^2(r(\tau(n))\Delta^2 w(\tau(n))}{G(w(\alpha(n)))}. \quad (3.9)$$

Using Lemma 2.4 in (3.9), we obtain that

$$\lambda Q(n) \leq -\frac{\Delta^2(r(n)\Delta^2 w(n)) + G(a)\Delta^2(r(\tau(n))\Delta^2 w(\tau(n))}{G(R_N(\alpha(n)-1)\Delta(r(\alpha(n))\Delta^2 w(\alpha(n)))},$$

that is,

$$\lambda Q(n)(R_N(\alpha(n)-1) \leq -\frac{\Delta^2(r(n)\Delta^2 w(n)) + G(a)\Delta^2(r(\tau(n))\Delta^2 w(\tau(n))}{G(\Delta(r(\alpha(n))\Delta^2 w(\alpha(n))))},$$

for $n \geq N > n_1$. In what follows,

$$\lambda Q(n)(R_N(\alpha(n)-1) \leq -\frac{\Delta^2(r(n)\Delta^2 w(n))}{G(\Delta(r(n)\Delta^2 w(n)))} - \frac{G(a)\Delta^2(r(\tau(n))\Delta^2 w(\tau(n))}{G(\Delta(r(\tau(n))\Delta^2 w(\tau(n))))},$$

for $n \geq n_2 > N$. Let $\ell = r(n)\Delta^2 w(n)$. Then the above inequality yields

$$\begin{aligned}\lambda Q(n)(R_N(\alpha(n)-1) \leq & \int_{\Delta\ell(n+1)}^{\Delta\ell(n)} \frac{du}{G(\Delta(r(n)\Delta^2 w(n)))} \\ & + G(a)\int_{\Delta\ell(\tau(n+1))}^{\Delta\ell(\tau(n))} \frac{dv}{G(\Delta(r(\tau(n))\Delta^2 w(\tau(n))))},\end{aligned}$$

where $\Delta\ell(n+1) \leq u \leq \Delta\ell(n)$ and $\Delta\ell(\tau(n+1)) \leq v \leq \Delta\ell(\tau(n))$. Hence for $n \geq n_2$, we get

$$\lambda Q(n)(R_N(\alpha(n)-1) \leq \int_{\Delta\ell(n+1)}^{\Delta\ell(n)} \frac{du}{G(u)} + G(a)\int_{\Delta\ell(\tau(n+1))}^{\Delta\ell(\tau(n))} \frac{dv}{G(v)},$$

that is,

$$\begin{aligned}\lambda \sum_{n=n_2}^{t-1} Q(n)(R_N(\alpha(n)-1) &\leq \sum_{n=n_2}^{t-1}\left[\int_{\Delta\ell(n+1)}^{\Delta\ell(n)} \frac{du}{G(u)} + G(a)\int_{\Delta\ell(\tau(n+1))}^{\Delta\ell(\tau(n))} \frac{dv}{G(v)}\right] \\ &= \int_{\Delta\ell(t)}^{\Delta\ell(n_2)} \frac{du}{G(u)} + G(a)\int_{\Delta\ell(\tau(t))}^{\Delta\ell(\tau(n_2))} \frac{dv}{G(v)}.\end{aligned}$$

Since $\Delta\ell(n)$ is decreasing, then

$$\begin{aligned}\lambda \sum_{n=n_2}^{\infty} Q(n)(R_N(\alpha(n)-1) &\leq \lim_{t\to\infty}\left[\int_{\Delta\ell(t)}^{\Delta\ell(n_2)} \frac{du}{G(u)} + G(a)\int_{\Delta\ell(\tau(t))}^{\Delta\ell(\tau(n_2))} \frac{dv}{G(v)}\right] \\ &< \infty,\end{aligned}$$

a contradiction to (H_{11}) due to (H_5). Ultimately, $w(n) < 0$, for $n \geq n_1$. The rest of the proof follows from Theorem 3.2 and hence the details are omitted. Thus the theorem is proved. □

Theorem 3.6. *Let* $0 \leq p(n) \leq a < \infty$*. Assume that* $\tau(n) \geq \alpha(n)$ *for all* $n \in N(n_0)$*. If* (H_0)–(H_6) *and* (H_{11}))*hold, then every unbounded solution of* (1.1) *is oscillatory .*

Proof. The proof follows from the proof of Theorem 3.5 and Remark 3.3. Hence the details are omitted. □

Theorem 3.7. *Let* $0 \leq p(n) \leq a < \infty$ *and* $\tau(n) \geq \alpha(n)$*, for all* $n \in N(n_0)$*. If* (H_0)–(H_4)*,* (H_6)*,* (H_7)*,* (H_{12}) *and* (H_{13}) *hold, then every solution of* (1.1) *either oscillates or converges to zero as* $n \to \infty$*.*

Proof. Proceeding as in Theorem 3.5, we obtain (3.8) for $n \geq n_1$. Since $w(n)$ is increasing, then there exist $n_2 > n_1$ and $\eta > 0$ such that $w(n) \geq \eta$, for $n \geq n_2$. Using (H_{12}) and Lemma 2.4, we get

$$\begin{aligned} G(w(\alpha(n))) &= \frac{G(w(\alpha(n)))}{w^\sigma(\alpha(n))} w^\sigma(\alpha(n)) \\ &\geq \frac{G(\eta)}{\eta^\sigma} w^\sigma(\alpha(n)) \\ &\geq \frac{G(\eta)}{\eta^\sigma} R_N^\sigma(\alpha(n)-1)(\Delta(r(\alpha(n))\Delta^2 w(\alpha(n))))^\sigma \end{aligned}$$

and hence the inequality (3.8) yields

$$\begin{aligned} \lambda \frac{G(\eta)}{\eta^\sigma} R_N^\sigma(\alpha(n)-1)Q(n) &\leq -\frac{\Delta^2(r(n)\Delta^2 w(n)) + G(a)\Delta^2(r(\tau(n))\Delta^2 w(\tau(n))}{(\Delta(r(\alpha(n))\Delta^2 w(\alpha(n))))^\sigma} \\ &\leq -\frac{\Delta^2(r(n)\Delta^2 w(n))}{(\Delta(r(n)\Delta^2 w(n)))^\sigma} - G(a)\frac{\Delta^2(r(\tau(n))\Delta^2 w(\tau(n))}{(\Delta(r(\tau(n))\Delta^2 w(\tau(n))))^\sigma} \end{aligned}$$

due to $\tau(n) \geq \alpha(n)$. Denoting $\ell = r(n)\Delta^2 w(n)$, it follows that

$$\begin{aligned} \lambda \frac{G(\eta)}{\eta^\sigma} R_N^\sigma(\alpha(n)-1)Q(n) \leq &-\int_{\Delta\ell(n)}^{\Delta\ell(n+1)} \frac{du}{(\Delta(r(n)\Delta^2 w(n)))^\sigma} \\ &-G(a)\int_{\Delta\ell(\tau(n))}^{\Delta\ell(\tau(n+1))} \frac{dv}{(\Delta(r(\tau(n))\Delta^2 w(\tau(n))))^\sigma}, \end{aligned}$$

for $n \geq n_2$. Using the same type of reasoning as in the proof of Theorem 3.5, we obtain

$$\lambda \frac{G(\eta)}{\eta^\sigma} \sum_{n=n_2}^{\infty} R_N^\sigma(\alpha(n)-1)Q(n) \leq \lim_{t\to\infty} \left[\int_{\Delta\ell(t)}^{\Delta\ell(n_2)} \frac{du}{u^\sigma} + G(a)\int_{\Delta\ell(\tau(t))}^{\Delta\ell(\tau(n_2))} \frac{dv}{v^\sigma} \right] < \infty,$$

a contradiction to (H_{13}), where we have used the fact that $\lim_{t\to\infty} \Delta\ell(t)$ exists. The rest of the proof follows from the proof of Theorem 3.5. Hence the theorem is proved. □

Remark 3.8. The prototype of G satisfying (H_1) and (H_2) is

$$G(u) = (c + d|u|^m)|u|^\mu \text{sgn} u,$$

where $c > 0, d > 0, m \geq 0$ and $\mu \geq 0$ such that $c + d = 1$.

Theorem 3.9. *Let* $0 \leq p(n) \leq a < \infty$ *and* $\tau(n) \geq \alpha(n)$*, for all* $n \in N(n_0)$*. If* (H_0)*–*(H_4)*,* (H_6)*,* (H_{12}) *and* (H_{13}) *hold, then every unbounded solution of* (1.1) *oscillates.*

Proof. The proof follows from the proof of Theorem 3.7 and Remark 3.3. Hence the details are omitted. □

Example 3.10. Consider

$$\Delta^2[n\Delta^2((y(n) + p(n)y(\tau(n))))] + q(n)y^3(\alpha(n)) - h(n)y^5(\beta(n)) = 0, \tag{3.10}$$

where $n \geq 5, 0 \leq p(n) = (1 + (-1)^n) \leq 2, \tau(n) = n - 3, \alpha(n) = n - 5 = \beta(n)$, $G(u) = u^3, H(v) = v^5, q(n) = (\frac{48(n+1)}{(n-1)^3} + \frac{1}{(n-1)}), h(n) = \frac{1}{(n-1)^3}$ and $r(n) = n$. Clearly, all the conditions of Theorem 3.9 are satisfied. Hence every unbounded solution of (3.10) oscillates. In particular, $y(n) = (n+4)(-1)^n$ is an oscillatory solution of (3.10).

Theorem 3.11. *Let* $-1 < b \leq p(n) \leq 0$*. If* (H_0)*,* (H_2)*,* (H_5)*,* (H_6) *and* (H_9) *hold, then every solution of* (1.1) *either oscillates or converges to zero as* $n \to \infty$*.*

Proof. Let $y(n)$ be a nonoscillatory solution of (1.1) on $[n_0, \infty)$, $n_0 \geq 0$. Because of (H_2), without loss of generality we may suppose that $y(n) > 0$, for $n \geq n_0$. Setting as in (3.1),(3.2) and (3.3), we obtain (3.4) for $n \geq n_1 > n_0$. Hence $w(n)$ is monotonic on $[n_1, \infty)$. If $w(n) > 0$, for $n \geq n_1$, then any one of cases (a) and (b) of Lemma 2.3 holds. Consequently, $w(n) > R_N(n-1)\Delta(r(n)\Delta^2 w(n))$, for $n \geq n_2 > n_1 + N$ due to Lemma 2.4. Moreover, $w(n) \leq y(n)$ implies that $y(n) > R_N(n-1)\Delta(r(n)\Delta^2 w(n))$, for $n \geq n_2$ and hence (3.4) becomes

$$\Delta^2(r(n)\Delta^2 w(n)) + q(n)G(R_N(\alpha(n) - 1)\Delta(r(n)\Delta^2 w(n))) \leq 0.$$

Using (H_2) and (H_5), and proceeding as in Theorem 3.5, we get from the above inequality that

$$\sum_{n=n_2}^{\infty} q(n)G(R_N(\alpha(n) - 1)) < \infty,$$

a contradiction to (H_5), where we are using the fact that $R_N(n)$ is nondecreasing. Ultimately, $w(n) < 0$, for $n \geq n_1$. Then any one of cases (b)-(e) of Lemma 2.3 holds. Due to Remark 3.3, $z(n)$ is bounded also is $w(n)$ and hence we assert that $y(n)$ is bounded.

Suppose there exists a subsequence $\{n_j^{'}\}$ of $\{n\}$ such that $\{n_j^{'}\} \to \infty$ and $y(n_j^{'}) \to \infty$ as $j \to \infty$ and

$$y(n_j^{'}) = \max\{y(n) : n_1 \le n \le n_j^{'}\}.$$

Since $\tau(n) \le n-1 < n$, then $y(\tau(n_j^{'})) < y(n_j^{'})$ implies that

$$\begin{aligned} w(n_j^{'}) &= y(n_j^{'}) + p(n_j^{'})y(\tau(n_j^{'})) - K(n_j^{'}) \\ &\ge (1+p(n_j^{'}))y(n_j^{'}) - K(n_j^{'}) \\ &\to \infty \ \ as\ j \to \infty, \end{aligned}$$

which is absurd, where we have used the fact that $(1+p(n_j^{'})) > 0$ and $\lim_{j\to\infty} K(n_j^{'})$ exists. Therefore, our assertion holds. It is easy to verify the cases (c),(d) and (e) following to Theorem 3.2. Also, proceeding as in Theorem 3.2, we obtain $\liminf_{n\to\infty} y(n) = 0$. Hence $\lim_{n\to\infty} z(n) = 0$ due to Lemma 2.1. As a result

$$\begin{aligned} 0 = \lim_{n\to\infty} z(n) &\ge \limsup_{n\to\infty}(y(n) + by(\tau(n))) \\ &\ge \limsup_{n\to\infty} y(n) + \liminf_{n\to\infty}(by(\tau(n))) \\ &= \limsup_{n\to\infty} y(n) + b\limsup_{n\to\infty} y(\tau(n)) \\ &= (1+b)\limsup_{n\to\infty} y(n) \end{aligned}$$

implies that $\limsup_{n\to\infty} y(n) = 0$. This completes the proof of the theorem. □

Theorem 3.12. *Let* $-1 < b \le p(n) \le 0$. *If all the conditions of Theorem 3.11 hold, then every unbounded solution of* (1.1) *oscillates.*

The proof of the theorem follows from Theorem 3.11 and hence the details are omitted.

Example 3.13. Consider

$$\Delta^2[e^{-n}\Delta^2((y(n)+p(n)y(\tau(n))))] + q(n)y^3(\alpha(n)) - h(n)y^5(\beta(n)) = 0, \quad (3.11)$$

$n \ge 0$, where $p(n) = (\frac{-1}{e})$, $q(n) = e^{-9}(2(e^{-1}+1)^4e^{3n} + e^{-n})$, $h(n) = e^{-(n+9)}$, $\tau(n) = n-1, \alpha(n) = n-3 = \beta(n)$ and $r(n) = e^{-n}$. Clearly, all the conditions of Theorem 3.11 are satisfied. Hence every solution of (3.11) either oscillates or converges to zero as $n \to \infty$. In particular, $y(n) = (-1)^n e^{-n}$ is such a solution of (3.11).

Theorem 3.14. *Let* $-\infty < b \le p(n) \le d < -1$. *If all the conditions of Theorem 3.11 hold, then every bounded solution of* (1.1) *either oscillates or converges to zero as* $n \to \infty$.

Proof. The proof follows from the proof of Theorem 3.11 and hence the details are omitted. □

4 Oscillation Criteria for (1.2)

This section is devoted to study the oscillatory behaviour of solutions of (1.2) with suitable forcing functions. Our attention is restricted to the forcing functions which are changing sign eventually. We have the following hypotheses regarding the forcing function $f(n)$:

(H_{14}) there exists a real valued function $F(n)$ such that $F(n)$ changes sign with

$$-\infty < \liminf_{n\to\infty} F(n) < 0 < \limsup_{n\to\infty} F(n) < \infty$$

and $\Delta^2(r(n)\Delta^2 F(n)) = f(n)$;

(H_{15}) there exists a real valued function $F(n)$ such that $F(n)$ changes sign with

$$\liminf_{n\to\infty} F(n) = -\infty, \quad \limsup_{n\to\infty} F(n) = \infty$$

and $\Delta^2(r(n)\Delta^2 F(n)) = f(n)$.

Theorem 4.1. *Let* $0 \le p(n) \le b < \infty$. *Assume that* ($H_0$)–($H_4$) *and* ($H_6$) *hold. If* ($H_{15}$) *holds and*

$$(H_{16}) \quad \sum_{n=n_0}^{\infty} Q(n)G(F^+(\alpha(n))) = \infty = \sum_{n=n_0}^{\infty} Q(n)G(F^-(\alpha(n))),$$

where $F^+(n) = \max\{0, F(n)\}$ *and* $F^-(n) = \max\{-q(n), 0\}$, *then* (1.2) *is oscillatory.*

Proof. Let $y(n)$ be a nonoscillatory solution of (1.2) on $[n_0, \infty)$. Suppose that $y(n) > 0$, for $n \ge n_0$. Setting as in (3.1),(3.2) and (3.3), let

$$v(n) = w(n) - F(n) = z(n) - K(n) - F(n). \tag{4.1}$$

Hence for $n \ge n_1 > n_0$, Eq.(1.2) becomes

$$\Delta^2(r(n)\Delta^2 v(n)) = -q(n)G(y(\alpha(n))) \le 0, \ but \ \neq 0. \tag{4.2}$$

Thus $v(n)$ is monotonic on $[n_1, \infty)$. Since $F(n)$ changes sign, then $v(n) > 0$ implies that $z(n) - K(n) > F(n)$. In what follows, $z(n) - K(n) < 0$ unlikely holds due to (H_{15}). Hence $z(n) - K(n) > 0$ implies that $z(n) - K(n) > F^+(n)$, that is,

$$z(n) > K(n) + F^+(n) > F^+(n), \tag{4.3}$$

for $n \geq n_1$. Upon using (1.2) for $n \geq n_2 > n_1$, we have

$$\begin{aligned}
0 &= \Delta^2(r(n)\Delta^2 v(n)) + q(n)G(y(\alpha(n))) + G(b)\Delta^2(r(\tau(n))\Delta^2 v(\tau(n)) \\
&\quad + G(b)q(\tau(n))G(y(\alpha(\tau(n)))) \\
&\geq \Delta^2(r(n)\Delta^2 v(n)) + G(b)\Delta^2(r(\tau(n))\Delta^2 v(\tau(n)) + Q(n)[G(y(\alpha(n))) \\
&\quad + G(b)G(y(\alpha(\tau(n))))] \\
&= \Delta^2(r(n)\Delta^2 v(n)) + G(b)\Delta^2(r(\tau(n))\Delta^2 v(\tau(n)) + Q(n)[G(y(\alpha(n))) \\
&\quad + G(by(\tau(\alpha(n))))] \\
&\geq \Delta^2(r(n)\Delta^2 v(n)) + G(b)\Delta^2(r(\tau(n))\Delta^2 v(\tau(n)) + \lambda Q(n)[G(y(\alpha(n)) \\
&\quad + by(\tau(\alpha(n))))] \\
&\geq \Delta^2(r(n)\Delta^2 v(n)) + G(b)\Delta^2(r(\tau(n))\Delta^2 v(\tau(n)) + \lambda Q(n)G(z(\alpha(n)))
\end{aligned}$$

due to (H_1), (H_2), (H_3) and (H_4). When $v(n) > 0$, any one of cases (a) and (b) of Lemma 2.3 holds. Using (4.3) in the last inequality, we obtain

$$\lambda Q(n)G(F^+(\alpha(n))) \geq -\Delta^2(r(n)\Delta^2 v(n)) - G(b)\Delta^2(r(\tau(n))\Delta^2 v(\tau(n)),$$

that is,

$$\lambda \sum_{n=n_2}^{\infty} Q(n)G(F^+(\alpha(n))) < \infty,$$

a contradiction to (H_{16}). Consequently, $v(n) < 0$, for $n \geq n_1$. Thus any one of cases (b)-(e) of Lemma 2.3 holds. Because $\lim_{n\to\infty} v(n)$ exists, then for each of the cases $z(n) = v(n) + K(n) + F(n)$ implies that

$$\begin{aligned}
\liminf_{n\to\infty} z(n) &= \liminf_{n\to\infty}(v(n) + K(n) + F(n)) \\
&\leq \limsup_{n\to\infty} v(n) + \liminf_{n\to\infty}(K(n) + F(n)) \\
&\leq \limsup_{n\to\infty} v(n) + \limsup_{n\to\infty} K(n) + \liminf_{n\to\infty} F(n) \\
&\to -\infty,
\end{aligned}$$

a contradiction to the fact that $z(n) > 0$, for $n \geq n_1$.

If $y(n) < 0$ for $n \geq n_0$, then we set $x(n) = -y(n)$ to obtain $x(n) > 0$ for $n \geq n_0$ and

$$\Delta^2(r(n)\Delta^2((x(n) + p(n)x(\tau(n))))) + q(n)G(x(\alpha(n))) - h(n)H(x(\beta(n))) = \bar{f}(n),$$

where $\bar{f} = -f$. If $\bar{F} = -F$, then $\bar{F}$ changes sign, $\bar{F}^+ = F^-$ and $\Delta^2(r(n)\Delta^2\bar{F}(n)) = \bar{f}(n)$. Proceeding as above we have a contradiction. Thus the theorem is proved. □

Theorem 4.2. *Let $-1 < p(n) \leq 0$. Suppose that (H_0), (H_2), (H_6) and (H_{15}) hold. If*

(H_{17}) $\sum_{n=n_0}^{\infty} q(n)G(F^+(\alpha(n))) = \infty = \sum_{n=n_0}^{\infty} q(n)G(F^-(\alpha(n))),$

then every solution of (1.2) *oscillates.*

Proof. Proceeding as in the proof of Theorem 4.1, we obtain $v(n) > 0$ or < 0, for $n \geq n_1$. If $v(n) > 0$, then any one of cases (a) and (b) of Lemma 2.3 holds for $n \geq n_1$. Using the same type of reasoning as in Theorem 4.1, $z(n) - K(n) > 0$. Because $K(n) > 0$, then $z(n) > 0$. Hence there exists $n_2 > n_1$ such that $v(n) > 0$ implies that $z(n) - K(n) > F^+(n)$, for $n \geq n_2$. Consequently,

$$y(n) > z(n) > z(n) - K(n) > F^+(n), \quad n \geq n_2.$$

Therefore, (4.2) becomes

$$q(n)G(F^+(\alpha(n))) \leq -\Delta^2(r(n)\Delta^2 v(n)), \quad n \geq n_2,$$

that is,

$$\sum_{n=n_2}^{\infty} q(n)G(F^+(\alpha(n))) < \infty,$$

a contradiction to (H_{17}). Ultimately, $v(n) < 0$, for $n \geq n_1$. As a result, $z(n) - K(n) < F(n)$ yields that $z(n) - K(n) < 0$ due to (H_{15}). In what follows, either $z(n) > 0$ or < 0, for $n \geq n_1$. Assume that the former holds. Similar to Theorem 4.1, it happens that $\liminf_{n\to\infty} z(n) < -\infty$, a contradiction to the fact that $z(n) > 0$, for $n \geq n_1$. Hence the later holds. Proceeding as in Theorem 3.11, it is easy to show that $y(n)$ is bounded and hence $z(n)$ is bounded. Using the same type of reasoning as above, it follows that $\liminf_{n\to\infty} z(n) = -\infty$, which is contradictory to the fact that $z(n)$ is bounded.

For the case $y(n) < 0$, for $n \geq n_0$, we can proceed as in Theorem 4.1 to obtain the desired contradiction. This completes the proof of the theorem. □

Theorem 4.3. *Let* $-\infty < p(n) \leq -1$. *If all the conditions of Theorem 4.2 are satisfied, then every bounded solution of* (1.2) *oscillates.*

Proof. The proof follows from the proof of Theorem 4.2. Hence the details are omitted. □

Theorem 4.4. *Let* $0 \leq p(n) \leq b < \infty$. *If* ($H_0$)–($H_4$), ($H_6$), ($H_{14}$) *and* ($H_{16}$) *hold, then every unbounded solution of* (1.2) *is oscillatory.*

Proof. Suppose on the contrary that $y(n)$ is an unbounded nonoscillatory solution of (1.2) such that $y(n) > 0$, for $n \geq n_0$. Setting as in (3.1),(3.2),(3.3) and (4.1), we obtain (4.2) and hence $v(n)$ is monotonic on $[n_1, \infty)$, $n_1 > n_0$. The case $v(n) > 0$, for $n \geq n_1$ can be followed from the proof of Theorem 4.1. Let $v(n) < 0$, for $n \geq n_1$. Then any one of cases (b)-(e) of Lemma 2.3 holds. In the case (b), $\lim_{n\to\infty} v(n)$ exists(finite) and hence

$$z(n) = v(n) + K(n) + F(n)$$

implies that

$$y(n) \le v(n) + K(n) + F(n), \tag{4.4}$$

that is, $y(n)$ is bounded due to (H_{14}), a contradiction. For each of the cases (c),(d) and (e), $v(n)$ is nonincreasing on $[n_1, \infty)$. Let $\lim_{n\to\infty} v(n) = c_1$, $c_1 \in [-\infty, 0)$. If $c_1 = -\infty$, then (4.4) yields

$$\begin{aligned}\liminf_{n\to\infty} y(n) &\le \liminf_{n\to\infty}(v(n) + K(n) + F(n))\\ &\le \limsup_{n\to\infty} v(n) + \liminf_{n\to\infty}(K(n) + F(n))\\ &\le \lim_{n\to\infty} v(n) + \liminf_{n\to\infty} K(n) + \limsup_{n\to\infty} F(n)\\ &\to -\infty,\end{aligned}$$

which is absurd. The contradiction is obvious, when $-\infty < c_1 < 0$.

The case $y(n) < 0$ is similar. Hence the theorem is proved. □

Theorem 4.5. *Let $-1 < b \le p(n) \le 0$. Assume that*

(H_{18}) $\tau(\tau^\ell(n)) = \tau^{\ell+1}(n)$ *and* $\lim_{\ell\to\infty} \tau^\ell(n) = c_1$, $c_1 > 0$

for all $n \in N(n_0)$. If (H_0), (H_2), (H_6), (H_{14}) and (H_{17}) hold, then every unbounded solution of (1.2) *oscillates.*

Proof. Let $y(n)$ be an unbounded nonoscillatory solution of (1.2) such that $y(n) > 0$, for $[n_0, \infty)$. Proceeding as in the proof of Theorem 4.2, we have a contradiction when $v(n) > 0$, for $n \ge n_1 > n_0 + \rho$.

Next, we suppose that $v(n) < 0$, for $n \ge n_1$. In what follows, $z(n) - K(n) < 0$ due to (H_{14}). Therefore, either $z(n) < 0$ or $z(n) > 0$, for $n \ge n_2 > n_1$. If $z(n) < 0$, for $n \ge n_2$, then $y(n) < y(\tau(n))$ and hence proceeding recursively, we obtain

$$y(n) < y(\tau(n)) < y(\tau(\tau(n))) = y(\tau^2(n)) < \ldots < y(\tau^\ell(n)) < \ldots,$$

that is, there exists a constant $c_2 > 0$ such that $y(n) < y(c_2)$, for any $n \ge n_2$ due to (H_{18}). Consequently, $y(n)$ is bounded for $n \ge n_2$, a contradiction to our hypothesis. Ultimately, $z(n) > 0$, for $n \ge n_2$ and hence $z(n) < K(n)$, for $n \ge n_2$ implies that $z(n)$ is bounded on $[n_2, \infty)$. On the other hand, $y(n)$ is unbounded implies that, there exists $\{\delta_j\}_{j=1}^{\infty} \subset \{n\}, n \in N(n_0)$ such that $\delta_j \to \infty$ as $j \to \infty$ and

$$y(\delta_j) = \max\{y(n) : n_2 \le n \le \delta_j\}.$$

Hence $\tau(n) \leq n-1 < n$ and $y(\tau(n)) < y(n)$ yields that

$$z(\delta_j) \geq (1+b)y(\delta_j) \to \infty \text{ as } j \to \infty,$$

a contradiction to the fact that $z(n)$ is bounded on $[n_2, \infty)$. This completes the proof of the theorem. □

Example 4.6. Consider

$$\Delta^2(e^{-n}\Delta^2(y(n)+p(n)y(n-1))) + q(n)y^3(n-2) - h(n)y^5(n-4) = f(n), \quad (4.5)$$

$n \geq 4$, where $p(n) = 2(1+(-1)^n)$, $q(n) = [e^n+(8e^{-1}+4e^{-2}+1)]$, $h(n) = e^{-n}$, $f(n) = (e^n - 4e^{-n})(-1)^n$, $\tau(n) = n-1$, $\alpha(n) = n-2$ and $\beta(n) = n-4$. Clearly, $Q(n) = e^{n-1} + (8+4e^{-1}+e)e^{-n}$. If we define

$$F(n) = \left[\frac{e^{2n}}{(e+1)^2(e^2+1)^2} - \frac{1}{(e^{-1}+1)^2}\right](-1)^n,$$

then $\Delta^2(e^{-n}\Delta^2 F(n)) = (e^n - 4e^{-n})(-1)^n$. Hence

$$F^+(n-2) = \begin{cases} \dfrac{e^{2n}}{(e+1)^2(e^2+1)^2} - \dfrac{1}{(e^{-1}+1)^2}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

$$F^-(n-2) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{e^{2n}}{(e+1)^2(e^2+1)^2} - \dfrac{1}{(e^{-1}+1)^2}, & \text{if } n \text{ is odd.} \end{cases}$$

Consequently,

$$\sum_{n=2}^{\infty} Q(n)G(F^+(n-2)) = \sum_{n=2}^{\infty}[e^{n-1} + (8+4e^{-1}+e)e^{-n}][F^+(n-2)]^3 = \infty$$

and

$$\sum_{n=2}^{\infty} Q(n)G(F^-(n-2)) = \sum_{n=2}^{\infty}[e^{n-1} + (8+4e^{-1}+e)e^{-n}][F^-(n-2)]^3 = \infty.$$

From Theorem 4.1, it follows that all solutions of (4.5) oscillate. In particular, $y(n) = (-1)^n$ is an oscillatory solution of (4.5).

5 Discussion

The solution space of (1.1)–(1.2) is divided for bounded and unbounded solutions. Due to the method incorporated here, we could not eliminate the bounded solutions of (1.1) as converges to zero. However, in case of unbounded solution, it oscillates. For Eq.(1.1), H could be linear, sublinear or superlinear.

It is interesting to notice the solution space of (1.2) pertaining (H_{14}) or (H_{15}). Emphasis will be given to the forcing function as compared to the results concerning (1.1). It reveals that every bounded solution of (1.2) oscillates, if (H_{15}) holds for all ranges of $p(n)$ and every unbounded solution of (1.2) oscillates, if (H_{14}) holds except $p(n) \leq -1$. Here is a question that "what happened to the behaviour of solutions of (1.2), if the solution is bounded and (H_{14}) holds". To this question, we state here the following results without proof.

Theorem 5.1. *Let* $0 \leq p(n) \leq b < \infty$. *If* ($H_0$)–($H_4$), ($H_6$), ($H_{14}$) *and* ($H_{16}$) *hold, then every solution of* (1.2) *either oscillates or converges to zero as* $n \to \infty$.

Theorem 5.2. *Let* $-1 < b \leq p(n) \leq 0$. *If* (H_0), (H_2), (H_6), (H_{14}), (H_{17}) *and* (H_{18}) *hold, then every solution of* (1.2) *either oscillates or converges to zero as* $n \to \infty$.

Theorem 5.3. *Let* $-\infty < p(n) \leq b < -1$. *If* ($H_0$), ($H_2$), ($H_6$), ($H_{14}$) *and* ($H_{17}$) *hold, then every bounded solution of* (1.2) *either oscillates or converges to zero as* $n \to \infty$.

Acknowledgment

The author is thankful to the referee for his helpful suggestions and necessary corrections in the completion of this paper.

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