International Journal of Difference Equations
ISSN 0973-6069, Volume 7, Number 1, pp. 61–77 (2012)
http://campus.mst.edu/ijde

Positivity of Outpayments according to the Annuity Scheme of Credit Repayment

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Abstract

Positivity of outpayments according to the annuity scheme of credit repayment is proved. Calculation of outpayments according to the annuity scheme of credit repayment has been implemented on the Java programming language in the framework of the IRBIS project which is developed by the company Integrated Banking Information Systems.

AMS Subject Classifications: 15A15, 91G80.
Keywords: Credit repayment, equation with lag, estimation of determinants.

1 Introduction

The calculation of effective interest rate is one of the important branches of mathematical finance.

In the present paper we prove the existence and uniqueness of the solution of certain system of linear algebraic equations which depends on several real parameters that obey some restrictions in the form of inequalities. We prove also positivity of the components of the solution of this linear system. These components have sense of payments on the main body of the bank credit according to the annuity scheme of repayment.

Calculation of outpayments according to the annuity scheme of credit repayment has been implemented on the Java programming language in the framework of the IRBIS project which is developed by the company Integrated Banking Information Systems.

The author expresses his gratitude to V. Fesunov and A. Kononov for statement of the problem.

Received April 29, 2012; Accepted June 30, 2012
Communicated by Martin Bohner

2 Problem Statement and Main Result

Let the credit in amount of V dollars (or some other monetary units) is given for some period of L days (for the beginning for sake of simplicity we shall consider the period which is enclosed in one calendar year with duration of 365 days). Let the rate of interest be $\alpha\%$ per year and let the period L be subdivided into n subperiods:

$$L = L_1 + L_2 + \ldots + L_n.$$

The calculation of the amount of repayment is carried out at the end of each period. In this moment the interest of that part of the main body of the credit which for this moment is not paid is charged, for the subperiod from the preceding moment of payment (or from the start date if the period is first) to the moment of charge plus the interest of that part of the main body of the credit which has not been paid at the preceding moment of charge (if a period is not first) for the subperiod from the preceding moment of charge to the preceding moment of payment and the current payment of the main part of the debt. The charged amount must be paid by the client at the nearest day of payment that follows this moment of charge. The repayment of the credit is carried out in such a way that at the end of each period the same amount of x dollars is paid.

This statement shows that the system of linear algebraic equations (2.3) (see below) can be treated as a difference equation with lag.

Let

$$r_i = L_i \cdot 0,01\alpha/365, \quad i = 1, \ldots, n. \tag{2.1}$$

Then let

$$L_i = L_{i1} + L_{i2}, \quad i = 2, \ldots, n,$$

where L_{i1} is the subperiod from the beginning of the i-th period to the moment of the $i-1$-th payment and L_{i2} is the subperiod from $i-1$-th payment to the end to the i-th period,

$$r_{ij} = L_{ij} \cdot \alpha/365, \quad i = 2, \ldots, n, \quad j = 1, 2.$$

Let us denote the amount of repayment of the main body at the end of the i-th period by A_i. Then the charged amount is

$$A_1 + r_1 V$$

(for the first period),

$$A_2 + r_{21}V + r_{22}(V - A_1)$$

(for the second period),

$$A_i + r_{i1}(V - A_1 - \ldots - A_{i-2}) + r_{i2}(V - A_1 - \ldots - A_{i-1})$$

(for the i-th period, $i = 3, \dots, n$). For the quantities A_i we have the following system of linear algebraic equations

$$\begin{cases} A_1 + r_1 V = A_2 + r_{i1} V_1 + r_{i2}(V - A_1) \\ A_2 + r_{21} V + r_{22}(V - A_1) = A_1 + r_{31}(V - A_1) + r_{32}(V - A_1 - A_2) \\ \qquad \dots \quad \dots \quad \dots \\ A_{n-1} + r_{n-1,n}(V - A_1 - \dots - A_{n-3}) + r_{n-1,2}(V - A_1 - \dots - A_{n-2}) \\ \qquad = A_n + r_{n1}(V - A_1 - \dots - A_{n-2}) + r_{n2}(V - A_1 - \dots - A_{n-1}) \\ A_1 + A_2 + \dots + A_n = V. \end{cases} \tag{2.2}$$

If the current year is bissextile or the period L covers several years some of which are bissextile, then the denominator of the formula (2.1) for all or several i must be 366.

Transposing all the terms that contain the quantities A_i ($i = 1, 2, \dots, n$) to the left side of the equalities and the free terms to their right sides, we arrive to the following system of equations:

$$\begin{cases} (1 + r_{22})A_1 - A_2 = (r_2 - r_1)V \\ (r_3 - r_{22})A_1 + (1 + r_{32})A_2 - A_3 = (r_3 - r_2)V \\ \qquad \dots \quad \dots \quad \dots \\ (r_n - r_{n-1})A_1 + (r_n - r_{n-1})A + 2 + \dots + (r_n - r_{n-1})A_{n-3} \\ \qquad +(r_n - r_{n-1,2})A_{n-2} + (1 + r_{n,2})A_{n-1} - A_n = (r_n - r_{n-1})V \\ A_1 + A_2 + \dots + A_n = V. \end{cases}$$

Dividing each equation of this system by V and introducing the notation

$$x_i = \frac{A_i}{V}, \quad i = 1, 2, \dots, n,$$

we arrive to the system of linear algebraic equations:

$$\begin{cases} (1 + r_{22})x_1 - x_2 = r_2 - r_1 \\ (r_3 - r_{22})x_1 + (1 + r_{22})x_2 - x_3 = r_3 - r_2 \\ \dots \quad \dots \quad \dots \\ (r_n - r_{n-1})x_1 + (r_{n1} - r_{n-1})x_2 + \dots + (r_n - r_{n-1})x_{n-3} \\ +(r_n - r_{n-1,2})x_{n-2} + (1 + r_{n2})x_{n-1} - x_n = r_n - r_{n-1} \\ x_1 + x_2 + \dots + x_n = 1. \end{cases} \tag{2.3}$$

Now we are in a position to formulate our main result.

Theorem 2.1. *Let $n \geq 2$ and let the inequalities*

$$0 < r_1 < \frac{1}{n+1},$$

$$0 < r_{i2} < r_i < \frac{1}{n+1}, \quad i = 2, 3, \dots, n$$

hold. Then the system of equations (2.3) *has a unique solution*

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

and all the components of this solution are positive.

Remark 2.2. The case when $r_{i,2} = r_i$ $(i = 2, \dots, n)$ has been considered by the author in the paper [1].

3 Auxiliary Proposition

Let

$$\Delta_n(r_1, \dots, r_n) = \det \begin{pmatrix} 1 + r_{22} & -1 & \dots & 0 & 0 \\ r_2 - r_{22} & 1 + r_{32} & \dots & 0 & 0 \\ \dots r_n - r_{n-1} & r_n - r_{n-1} & \dots & 1 + r_{n2} & -1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix} \tag{3.1}$$

be the determinant of the system (2.3) (the index n points at the order of the determinant) and let

$$\Delta_n^{(1)}(r_1, \dots, r_n) = \det \begin{pmatrix} r_2 - r_1 & 1 & \dots & 0 & 0 \\ r_3 - r_2 & 1 + r_{32} & \dots & 0 & 0 \\ \dots & & & & \\ r_n - r_{n-1} & r_n - r_{n-1} & \dots & 1 + r_{n2} & -1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix} \tag{3.2}$$

be the determinant that is obtained from the determinant $\Delta_n(r_1, \dots, r_n)$ by substitution of the column of the right-hand members of the system (2.3) for the first column. Then by the Cramer's rule

$$x_1 = \frac{\Delta_n^{(1)}(r_1, \dots, r_n)}{\Delta_n(r_1, \dots, r_n)} \tag{3.3}$$

in the case when the denominator is not equal to zero.

Lemma 3.1. *Let* $n \geq 2$ *and let the inequalities*

$$0 < r_{22} < r_2 < \frac{1}{n+1},$$
$$\dots$$
$$0 < r_{n2} < r_n < \frac{1}{n+1}$$

hold. Then

1. *If* $0 < r_1 < \dfrac{1}{n-1}$, *then the determinant of the system* (2.3) *is positive:*

$$\Delta_n(r_1, \ldots, r_n) > 0. \tag{3.4}$$

2. *If* $0 < r_1 < \dfrac{1}{n-1}$, *then*

$$\Delta_n^{(1)}(r_1, \ldots, r_n) > 0. \tag{3.5}$$

3. *If* $0 < r_1 < \dfrac{1}{n}$, *then*

$$(nr_1 - 1)\Delta_n(r_1, \ldots, r_n) + n\Delta_n^{(1)}(r_1, \ldots, r_n) > 0. \tag{3.6}$$

4. *If* $n \geq 5$, $0 < r_1 < \dfrac{1}{n-1}$, *then*

$$\Delta_n^{(1)}(r_1, \ldots, r_n) < \frac{2}{n-1}\Delta_n(r_1, \ldots, r_n). \tag{3.7}$$

Proof. We prove the statement by induction. The basis of induction. For $n = 2$ one has

$$\Delta_2(r_1, r_2) = \det \begin{pmatrix} 1 + r_{22} & -1 \\ 1 & 1 \end{pmatrix} = 2 + r_{22} > 0,$$

$$\Delta_2^{(1)}(r_1, r_2) = \det \begin{pmatrix} r_2 - r_1 & -1 \\ 1 & 1 \end{pmatrix} = 1 + r_2 - r_1 > 0,$$

$$\begin{aligned} (2r_2 - 1)\Delta_2(r_1, r_2) + 2\Delta_2^{(1)}(r_1, r_2) &= (2r_1 - 1)(2 + r_{22}) + 2(1 + r_2 - r_1) \\ &= 4r_1 + 2r_1r_{22} - 2 - r_{22} + 2 + 2r_2 - 2r_1 \\ &= 2r_1 + 2r_1r_{22} + 2r_2 - r_{22} > 0. \end{aligned}$$

For foundation of induction we shall also need the inequalities (3.5) and (3.4) for $n = 3$:

$$\begin{aligned} \Delta_3^{(1)}(r_1, r_2, r_3) &= \text{(by the rule of Sarrus)} \\ &= (r_2 - r_1)(1 + r_{32}) \cdot 1 + (-1)(-1) \cdot 1 + (r_3 - r_2) \cdot 1 \cdot 0 \\ &\quad -0 \cdot (1 + r_{32}) \cdot 1 - (-1) \cdot 1(r_2 - r_1) - (-1)(r_3 - r_2) \cdot 1 \\ &= r_2 - r_1 + r_2 \cdot r_{32} + 1 + r_2 - r_1 + r_3 - r_2 \\ &= r_3 + r_{32}(r_2 - r_1) + 1 + r_2 - 2r_1 \\ &> r_{32}(1 + r_2 - r_1) + 1 + r_2 - 2r_1 > 0. \end{aligned}$$

Note that

$$\Delta_3^{(1)}(r_1, r_2, r_3) < (1 + r_2)(1 + r_3) < 2. \tag{3.8}$$

Finally,

$$\begin{aligned}
\Delta_3(r_1, r_2, r_3) &= \det\begin{pmatrix} 1+r_{22} & -1 & 0 \\ r_3 - r_{22} & 1+r_{32} & -1 \\ 1 & 1 & 1 \end{pmatrix} = \text{(by the rule of Sarrus)} \\
&= (1+r_{22})(1+r_{32})\cdot 1 + (-1)\cdot(-1)\cdot 1 + (r_3 - r_{22})\cdot 1\cdot 0 \\
&\quad -0\cdot(1+r_{32})\cdot 1 - (-1)(r_3 - r_{22})\cdot 1 - (-1)\cdot 1\cdot(1+r_{22}) \\
&= (1+r_{22})(1+r_{32}) + 2 + r_2 > 3. \qquad (3.9)
\end{aligned}$$

The induction step. Let $n \geq 3$ and let the inequalities (3.4), (3.5) hold for $k = n-1$. Expanding the determinant $\Delta_n(r_1, r_2, \dots, r_n)$ upon the first row, we find:

$$\Delta_n(r_1, r_2, \dots, r_n) = (1+r_{22})\Delta_{n-1}(r_2, r_3, \dots, r_n) + \Delta^{(1)}_{n-1}(r_2, r_3, \dots, r_n). \qquad (3.10)$$

According to the hypothesis of induction, the expression at the right side of this equality is strictly positive which proves the inequality (3.4).

Now let us turn to the inequality (3.6). Since the determinants $\Delta_n(r_1, r_2, \dots, r_n)$ and $\Delta^{(1)}_n(r_1, r_2, \dots, r_n)$ differ only by the first column, their linear combination that is written in the left side of this inequality can be calculated with the help of substitution of the first column of the determinant $\Delta_n(r_1, r_2, \dots, r_n)$ by the corresponding linear combination of the first columns of the both determinants:

$$\begin{aligned}
&(nr_1 - 1)\Delta_n(r_1, r_2, \dots, r_n) + n\Delta^{(1)}_n(r_1, r_2, \dots, r_n) \\
&= \det\begin{pmatrix} (nr_1-1)(1+r_{22}) + n(r_2 - r_1) & -1 & \dots & 0 & 0 \\ (nr_1-1)(r_3 - r_{22}) + n(r_3 - r_2) & 1+r_{32} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ ((nr_1-1)+n)(r_n - r_{n-1}) & r_n - r_{n-1} & \dots & 1+r_{n2} & -1 \\ (nr_1-1)+n & 1 & \dots & 1 & 1 \end{pmatrix}. \qquad (3.11)
\end{aligned}$$

The partial derivative of this determinant respectively to the variable r_{22} equals to

$$\begin{aligned}
&(nr_1 - 1)\det\begin{pmatrix} 1 & -1 & \dots & 0 & 0 \\ -1 & 1+r_{32} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & r_n - r_{n-1} & \dots & 1+r_{n-2} & -1 \\ 0 & 1 & \dots & 1 & 1 \end{pmatrix} \\
&= \text{(expand respectively to the first row)} \\
&= (nr_1 - 1)(\Delta_{n-1}(r_2, r_3, \dots, r_n) - \Delta_{n-2}(r_3, r_4, \dots, r_n) \\
&= \text{(by the formula (3.10))} \\
&= (nr_1 - 1)(r_{32}\Delta_{n-2}(r_3, r_4, \dots, r_n) + \Delta^{(1)}_{n-2}(r_3, r_4, \dots, r_n)).
\end{aligned}$$

In this product the second efficient is positive by the induction hypothesis and the first efficient is negative in view of the assumption $r_1 < \frac{1}{n}$. Consequently, the required partial derivative is negative.

Thus substituting the quantity r_{22} by r_2 we decrease the determinant (3.11). The obtained determinant (after transformation of the expression in the left-upper corner) can be written in a form

$$\det \begin{pmatrix} (n-1)r_2 - 1 + nr_1(1+r_2) & -1 & \dots & 0 & 0 \\ ((n-1)+nr_1)(r_3 - r_2) & 1 + r_{32} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ ((n-1)+nr_1)(r_n - r_{n-1}) & r_n - r_{n-1} & \dots & 1 + r_{n2} & -1 \\ (n-1)+nr_1 & 1 & \dots & 1 & 1 \end{pmatrix} \tag{3.12}$$

Expanding the latter determinant with respect to the first row, we obtain:

$$((n-1)r_2 - 1 + nr_1(1+r_2))\Delta_{n-1}(r_2, r_3, \dots, r_n) + ((n-1)+nr_1)\Delta_{n-1}^{(1)}(r_2, r_3, \dots, r_n).$$

By the induction hypothesis, the determinants

$$\Delta_{n-1}(r_2, r_3, \dots, r_n) \quad \text{and} \quad \Delta_{n-1}^{(1)}(r_2, r_3, \dots, r_n)$$

are positive and so the latter expression is not less than

$$((n-1)r_2 - 1)\Delta_{n-1}(r_2, r_3, \dots, r_n) + (n-1)\Delta_{n-1}^{(1)}(r_2, r_3, \dots, r_n).$$

Then latter expression is, in one's turn, strictly positive.

Thus the determinant (3.12) along with the determinant (3.11) is positive. The inequality (3.6) is proved.

Now let us turn to the point 2) of our lemma. First of all, note that the partial derivative

$$\frac{\partial \Delta_n^{(1)}(r_1, r_2, \dots, r_n)}{\partial r_1}$$

is negative. This follows from the fact that this derivative equals to the quantity

$$-\Delta_{n-1}(r_2, r_3, \dots, r_n)$$

which is negative by the induction hypothesis. Therefore increasing if necessary the quantity r_1 we assume that the inequality

$$r_1 > r_2 \tag{3.13}$$

holds. Under this assumption the partial derivative

$$\frac{\partial \Delta_n^{(1)}(r_1, r_2, \dots, r_n)}{\partial r_{32}}$$

is negative. Indeed, if $n \geq 5$, then this derivative equals to

$$
\begin{aligned}
&\det \begin{pmatrix} r_2 - r_1 & 0 & 0 & \dots & 0 & 0 \\ r_3 - r_2 & 1 & -1 & \dots & 0 & 0 \\ r_4 - r_3 & -1 & 1 + r_{42} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ r_n - r_{n-1} & 0 & r_n - r_{n-1} & \dots & 1 + r_{n2} & -1 \\ 1 & 0 & 1 & \dots & 1 & 1 \end{pmatrix} \\
= \ & (r_2 - r_1) \det \begin{pmatrix} 1 & -1 & \dots & 0 & 0 \\ -1 & 1 + r_{42} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & r_n - r_{n-1} & \dots & 1 + r_{n2} & -1 \\ 0 & 1 & \dots & 1 & 1 \end{pmatrix} \\
= \ & \text{(expand the determinant respectively to the first row)} \\
= \ & (r_2 - r_1)(\Delta_{n-2}(r_3, r_4, \dots, r_n) - \Delta_{n-3}(r_4, r_5, \dots, r_n)) \\
= \ & \text{(by the formula (13))} = (r_2 - r_1)(r_{42}\Delta_{n-3}(r_4, r_3, \dots, r_n))) \\
& + \Delta^{(1)}_{n-3}(r_4, r_5, \dots, r_n)).
\end{aligned}
$$

In the obtained product the first efficient is negative in view of the inequality (3.13) and the second efficient is positive by the hypothesis of induction. Consequently, the considered partial derivative is negative.

As to the case $n = 4$, then this partial derivative equals to

$$
\begin{aligned}
&\det \begin{pmatrix} r_2 - r_1 & 0 & 0 & \dots & 0 & 0 \\ r_3 - r_2 & 1 & -1 & \dots & 0 & 0 \\ r_4 - r_3 & -1 & 1 + r_{42} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ r_n - r_{n-1} & 0 & r_n - r_{n-1} & \dots & 1 + r_{n2} & -1 \\ 1 & 0 & 1 & \dots & 1 & 1 \end{pmatrix} \\
= \ & (r_2 - r_1)(\det \begin{pmatrix} 1 + r_{42} & -1 \\ 1 & 1 \end{pmatrix} - 1) = (r_2 - r_1)(1 + r_{42}) < 0.
\end{aligned}
$$

Expanding the determinant $\Delta^{(1)}_n(r_1, r_2, \dots, r_n)$ respectively to the first row, we obtain:

$$\Delta^{(1)}_n(r_1, r_2, \dots, r_n) = (r_2 - r_1)\Delta_{n-1}(r_2, r_3, \dots, r_n) + \Delta^{(1)}_{n-1}(r_2, r_3, \dots, r_n).$$

Since the determinants $\Delta_{n-1}(r_2, r_3, \dots, r_n)$ and $\Delta^{(1)}_{n-1}(r_2, r_3, \dots, r_n)$ differ only by the first column, their linear combination that is written in the right side of this equality can be calculated with the help of substitution of the first column of the determinant $\Delta_{n-1}(r_2, r_3, \dots, r_n)$ by the corresponding linear combination of the first columns of the both determinants:

$$\Delta_n^{(1)}(r_1, r_2, \ldots, r_n)$$
$$= \det \begin{pmatrix} (r_2 - r_1)(1 + r_{32} + (r_3 - r_2) & -1 & \ldots & 0 & 0 \\ (r_2 - r_1)(r_4 - r_{32}) + (r_4 - r_3) & 1 + r_{42} & \ldots & 0 & 0 \\ (r_2 - r_1 + 1)(r_5 - r_4) & 1 + r_{42} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (r_2 - r_1 + 1)((r_n - r_{n-1})) & (r_n - r_{n-1}) & \ldots & 1 + r_{n2} & -1 \\ (r_2 - r_1 + 1) & 1 & \ldots & 1 & 1 \end{pmatrix}.$$

Since

$$\frac{\partial \Delta_n^{(1)}(r_1, r_2, \ldots, r_n)}{\partial r_{32}} < 0,$$

substituting the quantity r_{32} by r_3 we decrease the determinant $\Delta_n^{(1)}(r_1, r_2, \ldots, r_n)$. As a result of such substitution the determinant which stands in the right side of the latter equality will take a form

$$\det \begin{pmatrix} (r_2 - r_1)(1 + r_3) + (r_3 - r_2) & -1 & \ldots & 0 & 0 \\ (1 + r_2 - r_1)(r_4 - r_3) & 1 + r_{42} & \ldots & 0 & 0 \\ (1 + r_2 - r_1)(r_5 - r_4) & 1 + r_{42} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (1 + r_2 - r_1)(r_n - r_{n-1}) & r_n - r_{n-1} & \ldots & 1 + r_{n2} & -1 \\ (1 + r_2 - r_1) & 1 & \ldots & 1 & 1 \end{pmatrix}.$$

Expanding this determinant upon the first row, we shall obtain:

$$((r_2 - r_1)(1 + r_3) + (r_3 - r_2))\Delta_{n-2}(r_3, r_4, \ldots, r_n) + (1 + r_2 - r_1)\Delta_{n-2}^{(1)}(r_3, r_4, \ldots, r_n) \tag{3.14}$$

Let us a transform the coefficient at the determinant in the first item of the expression (3.13):

$$(r_2 - r_1)(1 + r_3) + (r_3 - r_2) = r_2 - r_1 + r_2 r_3 - r_1 r_3 + r_3 - r_2 = r_3(1 + r_2 - r_1) - r_1.$$

From this, using the induction hypothesis concerning the inequalities (3.4), (3.5) and the assumption $r_1 < \dfrac{1}{n-1}$, we obtain that the expression (3.14) not less than

$$\frac{1}{n-1}\Big(((n-2)r_3 - 1)\ \Delta_{n-2}(r_3, r_4, \ldots, r_n) + (n-2)\ \Delta_{n-2}^{(1)}(r_3, r_4, \ldots, r_n).\Big)$$

Now, using the hypothesis of induction concerning the inequality (3.6) and taking into account the inequality $r_3 < \dfrac{1}{n-2}$, we finally find that the expression (3.14) is strictly positive, which proves the inequality (3.5).

To prove the inequality (3.7), note first of all that if

$$0 < r_1 < \frac{1}{5},$$
$$0 < r_{22} < r_2 < \frac{1}{5},$$
$$0 < r_{32} < r_3 < \frac{1}{5},$$

then

$$\Delta_3^{(1)}(r_1, r_2, r_3) < \frac{1}{2}\Delta_3(r_1, r_2, r_3). \tag{3.15}$$

Indeed, as we already know, the inequality (3.9) valid while

$$\Delta_3^{(1)}(r_1, r_2, r_3) < (1 + r_2)(1 + r_3) < \left(\frac{6}{5}\right)^2 < \frac{3}{2}.$$

Then let

$$0 < r_1 < \frac{1}{3},$$
$$0 < r_{22} < r_2 < \frac{1}{5},$$
$$0 < r_{32} < r_3 < \frac{1}{5},$$
$$0 < r_{42} < r_4 < \frac{1}{5}.$$

Then

$$\begin{aligned}
&\Delta_4^{(1)}(r_1,r_2,r_3,r_4) - \frac{1}{2}\,\Delta_4^{(1)}(r_1,r_2,r_3,r_4)\\
&= \det\begin{pmatrix} r_2-r_1 & -1 & 0 & 0\\ r_3-r_2 & 1+r_{32} & -1 & 0\\ r_4-r_3 & r_4-r_{32} & 1+r_{42} & -1\\ 1 & 1 & 1 & 1\end{pmatrix}\\
&\quad -\frac{1}{2}\det\begin{pmatrix} 1+r_{22} & -1 & 0 & 0\\ r_3-r_{22} & 1+r_{32} & -1 & 0\\ r_4-r_3 & r_4-r_{32} & 1+r_{42} & -1\\ 1 & 1 & 1 & 1\end{pmatrix}\\
&= \det\begin{pmatrix} (r_2-r_1)-\frac{1}{2}(1+r_{22}) & -1 & 0 & 0\\ (r_3-r_2)-\frac{1}{2}(r_3-r_{22}) & 1+r_{32} & -1 & 0\\ \frac{1}{2}(r_4-r_3) & r_4-r_{32} & 1+r_{42} & -1\\ \frac{1}{2} & 1 & 1 & 1\end{pmatrix}\\
&< \det\begin{pmatrix} -\frac{1}{4} & -1 & 0 & 0\\ (r_3-r_2)-\frac{1}{2}(r_3-r_{22}) & 1+r_{32} & -1 & 0\\ \frac{1}{2}(r_4-r_3) & r_4-r_{32} & 1+r_{42} & -1\\ \frac{1}{2} & 1 & 1 & 1\end{pmatrix}.
\end{aligned}$$

It is clear that the partial derivative of the latter determinant respectively to the variable r_{22} which equals to

$$\frac{1}{2}\det\begin{pmatrix} 1+r_{42} & -1\\ 1 & -1\end{pmatrix}$$

is positive. Therefore this determinant less than

$$\det\begin{pmatrix} -\frac{1}{4} & -1 & 0 & 0\\ \frac{1}{2}(r_3-r_2) & 1+r_{32} & -1 & 0\\ \frac{1}{2}(r_4-r_3) & r_4-r_{32} & 1+r_{42} & -1\\ \frac{1}{2} & 1 & 1 & 1\end{pmatrix} = \text{(expand respectively to the first row)}$$

$$= -\frac{1}{4}\Delta_3(r_2,r_3,r_4) + \frac{1}{2}\Delta_3^{(1)}(r_2,r_3,r_4).$$

The latter expression is negative by the inequality (3.15). Thus

$$\Delta_4^{(1)}(r_1, r_2, r_3, r_4) < \frac{1}{2}\,\Delta_4(r_1, r_2, r_3, r_4). \tag{3.16}$$

Note that all the more

$$\Delta_4^{(1)}(r_1, r_2, r_3, r_4) < \frac{2}{3}\,\Delta_4(r_1, r_2, r_3, r_4).$$

that is the inequality (3.7) is valid for $n = 4$.

Finally, let $n \geq 5$. Consider the difference

$$\Delta_n^{(1)}(r_1, r_2, \ldots, r_n) - \frac{2}{n-1}\,\Delta_n(r_1, r_2, \ldots, r_n).$$

Since the determinants in the minuend and in the subtrahend differ only by the first column, this difference can be written in a form

$$\det\begin{pmatrix} (r_2 - r_1) - \frac{2}{n-1}(1 + r_{22}) & -1 & \ldots & 0 & 0 \\ (r_3 - r_2) - \frac{2}{n-1}(r_3 - r_{22}) & 1 + r_{32} & \ldots & 0 & 0 \\ \frac{n-3}{n-1}(r_4 - r_3) & r_4 - r_{32} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{n-3}{n-1}(r_n - r_{n-1}) & r_n - r_{n-1} & \ldots & 1 + r_{n2} & -1 \\ \frac{n-3}{n-1} & 1 & \ldots & 1 & 1 \end{pmatrix}. \tag{3.17}$$

Consider the partial derivative of this determinant respectively to the variable r_{32}. It equals to

$$\det\begin{pmatrix} (r_2 - r_1) - \frac{2}{n-1}(1 + r_{22}) & 0 & \ldots & 0 & 0 \\ (r_3 - r_2) - \frac{2}{n-1}(r_3 - r_{22}) & 1 & \ldots & 0 & 0 \\ \frac{n-3}{n-1}(r_4 - r_3) & -1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{n-3}{n-1}(r_n - r_{n-1}) & 0 & \ldots & 1 + r_{n2} & -1 \\ \frac{n-3}{n-1} & 0 & \ldots & 1 & 1 \end{pmatrix}$$

$$= \left((r_2 - r_1) - \frac{2}{n-1}(1 + r_{22})\right)\left(\Delta_{n-2}(r_3, r_4, \ldots, r_n) - \Delta_{n-3}(r_4, r_5, \ldots, r_n)\right).$$

We have: The first efficient is negative. When $n \geq 5$ the second efficient is positive by the formula (3.10). Thus the partial derivative of the determinant (3.17) respectively to the variable r_{32} is negative and so putting $r_{32} = 0$ we strengthen the inequality (3.7).

It is clear that putting $r_1 = 0$, $r_{22} = 0$, we strengthen the inequality also.

Now let us write the determinant (3.17) in a form

$$\det \begin{pmatrix} -\dfrac{2}{n-1} & -1 & \dots & 0 & 0 \\ \dfrac{n-3}{n-1} r_3 & 1 & \dots & 0 & 0 \\ \dfrac{n-3}{n-1}(r_4 - r_3) & r_4 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{n-3}{n-1}(r_n - r_{n-1}) & r_n - r_{n-1} & \dots & 1 + r_{n2} & -1 \\ \dfrac{n-3}{n-1} & 1 & \dots & 1 & 1 \end{pmatrix} +$$

$$+r_2 \det \begin{pmatrix} 1 & -1 & \dots & 0 & 0 \\ -1 & 1 & \dots & 0 & 0 \\ 0 & r_4 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & r_n - r_{n-1} & \dots & 1 + r_{n2} & -1 \\ 0 & 1 & \dots & 1 & 1 \end{pmatrix}. \tag{3.18}$$

The first item which equals to

$$-\frac{2}{n-1}\,\Delta_{n-1}(0, r_3, \dots, r_n) + \frac{n-3}{n-1}\,\Delta^{(1)}_{n-1}(0, r_3, \dots, r_n)$$

by the induction hypothesis does not exceed

$$\begin{aligned} -\frac{2}{n-1}\,\Delta_{n-1}(0, r_3, \dots, r_n) + 2\,\frac{n-3}{(n-1)(n-2)}\,\Delta_{n-1}(0, r_3, \dots, r_n) \\ = -\frac{2}{(n-1)(n-2)}\,\Delta_{n-1}(0, r_3, \dots, r_n). \end{aligned} \tag{3.19}$$

The second item can be transformed as

$$r_2(\Delta_{n-1}(0, r_3, \dots, r_n) - \Delta_{n-2}(r_3, \dots, r_n))$$

(we expand the determinant respectively to the first row). Since $r_{32} = 0$, by the formula (3.10), the expression in the parentheses equals to

$$\begin{aligned} \Delta^{(1)}_{n-2}(r_3, r_4, \dots, r_n) &< \text{(by the induction hypothesis)} \\ &< \frac{2}{(n-3)}\,\Delta_{n-2}(r_3, r_4, \dots, r_n) \\ &\leq \text{(by the formula (3.10))} \\ &< \frac{2}{(n-3)}\,\Delta_{n-1}(0, r_3, \dots, r_n). \end{aligned}$$

Since $r_2 < \dfrac{1}{n+1}$ by the assumption, the second item of the sum (3.18) is less than

$$\frac{2}{(n+1)(n-3)}\,\Delta_{n-1}(0, r_3, \ldots, r_n)$$

Since, in view of the formula (3.19), the first item does not exceed

$$-\frac{2}{(n-1)(n-2)}\,\Delta_{n-1}(0, r_2, \ldots, r_n),$$

the sum (3.18) is strictly less than

$$\begin{aligned} &\left(-\frac{2}{(n-1)(n-2)} + \frac{2}{(n+1)(n-3)}\right)\,\Delta_{n-1}(0, r_3, \ldots, r_n) \\ = &\ \ 2\,\frac{-n+5}{(n-1)(n-2)(n+1)(n-3)}\,\Delta_{n-1}(0, r_3, \ldots, r_n). \end{aligned}$$

when $n \geq 5$ the latter expression does not exceed zero. The proof is complete. □

4 Proof of the Main Result

In this section we prove our main theorem, Theorem 2.1.

Proof of Theorem 2.1. Let $n \geq 5$. In view of the inequality (3.4), the determinant of the system (2.3) differs from zero. In view of the inequalities (3.4), (3.5) and (3.7),

$$0 < x_1 < \frac{2}{n-1}.$$

Considering the quantity x_1 known, let us drop the first equation of the system (2.3). We shall obtain the system of linear algebraic equalities respectively to the variables $x_2, \ldots, x_n$ with the matrix

$$\begin{pmatrix} 1+r_{32} & -1 & 0 & \ldots & 0 & 0 \\ r_4 - r_{32} & 1+r_{42} & -1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ r_n - r_{n-1} & r_n - r_{n-1} & r_n - r_{n-1} & \ldots & 1+r_{n2} & -1 \\ 1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix}. \tag{4.1}$$

The column of the right sides will be written in a form

$$\begin{pmatrix} (r_3 - r_2) - x_1(r_3 - r_{22}) \\ (1 - x_1)(r_4 - r_3) \\ \ldots \\ (1 - x_1)(r_n - r_{n-1}) \\ 1 - x_1 \end{pmatrix}.$$

Let us introduce the new variables

$$x'_2 = \frac{x_2}{1 - x_1}, \quad x'_3 = \frac{x_3}{1 - x_1}, \dots, x'_n = \frac{x_n}{1 - x_1}.$$

These variables satisfy the system of linear algebraic equations with the matrix (4.1) for which the column of the right sides equals to

$$\begin{pmatrix} \dfrac{(r_3 - r_2) - x_1(r_3 - r_{22})}{1 - x_1} \\ r_4 - r_3 \\ \dots \\ r_n - r_{n-1} \\ 1 \end{pmatrix}.$$

Let us write the first element of this column in a form

$$r_3 - (r_2 + \frac{x_1}{1 - x_1}(r_2 - r_{22})).$$

Since the function

$$f(x_1) = \frac{x_1}{1 - x_1} = -1 + \frac{1}{1 - x_1}$$

is increasing and $x_1 < \dfrac{2}{n-1}$, we have

$$\frac{x_1}{1 - x_1} < \frac{2}{n - 3}.$$

In one's turn,

$$0 < r_2 - r_{22} < \frac{1}{n + 1}.$$

Introducing the notation

$$r'_2 = r_2 + \frac{x_1}{1 - x_1}(r_2 - r_{22}),$$

we obtain:

$$0 < r'_2 < \frac{1}{n + 1} + \frac{2}{(n + 1)(n - 3)} = \frac{n - 1}{(n + 1)(n - 3)} \leq \frac{1}{n - 2}$$

when $n \geq 5$.

Thus if we substitute r_2 by r'_2, then we get to the conditions of Lemma 3.1 and consequently

$$0 < x'_2 < \frac{2}{n - 2}.$$

In particular,

$$x_2 = x'_2(1 - x_1) > 0.$$

Repeating this procedure $n-4$ times we shall prove the inequalities

$$x_1 > 0, \dots, x_{n-4} > 0.$$

Then we shall obtain the system of the fourth order of the form (2.3) for which

$$\begin{aligned} 0 < r_1 < \frac{1}{3}, \\ 0 < r_{22} < r_2 < \frac{1}{5}, \\ 0 < r_{32} < r_3 < \frac{1}{5}, \\ 0 < r_{42} < r_4 < \frac{1}{5}. \end{aligned}$$

By the inequalities (3.4) and (3.5), for this system $x_1 > 0$ and so for initial system of equations $x_{n-3} > 0$ and, by the inequality (3.16), $x_1 < \frac{1}{2}$ and so $\frac{x_1}{1-x_1} < 1$.

Passing on to the system of the third order, considering the value of x_1 in the system of the fourth order known, we shall obtain the system of the form (2.3) for which

$$\begin{aligned} r_1 < \frac{2}{5}, \\ r_2 < \frac{1}{4}, \\ r_3 < \frac{1}{4}. \end{aligned}$$

For such a system $x_1 > 0$ and so for the initial system $x_{n-2} > 0$ and in view of the inequalities (3.8) and (3.9), for the system of the third order $x_1 < \frac{2}{3}$ and $\frac{x_1}{1-x_1} < 2$.

Passing on to the system of the second order, considering the value of x_1 known, we shall obtain the system of the form (2.3) for which

$$\begin{aligned} r_1 < \frac{3}{4}, \\ r_2 < \frac{1}{3}. \end{aligned}$$

For this system

$$\begin{aligned} \Delta_2(r_1, r_2) &= \det \begin{pmatrix} 1 + r_{22} & -1 \\ 1 & 1 \end{pmatrix} = 2 + r_{22} > 0, \\ \Delta_2^{(1)}(r_1, r_2) &= \det \begin{pmatrix} r_2 - r_1 & -1 \\ 1 & 1 \end{pmatrix} = 1 + r_2 - r_1 > 0. \end{aligned}$$

The determinant $\Delta_2^{(2)}(r_1, r_2)$ that is obtained by substitution of the second column of the determinant $\Delta_2(r_1, r_2)$ by the column of the right sides equals to

$$\det \begin{pmatrix} 1 + r_{22} & r_2 - r_1 \\ 1 & 1 \end{pmatrix} = 1 + r_{22} - r_2 + r_1 > 0.$$

From this in the obtained system of the second order

$$x_1 > 0, \quad x_2 > 0$$

and, correspondingly, in the initial system of equations

$$x_{n-1} > 0, \quad x_n > 0.$$

This completes the proof. □

References

[1] Mark A. Nudel′man, *Positivity of outpayments according to the annuity scheme of credit repayment,* Reports of the Odessa Seminar in Discrete Mathematics, no. 5, 2007, p. 36–43 (Russian).