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Attenuance and Resonance in a Periodically Forced Sigmoid Beverton–Holt Model

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Abstract

In this paper we continue to study the global behavior of the periodically forced Sigmoid Beverton–Holt model

$$y_{n+1} = \frac{a_n y_n^\delta}{1 + y_n^\delta}, \quad n = 0, 1, \ldots,$$

where the initial condition y_0 is positive and $\{a_n\}$ is a positive sequence. First, we establish the conditions for the equivalence of the above equation and the equation of the form

$$x_{n+1} = \frac{r Q_n x_n^\delta}{Q_n^\delta + (r-1) x_n^\delta}, \quad n = 0, 1, \ldots,$$

where $\delta > 0, r > 1, \{Q\}_n$ is a sequence, and $\{x_n\}$ is a positive sequence. Then we prove that if $0 < \delta \leq 1$, the unique periodic cycle $\{\bar{x}_n\}$ (which exists) is g-attenuant, that is

$$\left(\prod_{i=0}^{p-1} \bar{x}_i\right)^{\frac{1}{p}} < \left(\prod_{i=0}^{p-1} K_i\right)^{\frac{1}{p}},$$

where $\{K_n\}$ represents the unique sequence of carrying capacities which will be defined in terms of Q_n . Also, we show that in the cases $r > \delta > 1$ or $\delta > r > 1$

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then the periodic cycle $\{\bar{x}_n\}$ (which exists) is g-attenuant with respect to carrying capacities $\{K_n\}$ and g-resonant with respect to Allee thresholds $\{T_n\}$ which will also be defined in terms of Q_n, that is, it satisfies

$$\left(\prod_{i=0}^{p-1} T_i\right)^{\frac{1}{p}} < \left(\prod_{i=0}^{p-1} \bar{x}_i\right)^{\frac{1}{p}} < \left(\prod_{i=0}^{p-1} K_i\right)^{\frac{1}{p}}.$$

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1 Introduction

In this paper we continue the study of the global asymptotic behavior of the periodically forced Sigmoid Beverton–Holt model initiated in [18]. More precisely, we will investigate the g-attenuance and g-resonance of periodic solutions of periodically forced Sigmoid Beverton–Holt model

$$y_{n+1} = \frac{a_n y_n^\delta}{1 + y_n^\delta}, \quad n = 0, 1, \ldots \tag{1.1}$$

with initial condition

$$y_0 > 0$$

and where $\{a_n\}$ is a positive p-periodic sequence such that

$$a_n > 0, \quad a_{n+p} = a_n, \quad n = 0, 1, \ldots \quad \text{and} \quad \delta > 0. \tag{1.2}$$

The autonomous case of equation (1.1)

$$y_{n+1} = \frac{a y_n^\delta}{1 + y_n^\delta}, \quad n = 0, 1, \ldots, \tag{1.3}$$

where $a, \delta > 0$ has been introduced by Thompson [44] as a "*depensatory generalization of the Beverton–Holt stock-recruitment relationship used to develop a set of constraints designed to safeguard against overfishing*". This model has been used in the study of fish population dynamics, particularly when overfishing is present [16, 35–37, 43]. A very important feature of the Sigmoid Beverton–Holt model is that, in the case $\delta > 1,$ it exhibits the so-called "Allee effect".

The "Allee effect" is originally attributed to W. C. Allee [1, 7, 42], who broadly defined it as a "… *positive relationship between any component of individual fitness and either numbers or density of conspecifics*". Practically speaking, the Allee effect causes, at low population densities, per capita birth rate declines. Under such a scenario, at low population densities, the population may slide into extinction. There are various scenarios in which the Allee effect appears in nature (see [7, 8] and references cited therein) and can be attributed, for example, to difficulties in finding mates when the population size is small, a higher mortality rate in juveniles when there are not enough adults to protect them from predation, or uncontrollable harvesting, as in overfishing. On the other hand the Allee effect can be beneficial in some situations such as in controlling a population of fruit flies which are considered to be one of the worst insect pests in agriculture [8]. The techniques used to control them is the release of sterile males to create an Allee effect.

Several discrete mathematical models exhibiting the Allee effect are known and studied in the literature ([2, 16, 35–37, 43] and references cited therein). They all have the following features in common: (i) the existence of three equilibrium points: 0, T - Allee threshold, and K - carrying capacity of the environment ($0 < T < K$); (ii) equilibria 0 and K are stable, while T is unstable; (iii) if the population size drops below T, then the population slides into extinction, so it approaches 0.

Periodically forced population models exhibiting the Allee effect are relatively new in the literature [15, 18, 34, 45]. In [34] several periodically forced discrete models exhibiting the Allee effect are studied, while a class of general unimodal maps with such properties has been investigated in [15]. The effects of harvesting in some periodically forced population models with Allee effect are studied in [45].

In the special case when $\delta = 1$, equation (1.1) reduces to the well-known periodically forced Pielou logistic equation

$$y_{n+1} = \frac{a_n y_n}{1 + y_n}, \quad n = 0, 1, \ldots$$

($\{a_n\}$ is a positive p-periodic sequence) which is also equivalent to the periodically forced Beverton–Holt equation

$$x_{n+1} = \frac{r_n x_n}{1 + (r_n - 1)\frac{x_n}{K_n}}, \qquad n = 0, 1, \ldots \tag{1.4}$$

($\{K_n\}$ and $\{r_n\}$ are positive p-periodic sequences, $r_n > 1$). The dynamics and properties of both equations have been studied in great detail in recent literature (see for example [3–6, 10–14, 17, 19–28, 32, 38, 41]).

The concept of the attenuance and resonance of periodic cycles was originally introduced by Cushing and Henson when they formulated the well-known conjecture for the periodically forced Beverton–Holt model (1.4) in the case $r_n = r > 1$, $n = 0, 1, \ldots$ (see [10]); namely we have

Definition 1.1. A p-periodic cycle of a difference equation (if such exists) $\{\bar{x}_n\}$ is attenuant (resonant) if its average value is less (greater) than the average value of the carrying capacities $\{K_n\}$, where $\{K_n\}$ is a q-periodic sequence; that is,

$$av(\bar{x}_n) < av(K_n) \qquad (av(\bar{x}_n) > av(K_n))$$

where $av(\bar{x}_n) = \dfrac{1}{p}\displaystyle\sum_{k=0}^{p-1} \bar{x}_k$ and $av(K_n) = \dfrac{1}{q}\displaystyle\sum_{k=0}^{q-1} K_k$.

In [22, 23] the geometric mean was used as the average of the periodic cycle and the carrying capacity instead of the arithmetic mean. That led to the concept of g-attenuance and g-resonance:

Definition 1.2. A p-periodic cycle of a difference equation (if such exists) $\{\bar{x}_n\}$ is g-attenuant (g-resonant) if its geometric mean is less (greater) than the geometric mean of the carrying capacities $\{K_n\}$, where $\{K_n\}$ is a q-periodic sequence; that is,

$$\left(\prod_{i=0}^{p-1} \bar{x}_i\right)^{1/p} < \left(\prod_{i=0}^{q-1} K_i\right)^{1/q} \quad \left(\left(\prod_{i=0}^{p-1} \bar{x}_i\right)^{1/p} > \left(\prod_{i=0}^{q-1} K_i\right)^{1/q}\right).$$

Our motivation for using the geometric mean comes from population dynamics, where it is quite common practice (particularly when deriving population models by applying correlation and regression techniques [40]) to use a logarithm of the population density (known as "log population density") instead of the population density itself. If $\{x_n\}$ denotes a population density then the corresponding log population density $\{X_n\}$ is defined as

$$X_n = \ln(x_n), \quad n = 0, 1, \ldots.$$

Assume that a given population model has a p-periodic cycle $\{\bar{x}_n\}$. Then $\{\bar{X}_n\}$, where

$$\bar{X}_n = \ln(\bar{x}_n), \quad n = 0, 1, \ldots$$

is a corresponding p-periodic cycle for the log population density. So, we have

$$av(\bar{X}_n) = \frac{1}{p}\sum_{k=0}^{p-1} \bar{X}_k = \frac{1}{p}\sum_{k=0}^{p-1} \ln(\bar{x}_k) = \ln\left(\prod_{i=0}^{p-1} \bar{x}_i\right)^{\frac{1}{p}},$$

that is, the average of a p-periodic cycle of the log population density equals the logarithm of the geometric mean of a p-periodic cycle of the population density. Therefore, the g-attenuance or g-resonance can be interpreted as the attenuance or resonance associated with the log population density, respectively.

Next, we summarize some results from [18] that will be useful in the sequel:

Lemma 1.3. *Assume that $\{a_n\}$ is a positive periodic sequence with period p and $\delta > 0$. Let the sequence of functions $\{f_n\}$ be defined by*

$$f_n(x) = \frac{a_n x^\delta}{1 + x^\delta}, \qquad x > 0, \ n = 0, 1, \ldots \tag{1.5}$$

and let $a_{crit} = \delta(\delta - 1)^{1/\delta - 1}$, provided $\delta > 1$. Then the following statements are true:

(i) If $\delta \in (0, 1)$, then function f_n has the unique positive fixed point k_n.

(ii) If $\delta > 1$ and $a_n < a_{crit}$, then the function f_n does not have positive fixed points and for all $x > 0$, $f_n(x) < x$.

(iii) If $a_n = a_{crit}$, then the function f_n has the unique positive fixed point k_n.

(iv) If $a_n > a_{crit}$, then the function f_n has two positive fixed points t_n and k_n ($0 < t_n < a_n(\delta - 1)/\delta < k_n$).

In the case when $\delta > 1$ and $\min\{a_1, \ldots, a_p\} > a_{crit}$ we have $a_n > a_{crit}, \quad n = 0, 1, \ldots,$ so for every $n = 0, 1, \ldots$ f_n has two positive fixed points t_n and k_n, respectively, such that $0 < t_n < a_n(\delta - 1)/\delta < k_n$. Each of the sequences $\{t_n\}$ and $\{k_n\}$ is p-periodic. We call $\{k_n\}$ the *sequence of carrying capacities*, and $\{t_n\}$ the *sequence of Allee thresholds.* Similarly, in the case $\delta \in (0, 1)$, each of the functions f_n has a unique fixed point k_n and the sequence $\{k_n\}$ represents the *sequence of carrying capacities.*

Theorem 1.4. *Assume that $\{a_n\}$ is a positive periodic sequence with period p and $\delta \in (1, \infty)$. Let*

$$\min\{a_1, \ldots, a_p\} > a_{crit}.$$

Then the following statements are true:

(i) Equation (1.1) has an invariant interval $[\alpha, \beta]$ where

$$\max\{t_1, \ldots, t_p\} < \alpha < \min\{k_1, \ldots, k_p\}, \quad \beta \geq \max\{a_1, \ldots, a_p\}$$

and where $\{k_n\}$ and $\{t_n\}$ are p-periodic sequences of carrying capacities and Allee thresholds of (1.1), respectively.

(ii) In the interval $[\alpha, \beta]$, there exists a unique p-periodic solution $\{\bar{y}_n\}$ of equation (1.1) ($\bar{y}_n \in [\alpha, \beta]$ for $n = 0, 1, \ldots$) which attracts all positive solutions of equation (1.1) with initial conditions in $[\alpha, \infty)$.

(iii) In the interval $[\gamma, \varepsilon]$ where

$$0 < \gamma < \min\{t_1, \ldots, t_p\},$$
$$\max\{t_1, \ldots, t_p\} < \varepsilon < \min\{a_1, \ldots, a_p\}(\delta - 1)/\delta < \min\{k_1, \ldots, k_p\}.$$

there exists a unique p-periodic solution $\{\tilde{y}_n\}$ ($\tilde{y}_n \in [\gamma, \varepsilon]$) of equation (1.1), which repels all positive solutions with initial conditions in $[\gamma, \varepsilon]$.

Theorem 1.5. *Let $\{a_n\}$ be a positive periodic sequence with period p and let $\delta \in (0, 1)$. Then equation (1.1) has a unique p-periodic positive solution $\{\bar{y}_n\}$ which is a global attractor of all positive solutions of (1.1).*

2 Preliminaries

In this section we establish the condition for the equivalence of the Sigmoid Beverton–Holt equation (1.1) and the equation of the form

$$x_{n+1} = \frac{rQ_n x_n^\delta}{Q_n^\delta + (r-1)x_n^\delta} \tag{2.1}$$

where $\delta > 0, r > 1, \{Q_n\}$ is a p-periodic sequence, and $\{x_n\}$ is a positive sequence.

Lemma 2.1. *Consider the difference equation* (1.1)*, where* $\delta > 0$*,* $\{a_n\}$ *is a positive* p*-periodic sequence, and the initial condition* $y_0 > 0$*. Assume that the equation*

$$A = r(r-1)^{\frac{1-\delta}{\delta}}, \tag{2.2}$$

where

$$A = \left(\prod_{i=0}^{p-1} a_i\right)^{\frac{1}{p}}, \tag{2.3}$$

has a solution $r > 1$*. Let the sequence* $\{Q_n\}$ *be defined by*

$$Q_n = \begin{cases} Q > 0 & \text{for } n = 0 \\ \dfrac{\left(r(r-1)^{\frac{1-\delta}{\delta}}\right)^n Q}{\left(\prod_{i=0}^{n-1} a_i\right)} & \text{for } n = 1, 2, \ldots. \end{cases} \tag{2.4}$$

Then the sequence $\{Q_n\}$ *is* p*-periodic and the substitution*

$$y_n = (r-1)^{\frac{1}{\delta}} \frac{x_n}{Q_n}, \quad n = 0, 1, \ldots \tag{2.5}$$

transforms equation (1.1) *into* (2.1)*. In addition, if* $\{y_n\}$ *is a* p*-periodic solution of equation* (1.1)*, then the sequence* $\{x_n\}$*, defined by* (2.5)*, is also a* p*-periodic solution of equation* (2.1)*.*

Proof. First we will show that the sequence $\{Q_n\}$ satisfies the equation:

$$Q_{n+1} = \frac{r(r-1)^{\frac{1}{\delta}-1}}{a_n} Q_n. \tag{2.6}$$

Clearly, for $n = 1, 2, \ldots$ we have

$$Q_{n+1} = \frac{\left(r(r-1)^{\frac{1}{\delta}-1}\right)^{n+1} Q}{\left(\prod_{i=0}^{n} a_i\right)} = \frac{\left(r(r-1)^{\frac{1}{\delta}-1}\right)^{n} Q}{\left(\prod_{i=0}^{n-1} a_i\right)} \frac{r(r-1)^{\frac{1}{\delta}-1}}{a_n} = \frac{r(r-1)^{\frac{1}{\delta}-1}}{a_n} Q_n$$

Next, we need to show that $\{Q_n\}$ is , that is

$$Q_{n+p} = Q_n, \qquad n = 0, 1, \ldots.$$

Clearly

$$\begin{aligned} Q_{n+p} &= \frac{\left(r(r-1)^{\frac{1}{\delta}-1}\right)^{n+p}}{\left(\prod_{i=0}^{n+p-1} a_i\right)} Q = \frac{\left(r(r-1)^{\frac{1}{\delta}-1}\right)^{p}}{\left(\prod_{i=n}^{n+p-1} a_i\right)} \frac{\left(r(r-1)^{\frac{1}{\delta}-1}\right)^{n}}{\left(\prod_{i=0}^{n-1} a_i\right)} Q \\ &= \frac{\left(r(r-1)^{\frac{1}{\delta}-1}\right)^{p}}{\left(\prod_{i=n}^{n+p-1} a_i\right)} Q_n. \end{aligned}$$

Since $\{a_n\}$ is , then

$$\prod_{i=n}^{n+p-1} a_i = \prod_{i=0}^{p-1} a_i = A^p$$

and since $r > 1$ is a solution of the equation

$$A = r(r-1)^{\frac{1}{\delta}-1}$$

we obtain

$$Q_{n+p} = \frac{\left(r(r-1)^{\frac{1}{\delta}-1}\right)^{p}}{A^p} Q_n = Q_n.$$

Furthermore, by substituting (2.5) into (1.1) we get

$$(r-1)^{\frac{1}{\delta}} \frac{x_{n+1}}{Q_{n+1}} = \frac{a_n \left((r-1)^{\frac{1}{\delta}} \frac{x_n}{Q_n}\right)^{\delta}}{1 + \left((r-1)^{\frac{1}{\delta}} \frac{x_n}{Q_n}\right)^{\delta}}$$

and then by applying equation (2.6) we obtain

$$\begin{aligned} x_{n+1} &= \frac{a_n Q_{n+1} (r-1)^{1-\frac{1}{\delta}} \left(\frac{x_n}{Q_n}\right)^{\delta}}{1 + (r-1)\left(\frac{x_n}{Q_n}\right)^{\delta}} \\ &= \frac{a_n \frac{r(r-1)^{\frac{1}{\delta}-1}}{a_n} Q_n (r-1)^{1-\frac{1}{\delta}} \left(\frac{x_n}{Q_n}\right)^{\delta}}{1 + (r-1)\left(\frac{x_n}{Q_n}\right)^{\delta}} = \frac{r Q_n x_n^{\delta}}{Q_n^{\delta} + (r-1) x_n^{\delta}} \end{aligned}$$

which completes the proof. □

In the previous lemma the key factor for the transformation of equation (1.1) into (2.1) is that equation (2.2) has a solution $r > 1$. The next lemma establishes the conditions when such solution exists.

Lemma 2.2. *Consider equation* (2.2) *where* $A, \delta > 0$. *Let* $a_{crit} = \delta(\delta-1)^{1/\delta - 1}$. *Then the following statements are true:*

(i) If $0 < \delta < 1$, *then equation* (2.2) *has a unique solution* $r > 1$.

(ii) If $\delta = 1$ *and* $A > 1$, *then equation* (2.2) *has a unique solution* $r = A > 1$.

(iii) If $\delta > 1$ *and* $A < a_{crit}$, *then equation* (2.2) *has no solutions* $r > 1$.

(iv) If $\delta > 1$ *and* $A = a_{crit}$, *then equation* (2.2) *has a unique solution* $r = \delta > 1$.

(v) If $\delta > 1$ *and* $A > a_{crit}$, *then equation* (2.2) *has two solutions* r_1, r_2 *such that*

$$1 < r_1 < 1 + \left[\frac{(\delta-1)}{\delta}A\right]^{\delta} < r_2.$$

Proof. (i) Let $u = (r-1)^{\frac{1}{\delta}}$. Then $r = 1 + u^{\delta}$ so equation (2.2) becomes

$$A = (1+u^{\delta})u^{1-\delta} \tag{2.7}$$

or

$$u + u^{1-\delta} - A = 0.$$

Let $\phi(u) = u + u^{1-\delta} - A$. Then $\phi(0) = -A < 0$, $\phi(\infty) = \infty$ and, since $\delta < 1$, $\phi'(u) = 1 + (1-\delta)u^{-\delta} > 0$. Therefore there exists a unique $\bar{u} > 0$ such that $\phi(\bar{u}) = 0$. So

$$r = 1 + \bar{u}^{\delta} > 1$$

and the proof of part (i) is complete.
(ii) In the case $\delta = 1$, equation (2.2) becomes $A = r$, and so it is trivial.
(iii-v) As in part (i) we introduce $u = (r-1)^{\frac{1}{\delta}}$ so equation (2.2) becomes (2.7) which can be written as

$$u^{\delta} - Au^{\delta-1} + 1 = 0. \tag{2.8}$$

Consider the function $\psi(u) = u^{\delta} - Au^{\delta-1} + 1$. Then $\psi(0) = 1$, $\psi(\infty) = \infty$ and

$$\psi'(u) = \delta u^{\delta-1} - A(\delta-1)u^{\delta-2} = u^{\delta-2}(\delta u - A(\delta-1)).$$

Since $\psi' < 0$ on $\left(0, \dfrac{A(\delta-1)}{\delta}\right)$ and $\psi' > 0$ on $\left(\dfrac{A(\delta-1)}{\delta}, \infty\right)$ the function ψ has a minimum at $u' = \dfrac{A(\delta-1)}{\delta}$. Then

$$\psi(u') = \left(\frac{A(\delta-1)}{\delta}\right)^{\delta} - A\left(\frac{A(\delta-1)}{\delta}\right)^{\delta-1} + 1 = 1 - \frac{A^{\delta}(\delta-1)^{\delta-1}}{\delta^{\delta}}.$$

If $A < \delta(\delta-1)^{1/\delta-1}$, then

$$\psi(u') = 1 - \frac{A^\delta(\delta-1)^{\delta-1}}{\delta^\delta} > 1 - \frac{\left(\delta(\delta-1)^{\frac{1}{\delta}-1}\right)^\delta (\delta-1)^{\delta-1}}{\delta^\delta} = 0.$$

So $\psi(u) \geq \psi(u') > 0$ for $u > 0$ and equation (2.8) has no solutions. Therefore equation (2.2) also has no solutions.

If $A = \delta(\delta-1)^{1/\delta-1}$, then $\psi(u') = 0$ so the only solution of equation (2.8) is

$$u = u' = \frac{A(\delta-1)}{\delta} = (\delta-1)^{\frac{1}{\delta}}.$$

Therefore, equation (2.2) has a solution

$$r = \delta > 1.$$

Finally, if $A > \delta(\delta-1)^{\frac{1}{\delta}-1}$ we find

$$\psi(u') < 0.$$

Thus there are two solutions u_1 and u_2 such that

$$0 < u_1 < \frac{A(\delta-1)}{\delta} < u_2$$

and so equation (2.2) has two solutions

$$1 < r_1 < 1 + \left[\frac{A(\delta-1)}{\delta}\right]^\delta < r_2$$

which completes the proof. □

The next lemma is the converse of Lemma 2.1.

Lemma 2.3. *Consider equation* (2.1)*, where* $\delta > 0, r > 1$, $\{Q_n\}$ *is a positive* p*-periodic sequence, and* $x_0 > 0$. *Then the substitution*

$$x_n = Q_n(r-1)^{-1/\delta} y_n, \quad n = 0, 1, \ldots \tag{2.9}$$

transforms (2.1) *into* (1.1)*, where* $\{a_n\}$ *is a positive* p*-periodic sequence defined by*

$$a_n = r(r-1)^{1/\delta-1}\frac{Q_n}{Q_{n+1}}, \quad n = 0, 1, \ldots. \tag{2.10}$$

In addition, if $\{x_n\}$ *is a* p*-periodic solution of equation* (2.1)*, then the sequence* $\{y_n\}$, *defined by* (2.9)*, is also a* p*-periodic solution of equation* (1.1).

Proof. The proof is trivial and will be omitted. □

Next we will focus on the existence of sequences of carrying capacities and Allee thresholds for equation (2.1). So, to find the carrying capacities and Allee thresholds we put $x_{n+1} = x_n = x$ and solve for x :

$$x = \frac{rQ_n x^\delta}{Q_n^\delta + (r-1)x^\delta}.$$

The following technical lemma will be useful in the sequel.

Lemma 2.4. *Consider the function* ξ *defined by*

$$\xi(t) = 1 + (r-1)t^\delta - rt^{\delta-1}, \quad t > 0. \tag{2.11}$$

Then the following statements are true:

(i) If $0 < \delta \le 1 < r$*, then* $t = 1$ *is the unique positive zero of the function* ξ.

(ii) If $1 < \delta < r$*, then the function* ξ *has two positive zeros* $t = 1$ *and* $t = \bar{t}$*, where* $\bar{t} \in (0, r(\delta-1)/(\delta(r-1))) \subset (0,1)$.

(iii) If $r = \delta > 1$*, then* $t = 1$ *is the unique positive zero of the function* ξ.

(iv) If $1 < r < \delta$*, then the function* ξ *has two positive zeros* $t = 1$ *and* $t = \tilde{t}$ *where* $\tilde{t} \in (r(\delta-1)/(\delta(r-1)), \infty) \subset (1, \infty)$.

(v) If $1 < \delta < r$*, then*

$$\bar{t} < \left(\frac{\delta-1}{r-1}\right)^{1/\delta} < 1;$$

also if $1 < r < \delta$*, then*

$$\tilde{t} > \left(\frac{\delta-1}{r-1}\right)^{1/\delta} > 1.$$

Proof. Clearly $\xi(1) = 0$ so $t = 1$ is a positive zero of the function ξ for all $r > 1$ and $\delta > 0$. Next note $\xi(\infty) = \infty$ and

$$\xi(0+) = \begin{cases} 1 & , \quad \delta > 1 \\ 1-r & , \quad \delta = 1 \\ -\infty & , \quad 0 < \delta < 1 \end{cases}.$$

The derivative of the function ξ is given by $\xi'(t) = \delta(r-1)t^{\delta-1} - r(\delta-1)t^{\delta-2}$.

When $0 < \delta \le 1$, then $\xi'(t) > 0$ so $t = 1$ is the only positive zero of the function ξ which completes the proof of part (i).

On the other hand if $\delta > 1$ we have

$$\xi'(t) = \begin{cases} < 0 \quad , \quad 0 < t < \dfrac{r(\delta-1)}{\delta(r-1)} \\ = 0 \quad , \quad t = \dfrac{r(\delta-1)}{\delta(r-1)} \\ > 0 \quad , \quad t > \dfrac{r(\delta-1)}{\delta(r-1)} \end{cases},$$

and

$$\xi'(1) = \delta(r-1) - r(\delta-1) = r - \delta.$$

The following cases are possible:
Case 1. $r > \delta > 1$.Since $\xi'(1) > 0$, then there exists unique $\bar{t} \in (0, r(\delta-1)/((\delta(r-1))) \subset (0,1)$ such that $\xi(\bar{t}) = 0$.
Case 2. $r = \delta > 1$. Then $\xi'(1) = \xi(1) = 0$ so $t = 1$ is the only solution of the equation (2.18).
Case 3. $\delta > r > 1$. Then $\xi'(1) < 0$ so there exists $\tilde{t} \in (r(\delta-1)/((\delta(r-1)), \infty) \subset (1, \infty)$ such that $\xi(\tilde{t}) = 0$ which completes the proof of part (iv).
(v) First observe that $\xi(t) < 0$ for $t \in (\bar{t}, 1)$ and $t \in (1, \tilde{t})$, provided $r > \delta > 1$ and $\delta > r > 1$, respectively. To complete the proof it suffices to show

$$\xi\left(\left(\frac{\delta-1}{r-1}\right)^{1/\delta}\right) < 0. \tag{2.12}$$

Clearly

$$\begin{aligned} \xi\left(\left(\frac{\delta-1}{r-1}\right)^{1/\delta}\right) &= 1 + (r-1)\left(\frac{\delta-1}{r-1}\right) - r\left(\frac{\delta-1}{r-1}\right)^{(\delta-1)/\delta} \\ &= r\delta(r-1)^{(1-\delta)/\delta}\left(\frac{(r-1)^{(\delta-1)/\delta}}{r} - \frac{(\delta-1)^{(\delta-1)/\delta}}{\delta}\right) \\ &= r\delta(r-1)^{(1-\delta)/\delta}\left(\zeta(r) - \zeta(\delta)\right), \end{aligned}$$

where

$$\zeta(s) = \frac{(s-1)^{(\delta-1)/\delta}}{s}, \ s > 0.$$

Next, we find

$$\zeta'(s) = \frac{\delta - s}{\delta s^2 (s-1)^{1/\delta}} \begin{cases} > 0, & \text{if } 0 < s < \delta \\ = 0, & \text{if } s = \delta \\ < 0, & \text{if } s > \delta. \end{cases}$$

Since ζ decreases for $s > \delta$, for $r > \delta > 1$, we obtain $\zeta(r) < \zeta(\delta)$ and (2.12) holds. Also, ζ increases for $\delta > r > 1$, so we have $\zeta(r) < \zeta(\delta)$ and again (2.12) is satisfied which completes the proof. □

The following lemma establishes the existence of carrying capacities and Allee thresholds of equation (2.1).

Lemma 2.5. *Consider equation* (2.1)*, where* $\delta > 0, r > 1$, $\{Q_n\}$ *is a positive* p*-periodic sequence, and* $x_0 > 0$*. Then the following statements are true:*

(i) If $0 < \delta \leq 1$*, then equation* (2.1) *has a unique sequence of carrying capacities* $\{K_n\}$ *defined by*

$$K_n = Q_n, \quad n = 0, 1, \ldots \tag{2.13}$$

and no Allee thresholds.

(ii) If $r > \delta > 1$*, then equation* (2.1) *has a unique sequence of carrying capacities* $\{K_n\}$ *and a unique sequence of Allee thresholds* $\{T_n\}$ *defined by*

$$K_n = Q_n, \quad T_n = \bar{t}Q_n, \quad n = 0, 1, \ldots, \tag{2.14}$$

where $\bar{t}$ *is the unique solution of the equation*

$$1 + (r-1)t^{\delta} - rt^{\delta-1} = 0 \tag{2.15}$$

in the interval $(0, r(\delta-1)/((\delta(r-1)))$.

(iii) If $r = \delta > 1$*, then equation* (2.1) *has a unique sequence of carrying capacities* $\{K_n\}$ *defined by*

$$K_n = Q_n, \quad n = 0, 1, \ldots$$

and no Allee thresholds.

(iv) If $\delta > r > 1$*, then equation* (2.1) *has a unique sequence of carrying capacities* $\{K_n\}$ *and a unique sequence of Allee thresholds* $\{T_n\}$ *defined by*

$$K_n = \tilde{t}Q_n, \quad T_n = Q_n, \quad n = 0, 1, \ldots, \tag{2.16}$$

where $\tilde{t}$ *is a unique solution of equation* (2.15) *in the interval* $(r(\delta-1)/(\delta(r-1)), \infty)$.

Proof. Consider the equation

$$x_{n+1} = \frac{rx_n\left(\frac{x_n}{Q_n}\right)^{\delta-1}}{1 + (r-1)\left(\frac{x_n}{Q_n}\right)^{\delta}} \tag{2.17}$$

where $\delta > 0$, $r > 1$ and $\{Q_n\}$ is positive sequence and $x_0 > 0$.

To find carrying capacities and Allee thresholds we put $x_n = x_{n+1} = x$ into equation (2.17) and solve for x to obtain

$$x = \frac{rx\left(\frac{x}{Q_n}\right)^{\delta-1}}{1+(r-1)\left(\frac{x}{Q_n}\right)^{\delta}}.$$

Clearly $x = 0$ is one of the solutions, so to find other solutions we have

$$1+(r-1)\left(\frac{x}{Q_n}\right)^{\delta} - r\left(\frac{x}{Q_n}\right)^{\delta-1} = 0.$$

Let $t = x/Q_n$. Then the above equation becomes

$$\xi(t) = 1+(r-1)t^{\delta} - rt^{\delta-1} = 0 \tag{2.18}$$

where the function ξ is defined by (2.11). Using Lemma 2.4 we have the following four cases:
(i) If $\delta \leq 1$, then $t = 1$ is the unique solution of equation (2.18) and hence $\{K_n\}$, defined by $K_n = Q_n$, $n = 0, 1, \dots$, is the unique sequence of carrying capacities of equation (2.17).
(ii) If $r > \delta > 1$, then equation (2.18) has two positive solutions $t = 1$ and $t = \bar{t}$ where $0 < \bar{t} < \dfrac{r(\delta-1)}{\delta(r-1)} < 1$. Therefore equation (2.17) has carrying capacities $\{K_n\}$ and Allee thresholds $\{T_n\}$, defined by $K_n = Q_n$, $T_n = \bar{t}Q_n$, $n = 0, 1, \dots$.
(iii) If $r = \delta > 1$. Then $\xi'(1) = \xi(1) = 0$ so $t = 1$ is the only solution of equation (2.18). Therefore equation (2.17) has only carrying capacities (which coincide with Allee thresholds) $\{K_n\}$, where $K_n = Q_n$, $n = 0, 1, \dots$.
(iv) $\delta > r > 1$. Then $\xi'(1) < 0$ so there exists $\tilde{t} > \dfrac{r(\delta-1)}{\delta(r-1)} > 1$ such that $\xi(\tilde{t}) = 0$ so equation (2.17) has carrying capacities $\{K_n\}$ and Allee thresholds are $\{T_n\}$, defined by $K_n = \tilde{t}Q_n$ and $T_n = Q_n$, $n = 0, 1, \dots$, respectively.
Clearly, both carrying capacities $\{K_n\}$ and Allee thresholds $\{T_n\}$ are p-periodic sequences. $\square$

The next theorem establishes sufficient conditions for the existence of p-periodic solutions of equation (2.1).

Theorem 2.6. *Consider equation* (2.1)*, where* $\delta > 0, r > 1$, $\{Q_n\}$ *is a positive* p*-periodic sequence and* $x_0 > 0$*. Then the following statements are true:*

(i) If $0 < \delta < 1$*, then there exists a unique* p*-periodic solution* $\{\bar{x}_n\}$ *of equation* (2.1) *which attracts all positive solutions.*

(ii) If $\delta > 1$ *and*

$$\min\left\{\frac{Q_p}{Q_1}, \frac{Q_1}{Q_2}, \ldots, \frac{Q_{p-1}}{Q_p}\right\} > \frac{\delta(\delta-1)^{1/\delta-1}}{r(r-1)^{1/\delta-1}}, \tag{2.19}$$

then the equation (2.1) *has two positive* p*-periodic solutions* $\{\bar{x}_n\}$ *and* $\{\tilde{x}_n\}$, $\tilde{x}_n < \bar{x}_n$, *for* $n = 0, 1, \ldots$. *The solution* $\{\bar{x}_n\}$ *is an attractor while* $\{\tilde{x}_n\}$ *is a repeller.*

Proof. (i) It follows directly from Theorem 1.5 and Lemmas 2.1 and 2.2(i).
(ii) Condition (2.19)) implies that $\{a_n\}$ defined by (2.10) satisfies $\min\{a_1, \ldots, a_p\} > a_{crit}$. Then from Theorem 1.4 it follows that equation (1.1) has two p-periodic solution $\{\bar{y}_n\}$ and $\{\tilde{y}_n\}$ such that $\tilde{y}_n < \bar{y}_n$. Finally, the existence of two positive p-periodic solutions $\{\bar{x}_n\}$ and $\{\tilde{x}_n\}$, $\tilde{x}_n < \bar{x}_n$, for $n = 0, 1, \ldots$ follows from Lemmas 2.1 and 2.2(v) which completes the proof. □

3 G-attenuance and G-resonance

In this section we examine the question of g-attenuance of periodic cycles of equation (2.1). The following result is true:

Theorem 3.1. *Consider equation* (2.1)*, where* $\delta > 0, r > 1$, $\{Q_n\}$ *is a positive* p*-periodic sequence, and* $x_0 > 0$. *Let* $\{\bar{x}_n\}$ *be a* p*-periodic solution of equation* (2.1)*. Then the following statements are true:*

(i) If $0 < \delta \le 1$, *then the periodic cycle* $\{\bar{x}_n\}$ *is* g*-attenuant, that is*

$$\left(\prod_{i=0}^{p-1} \bar{x}_i\right)^{\frac{1}{p}} < \left(\prod_{i=0}^{p-1} K_i\right)^{\frac{1}{p}},$$

where $\{K_n\}$ *represents the unique sequence of carrying capacities, defined by* (2.13).

(ii) If $r > \delta > 1$ *or* $\delta > r > 1$, *then the periodic cycle* $\{\bar{x}_n\}$ *is* g*-attenuant with respect to carrying capacities and* g*-resonant with respect to Alee thresholds, that is, it satisfies*

$$\left(\prod_{i=0}^{p-1} T_i\right)^{\frac{1}{p}} < \left(\prod_{i=0}^{p-1} \bar{x}_i\right)^{\frac{1}{p}} < \left(\prod_{i=0}^{p-1} K_i\right)^{\frac{1}{p}},$$

where $\{K_n\}$ *represents the unique sequence of carrying capacities and the sequence* $\{T_n\}$ *is the unique sequence of Allee thresholds, defined by* (2.14) *and* (2.16)*, respectively.*

Proof. Consider the equation

$$x_{n+1} = \frac{r x_n \left(\frac{x_n}{Q_n}\right)^{\delta-1}}{1+(r-1)\left(\frac{x_n}{Q_n}\right)^{\delta}}. \tag{3.1}$$

Let $\{\bar{x}_n\}$ be a solution of equation (3.1) provided such solution exists. Then

$$\prod_{i=0}^{p-1} \bar{x}_{i+1} = \frac{r^p \left(\prod_{i=0}^{p-1} \bar{x}_i\right) \left(\prod_{i=0}^{p-1} \left(\frac{\bar{x}_i}{Q_i}\right)^{\delta-1}\right)}{\prod_{i=0}^{p-1} \left(1+(r-1)\left(\frac{\bar{x}_i}{Q_i}\right)^{\delta}\right)}$$

or equivalently

$$\prod_{i=0}^{p-1} \left(1+(r-1)\left(\frac{\bar{x}_i}{Q_i}\right)^{\delta}\right) = r^p \prod_{i=0}^{p-1} \left(\frac{\bar{x}_i}{Q_i}\right)^{\delta-1}. \tag{3.2}$$

We introduce the function G by

$$G(x_0, \ldots, x_{p-1}) = \sum_{i=0}^{p-1} \ln \left(1+(r-1)\left(\frac{x_i}{Q_i}\right)^{\delta}\right) - p \ln r - (\delta-1) \sum_{i=0}^{p-1} \ln \left(\frac{x_i}{Q_i}\right). \tag{3.3}$$

Then condition (3.2) is equivalent to

$$G(\bar{x}_0, \ldots, \bar{x}_{p-1}) = 0. \tag{3.4}$$

Let

$$F(x_0, \ldots, x_{p-1}) = \frac{1}{p} \sum_{i=0}^{p-1} \ln x_i \tag{3.5}$$

Our goal is to find the extrema of F under the constraint $G = 0$. We will apply the Lagrange multipliers method. Let

$$\begin{aligned} F + \lambda G = {} & \frac{1}{p} \sum_{i=0}^{p-1} \ln x_i \\ & + \lambda \left[\sum_{i=0}^{p-1} \ln \left(1+(r-1)\left(\frac{x_i}{Q_i}\right)^{\delta}\right) - p \ln r - (\delta-1) \sum_{i=0}^{p-1} \ln \left(\frac{x_i}{Q_i}\right) \right]. \end{aligned}$$

So, our first step is to solve the following system with respect to $x_0, \ldots, x_{p-1}, \lambda :$

$$\begin{aligned} \frac{\partial}{\partial x_i} (F + \lambda G) = 0, \quad & i = 0, \ldots, p-1 \\ G = 0. & \end{aligned}$$

The above system becomes

$$\frac{\partial}{\partial x_i}(F+\lambda G)=\frac{1-p\lambda(\delta-1)}{p\,x_i}+\lambda\frac{(r-1)\delta\left(\frac{x_i}{Q_i}\right)^{\delta-1}\frac{1}{Q_i}}{1+(r-1)\left(\frac{x_i}{Q_i}\right)^{\delta}}=0,\quad i=0,\dots,p-1 \tag{3.6}$$

$$G=\sum_{i=0}^{p-1}\ln\left(1+(r-1)\left(\frac{x_i}{Q_i}\right)^{\delta}\right)-p\ln r-(\delta-1)\sum_{i=0}^{p-1}\ln\left(\frac{x_i}{Q_i}\right)=0 \tag{3.7}$$

and we observe that the substitution $t=x_i/Q_i$ reduces system (3.6)-(3.7) to

$$\frac{\lambda(r-1)\delta t^{\delta}}{1+(r-1)t^{\delta}}=\lambda(\delta-1)-\frac{1}{p} \tag{3.8}$$

$$\ln\left(1+(r-1)t^{\delta}\right)-\ln r-(\delta-1)\ln t=0. \tag{3.9}$$

So instead of solving system (3.6)-(3.7) we have to find t and λ which satisfy (3.8)-(3.9). From (3.8) we find

$$\lambda(r-1)\delta t^{\delta}=\left(1+(r-1)t^{\delta}\right)\left(\lambda(\delta-1)-\frac{1}{p}\right)$$

and

$$\lambda=\frac{1+(r-1)t^{\delta}}{\left((\delta-1)-t^{\delta}(r-1)\right)p}. \tag{3.10}$$

On the other hand equation (3.9) can be written as:

$$\xi(t)=1+(r-1)t^{\delta}-rt^{\delta-1}=0,\ t>0 \tag{3.11}$$

where the function ξ was defined by (2.11).

We consider the following three cases:

Case 1: $\delta\le 1<r$. From Lemma 2.4(i) it follows $t=1$ is the only positive solution of the equation $\xi(t)=0$. Then from (3.10) we find $\lambda=r/(p(\delta-r))$so the system (3.6)-(3.7) has the unique solution

$$x_0=Q_0,\dots,x_{p-1}=Q_{p-1},\lambda=r/(p(\delta-r)). \tag{3.12}$$

Then $(Q_0,\dots,Q_{p-1})$ is the unique critical point of the function $F+\lambda G$.

Case 2: $1<\delta<r$. From Lemma 2.4(ii) the function ξ has two positive zeros $t=1$ and $t=\bar t$, where $\bar t\in(0,r(\delta-1)/(\delta(r-1)))\subset(0,1)$. Using (3.10) we find corresponding values for λ so the system (3.6)-(3.7) has two solutions

$$x_0=Q_0,\dots,x_{p-1}=Q_{p-1},\ \lambda=\frac{r}{p(\delta-r)},$$

$$x_0=\bar tQ_0,\dots,x_{p-1}=\bar tQ_{p-1},\ \lambda=\frac{1+(r-1)\bar t^{\delta}}{\left((\delta-1)-\bar t^{\delta}(r-1)\right)p}.$$

Then the function $F+\lambda G$ has two critical points $(Q_0,\ldots,Q_{p-1})$ and $(\bar{t}Q_0,\ldots,\bar{t}Q_{p-1})$, such that $\bar{t}\in(0,r(\delta-1)/(\delta(r-1)))$ is the positive zero of the function ξ.
Case 3: $r>\delta>1$. Again, by applying Lemma 2.4(iv) we find the function ξ has two positive zeros $t=1$ and $t=\tilde{t}$, where $\tilde{t}\in(r(\delta-1)/(\delta(r-1)),\infty)\subset(1,\infty)$. Similarly as in case 2, the system (3.6)-(3.7) has two solutions

$$x_0=Q_0,\ldots,x_{p-1}=Q_{p-1},\ \lambda=\frac{r}{p(\delta-r)},$$
$$x_0=\tilde{t}Q_0,\ldots,x_{p-1}=\tilde{t}Q_{p-1},\ \lambda=\frac{1+(r-1)\tilde{t}^{\delta}}{\left((\delta-1)-\tilde{t}^{\delta}(r-1)\right)p}.$$

Again, in this case the function $F+\lambda G$ has two critical points $(Q_0,\ldots,Q_{p-1})$ and $(\tilde{t}Q_0,\ldots,\tilde{t}Q_{p-1})$, where $\tilde{t}\in(r(\delta-1)/(\delta(r-1)),\infty)$ is the positive zero of the function ξ.

Next we will examine $d^2(F+\lambda G)$, at the critical points $(Q_0,\ldots,Q_{p-1})$, and also at points $(\bar{t}Q_0,\ldots,\bar{t}Q_{p-1})$ if $r>\delta>1$ and at $(\tilde{t}Q_0,\ldots,\tilde{t}Q_{p-1})$ when $\delta>r>1$
Consider

$$\frac{\partial^2(F+\lambda G)}{\partial x_i^2}=\frac{\lambda p(\delta-1)-1}{px_i^2}+\frac{\lambda(r-1)\delta}{Q_i^2}\frac{\left[(\delta-1)-(r-1)\left(\frac{x_i}{Q_i}\right)^{\delta}\right]\left(\frac{x_i}{Q_i}\right)^{\delta-2}}{\left(1+(r-1)\left(\frac{x_i}{Q_i}\right)^{\delta}\right)^2}$$

$$\frac{\partial^2(F+\lambda G)}{\partial x_i\partial x_j}=0,\quad i\neq j.$$

Furthermore we find that at the point $(Q_0,\ldots,Q_{p-1})$ with $\lambda=\dfrac{r}{p(\delta-r)}$

$$\begin{aligned}\left.\frac{\partial^2(F+\lambda G)}{\partial x_i^2}\right|_{(Q_0,\ldots,Q_{p-1})}&=\frac{\frac{r(\delta-1)p}{p(\delta-r)}-1}{pQ_i^2}+\frac{\frac{r}{p(\delta-r)}(r-1)\delta}{Q_i^2}\left[\frac{(\delta-1)-(r-1)}{(1+(r-1))^2}\right]\\&=\frac{r(\delta-1)-(\delta-r)}{pQ_i^2(\delta-r)}+\frac{(r-1)\delta}{rpQ_i^2}\\&=\frac{(r-1)\delta}{p\,Q_i^2(\delta-r)}+\frac{(r-1)\delta}{pQ_i^2r}\\&=\frac{(r-1)\delta^2}{p\,Q_i^2r(\delta-r)}\end{aligned}$$

So, for $\delta\le 1<r$ and for $r>\delta>1$

$$\left.\frac{\partial^2(F+\lambda G)}{\partial x_i^2}\right|_{x_i=Q_i}<0, \tag{3.13}$$

while for $\delta>r>1$

$$\left.\frac{\partial^2(F+\lambda G)}{\partial x_i^2}\right|_{x_i=Q_i}>0. \tag{3.14}$$

Next we will consider the case when $x_i = \bar{t}Q_i$ and $1 < \delta < r$ (or $x_i = \tilde{t}Q_i$). In this case we have

$$\lambda = \frac{1 + (r-1)\bar{t}^\delta}{((\delta - 1) - \bar{t}^\delta (r-1))p}$$

and

$$\left.\frac{\partial^2 (F + \lambda G)}{\partial x_i^2}\right|_{x_i = \bar{t}Q_i} = \frac{\lambda p(\delta - 1) - 1}{p\,\bar{t}^2 Q_i^2} + \frac{\lambda (r-1)\delta}{Q_i^2} \frac{\left[(\delta-1) - (r-1)\bar{t}^\delta\right]\bar{t}^{\delta-2}}{\left(1 + (r-1)\,\bar{t}^\delta\right)^2}.$$

Since

$$\frac{\lambda (r-1)\delta \bar{t}^\delta}{1 + (r-1)\bar{t}^\delta} = \frac{\lambda p(\delta - 1) - 1}{p}$$

we get

$$\begin{aligned}
\left.\frac{\partial^2 (F + \lambda G)}{\partial x_i^2}\right|_{x_i = \bar{t}Q_i} &= \frac{\lambda p(\delta - 1) - 1}{p\,\bar{t}^2 Q_i^2} + \frac{\lambda p(\delta - 1) - 1}{p\bar{t}^2 Q_i^2}\left[\frac{(\delta-1) - (r-1)\bar{t}^\delta}{1 + (r-1)\bar{t}^\delta}\right] \\
&= \frac{\lambda p(\delta - 1) - 1}{p\,\bar{t}^2 Q_i^2}\left[\frac{1 + (r-1)\bar{t}^\delta + (\delta - 1) - (r-1)\bar{t}^\delta}{1 + (r-1)\bar{t}^\delta}\right] \\
&= \frac{\frac{(1 + (r-1)\bar{t}^\delta)}{(\delta - 1 - (r-1)\bar{t}^\delta)\,p}\, p(\delta - 1) - 1}{p\,\bar{t}^2 Q_i^2} \cdot \frac{\delta}{1 + (r-1)\bar{t}^\delta} \\
&= \frac{\delta\left[(\delta - 1)(1 + (r-1)\bar{t}^\delta) - (\delta - 1) + (r-1)\bar{t}^\delta\right]}{p\,\bar{t}^2 Q_i^2\,(1 + (r-1)\bar{t}^\delta)\,(\delta - 1 - (r-1)\bar{t}^\delta)} \\
&= \frac{\delta^2 (r-1)\bar{t}^\delta}{p\,\bar{t}^2 Q_i^2\,(1 + (r-1)\bar{t}^\delta)\,(\delta - 1 - (r-1)\bar{t}^\delta)}.
\end{aligned}$$

In the case $r > \delta > 1$, from Lemma 2.4(v) we have $\bar{t}^\delta < (\delta - 1)/(r-1)$, so $(\delta - 1) - (r-1)\bar{t}^\delta > 0$ and we obtain

$$\left.\frac{\partial^2 (F + \lambda G)}{\partial x_i^2}\right|_{x_i = \bar{t}Q_i} > 0. \tag{3.15}$$

Similarly we find

$$\left.\frac{\partial^2 (F + \lambda G)}{\partial x_i^2}\right|_{x_i = \tilde{t}Q_i} = \frac{\delta^2 (r-1)\tilde{t}^\delta}{p\,\tilde{t}^2 Q_i^2\,\left(1 + (r-1)\tilde{t}^\delta\right)\left(\delta - 1 - (r-1)\tilde{t}^\delta\right)}$$

Then, in the case $\delta > r > 1$, from Lemma 2.4(v) we have $\tilde{t} > (\delta - 1)/(r-1)$ so $(\delta - 1) - (r-1)\tilde{t}^\delta < 0$ and we obtain

$$\left.\frac{\partial^2 (F + \lambda G)}{\partial x_i^2}\right|_{x_i = \tilde{t}Q_i} < 0. \tag{3.16}$$

To determine the character of critical points we will examine $d^2(F+\lambda G)$ and critical points. Since $\partial^2(F+\lambda G)/\partial x_i \partial x_j = 0$, $i,j = 0,\ldots,p-1$, $i \neq j$, we have

$$d^2(F+\lambda G) = \sum_{i=0}^{p-1} \frac{\partial^2(F+\lambda G)}{\partial x_i^2} dx_i^2.$$

Using (3.13) - (3.16) we obtain the following:

$$\begin{aligned}
d^2(F+\lambda G)\big|_{(Q_0,\ldots,Q_{p-1})} &\begin{cases} <0, & \text{if } \delta \le 1 < r \\ <0, & \text{if } r > \delta > 1 \\ >0, & \text{if } \delta > r > 1, \end{cases} \\
d^2(F+\lambda G)\big|_{(\bar{t}Q_0,\ldots,\bar{t}Q_{p-1})} &> 0, \text{ provided } r > \delta > 1, \\
d^2(F+\lambda G)\big|_{(\tilde{t}Q_0,\ldots,\tilde{t}Q_{p-1})} &< 0, \text{ provided } \delta > r > 1.
\end{aligned}$$

Therefore, in the case $\delta \le 1 < r$ the function F attains a maximum (subject to the constraint $G = 0$) at the point $(Q_0,\ldots,Q_{p-1})$, so

$$F(x_0,\ldots,x_{p-1}) \le F(Q_0,\ldots,Q_{p-1}).$$

Let $x_i = \bar{x}_i$, $i = 0,\ldots,p-1$, where $\{\bar{x}_n\}$ is the solution of equation (2.1). Since $\{Q_n\}$ can not be a solution of the equation (2.1), we have $(\bar{x}_0,\ldots,\bar{x}_{p-1}) \neq (Q_0,\ldots,Q_{p-1})$ and the above inequality is strict:

$$F(\bar{x}_0,\ldots,\bar{x}_{p-1}) < F(Q_0,\ldots,Q_{p-1}).$$

Finally, since $\{K_n\} = \{Q_n\}$ we obtain

$$\left(\prod_{i=0}^{p-1} \bar{x}_n\right)^{\frac{1}{p}} < \left(\prod_{i=0}^{p-1} K_n\right)^{\frac{1}{p}},$$

which completes the proof of part (i).

Next we consider the case $r > \delta > 1$. Since

$$d^2(F+\lambda G)\big|_{(Q_0,\ldots,Q_{p-1})} < 0, \text{ and } d^2(F+\lambda G)\big|_{(\bar{t}Q_0,\ldots,\bar{t}Q_{p-1})} > 0,$$

the function F attains the maximum and the minimum (subject to the constraint $G = 0$) at the critical points $(Q_0,\ldots,Q_{p-1})$ and $(\bar{t}Q_0,\ldots,\bar{t}Q_{p-1})$, respectively. Thus

$$F(\bar{t}Q_0,\ldots,\bar{t}Q_{p-1}) \le F(x_0,\ldots,x_{p-1}) \le F(Q_0,\ldots,Q_{p-1}).$$

Again, let $x_i = \bar{x}_i$, $i = 0,\ldots,p-1$, where $\{\bar{x}_n\}$ is the solution of equation (2.1). Since $\{Q_n\}$ and $\{\bar{t}Q_n\}$ can not be solutions of the equation (2.1), we have $(\bar{x}_0,\ldots,\bar{x}_{p-1}) \neq$

$(Q_0, \ldots, Q_{p-1})$ and $(\bar{x}_0, \ldots, \bar{x}_{p-1}) \neq (\bar{t}Q_0, \ldots, \bar{t}Q_{p-1})$ so the above inequalities are strict:

$$F(\bar{t}Q_0, \ldots, \bar{t}Q_{p-1}) < F(\bar{x}_0, \ldots, \bar{x}_{p-1}) < F(Q_0, \ldots, Q_{p-1}).$$

Finally, since $\{K_n\} = \{Q_n\}$ and $\{T_n\} = \{\bar{t}Q_n\}$, we obtain

$$\left(\prod_{i=0}^{p-1} T_i\right)^{\frac{1}{p}} < \left(\prod_{i=0}^{p-1} \bar{x}_i\right)^{\frac{1}{p}} < \left(\prod_{i=0}^{p-1} K_i\right)^{\frac{1}{p}}.$$

The proof in the remaining case $\delta > r > 1$ is similar to the previous case so it is omitted. This completes the proof of the theorem. □

Finally, we will address the question about g-attenuance and g-resonance of periodic cycles of equation (1.1). First we will introduce the autonomous Sigmoid-Beverton–Holt model

$$z_{n+1} = \frac{Az_n^\delta}{1 + z_n^\delta}, \quad n = 0, 1, \ldots, \tag{3.17}$$

where $z_0 > 0$, $\{a_n\}$ is a positive p-periodic sequence and

$$A = \left(\prod_{i=1}^{p} a_i\right)^{1/p}.$$

Equation (3.17) is associated with periodically forced Sigmoid Beverton–Holt model (1.1) in the way that its coefficient A represents the geometric mean of one cycle of the sequence $\{a_n\}$. Let K_A and T_A denote the carrying capacity and Allee threshold for equation (3.17), respectively.

Theorem 3.2. *Consider equation* (1.1) *and the associated autonomous equation* (3.17)*, where* $\delta > 0$*,* $\{a_n\}$ *is a positive* p*-periodic sequence,* $A = (a_i)^{1/p}$ *, and* $y_0, z_0 > 0$*. Let* $\{\bar{y}_n\}$ *be a* p*-periodic solution of equation* (1.1)*. Then the following statements are true:*

(i) If $0 < \delta < 1$*, then the periodic cycle* $\{\bar{y}_n\}$ *satisfies*

$$\left(\prod_{i=0}^{p-1} \bar{y}_i\right)^{1/p} < K_A,$$

where K_A *is the unique positive carrying capacity of the associated autonomous equation* (3.17)*.*

(ii) If $\delta > 1$ *and* $A > a_{crit}$*, then the periodic cycle* $\{\bar{y}_n\}$ *satisfies*

$$T_A < \left(\prod_{i=0}^{p-1} \bar{y}_i\right)^{\frac{1}{p}} < K_A,$$

where K_A *is the unique carrying capacity and* T_A *is the unique Allee threshold of the associated autonomous equation* (3.17)*.*

Proof. Since $\{\bar{y}_n\}$ is a p-periodic solution of (1.1), then, according to Lemmas 2.1 and 2.2, equation (2.1) has a p-periodic solution $\{\bar{x}_n\}$ defined by

$$\bar{x}_n = Q_n(r-1)^{-1/\delta}\bar{y}_n, \quad n = 0, 1, \ldots, \tag{3.18}$$

where r is a positive solution of (2.2) and $\{Q_n\}$ is given by (2.4).

First we consider the case (i). Then $0 < \delta < 1$, and, by using Lemma 2.5 and Theorem 3.1 we have $K_n = Q_n$ and

$$\begin{aligned}\left(\prod_{i=0}^{p-1} \bar{y}_i\right)^{1/p} &= \left(\prod_{i=0}^{p-1} \frac{\bar{x}_i(r-1)^{1/\delta}}{Q_i}\right)^{1/p} = (r-1)^{1/\delta} \frac{\left(\prod_{i=0}^{p-1} \bar{x}_i\right)^{1/p}}{\left(\prod_{i=0}^{p-1} Q_i\right)^{1/p}} \\ &< (r-1)^{1/\delta} \frac{\left(\prod_{i=0}^{p-1} K_i\right)^{1/p}}{\left(\prod_{i=0}^{p-1} Q_i\right)^{1/p}} = (r-1)^{1/\delta}.\end{aligned}$$

Let $z = (r-1)^{1/\delta}$. Then $r = 1 + z^\delta$ and since r is a positive solution of the equation

$$A = r(r-1)^{\frac{1-\delta}{\delta}}$$

we obtain that z satisfies the equation

$$A = z^{1-\delta} + z; \tag{3.19}$$

so z is a positive equilibrium of the associated autonomous equation (3.17). Therefore

$$z = (r-1)^{1/\delta} = K_A$$

and

$$\left(\prod_{i=0}^{p-1} \bar{y}_i\right)^{1/p} < K_A$$

which completes the proof of part (i).

Now, we consider the case (ii). Since $A > a_{crit}$, from Lemma 2.2 it follows that equation (2.2) has two solutions r_1, r_2 such that $1 < r_1 < 1 + \left[\frac{(\delta-1)}{\delta}A\right]^\delta < r_2$. So we will select $r = r_2$. Observe that $A > a_{crit} = \delta(\delta-1)^{1/\delta - 1}$ is equivalent to $1 + \left[\frac{(\delta-1)}{\delta}A\right]^\delta > \delta$ and we have $r > \delta > 1$. Then by Lemma 2.5 and Theorem 3.1 we have $K_n = Q_n$ and $T_n = \bar{t}Q_n$ $n = 0, 1, \ldots$, where $\bar{t}$ is a unique solution of

$$1 + (r-1)t^\delta - rt^{\delta-1} = 0 \tag{3.20}$$

such that $0 < \bar{t} < r(\delta-1)/((\delta(r-1)) < 1$. Next we will show that $z = (r-1)^{1/\delta}$ and $z = \bar{t}(r-1)^{1/\delta}$ are two positive equilibria of the associated autonomous equation (3.17) and therefore

$$K_A = (r-1)^{1/\delta} > \bar{t}(r-1)^{1/\delta} = T_A.$$

Since we already showed in part (i) that $z = (r-1)^{1/\delta}$ satisfies equation (3.19) it remains to show that $z = \bar{t}(r-1)^{1/\delta}$ is also a solution of the same equation. Since $\bar{t}$ is a solution of equation (3.20) it follows $\bar{t}^{1-\delta} = (1-r)\bar{t} + r$ and we have

$$\begin{aligned} A - \left(\bar{t}(r-1)^{1/\delta}\right)^{1-\delta} - \bar{t}(r-1)^{1/\delta} &= A - \bar{t}^{1-\delta}(r-1)^{(1-\delta)/\delta} - \bar{t}(r-1)^{1/\delta} \\ &= A - (r-1)^{1/\delta}\left(\frac{\bar{t}^{1-\delta}}{r-1} + \bar{t}\right) \\ &= A - (r-1)^{1/\delta}\left(\frac{(1-r)\bar{t}+r}{r-1} + \bar{t}\right) \\ &= A - r(r-1)^{1/\delta - 1} = 0. \end{aligned}$$

Again, as in the case (i) we use the fact that equation (2.1) has a p-periodic solution $\{\bar{x}_n\}$ defined by (3.18) and by applying Theorem 3.1 we get

$$\left(\prod_{i=0}^{p-1} \bar{y}_i\right)^{1/p} = (r-1)^{1/\delta}\frac{\left(\prod_{i=0}^{p-1} \bar{x}_i\right)^{1/p}}{\left(\prod_{i=0}^{p-1} Q_i\right)^{1/p}} < (r-1)^{1/\delta}\frac{\left(\prod_{i=0}^{p-1} K_i\right)^{1/p}}{\left(\prod_{i=0}^{p-1} Q_i\right)^{1/p}} = (r-1)^{1/\delta},$$

$$\left(\prod_{i=0}^{p-1} \bar{y}_i\right)^{1/p} = (r-1)^{1/\delta}\frac{\left(\prod_{i=0}^{p-1} \bar{x}_i\right)^{1/p}}{\left(\prod_{i=0}^{p-1} Q_i\right)^{1/p}} > (r-1)^{1/\delta}\frac{\left(\prod_{i=0}^{p-1} T_i\right)^{1/p}}{\left(\prod_{i=0}^{p-1} Q_i\right)^{1/p}} = \bar{t}(r-1)^{1/\delta}.$$

Finally

$$T_A < \left(\prod_{i=0}^{p-1} \bar{y}_i\right)^{1/p} < K_A$$

which completes the proof of the theorem □

4 Open Problems

In this section we formulate some open problems.

Clearly we can not determine g-attenuance or g-resonance of a periodic cycle of equation (2.1) in the case $r = \delta > 1$, given the tool used in the proof of Theorem 3.1. So we formulate the following open problem

Open Problem 4.1. Consider equation (2.1), where $r = \delta > 1$, $\{Q_n\}$ is a positive p-periodic sequence, and $x_0 > 0$. Let $\{\bar{x}_n\}$ be a p-periodic solution of equation (2.1). Determine the g-attenuance or g-resonance of $\{\bar{x}_n\}$ with respect to carrying capacities.

Theorem 3.2 establishes bounds of the geometric mean of a periodic cycle of equation (1.1) in terms of the carrying capacity K_A and the Allee threshold T_A of the associated autonomous equation (3.17). Clearly it does not represent the result about g-attenuance or g-resonance of a periodic cycle of equation (1.1). It is not clear how K_A and T_A are related to geometric means of carrying capacities and Allee thresholds of equation (1.1) so we formulate the following open problem.

Open Problem 4.2. Consider equation (1.1), where $\delta > 0$, $\{a_n\}$ is a positive p-periodic sequence, and $y_0 > 0$. Determine the relationship between the carrying capacity K_A and the Allee threshold T_A of the associated autonomous equation (3.17) and the averages (geometric or some other means) of carrying capacities and Allee thresholds of equation (2.1).

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