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On Rational Difference Equations with Nonnegative Periodic Coefficients

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Abstract

We investigate the global stability, periodic character, and the boundedness nature of the solutions of several special cases which are contained in the difference equation

$$x_{n+1} = \frac{\alpha_n + \beta_n x_n x_{n-1} + \gamma_n x_{n-1}}{A_n + B_n x_n x_{n-1} + C_n x_{n-1}}, n = 0, 1, \dots,$$

where the parameters $\alpha_n, \beta_n, \gamma_n, A_n, B_n, C_n$ are nonnegative periodic sequences, and the initial conditions x_{-1}, x_0 are nonnegative real numbers, such that the denominators are always positive.

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1 Introduction and Preliminaries

We investigate the global stability, periodic character, and the boundedness nature of the solutions of several special cases which are contained in the difference equation

$$x_{n+1} = \frac{\alpha_n + \beta_n x_n x_{n-1} + \gamma_n x_{n-1}}{A_n + B_n x_n x_{n-1} + C_n x_{n-1}}, n = 0, 1, \dots \tag{1.1}$$

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where the parameters $\alpha_n, \beta_n, \gamma_n, A_n, B_n, C_n$ are nonnegative periodic sequences, and the initial conditions x_{-1}, x_0 are nonnegative real numbers, such that the denominator is positive.

We will make use of the following known results. For the next three theorems, we consider the difference equation defined by

$$x_{n+1} = f(x_n, x_{n-1}), n = 0, 1, \dots. \tag{1.2}$$

Theorem 1.1 (Amleh, Camouzis, Ladas [1])**.** *Let I be a set of real numbers and let*

$$f : I \times I \to I$$

be a function $f(z_1, z_2)$ which increases in both variables. Then for every solution, $\{x_n\}_{n=-1}^{\infty}$, of Eq. (1.2)*, the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution do exactly one of the following:*

(i) Eventually they are both monotonically increasing.

(ii) Eventually they are both monotonically decreasing.

(iii) One of them is monotonically increasing and the other is monotonically decreasing.

Theorem 1.2 (El-Metwally, Grove, Ladas, Voulov [18,21])**.** *Let I be an interval of real numbers and let*

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), n = 0, 1, \dots, \tag{1.3}$$

where $F \in C(I^{k+1}, I)$. Assume that the following three conditions are satisfied:

1. *F is increasing in each of its arguments.*
2. *$F(z_1, \dots, z_{k+1})$ is strictly increasing in each of the arguments $z_{i_1}, z_{i_2}, \dots, z_{i_l}$ where $1 \le i_1 < i_2 < \dots < i_l \le k+1$, and the arguments $i_1, i_2, \dots, i_l$ are relatively prime.*
3. *Every point c in I is an equilibrium point of the Eq.* (1.3)*.*

Then every solution of Eq. (1.3) *has a finite limit.*

Theorem 1.3 (Camouzis, Ladas [13, 14])**.** *Let I be a set of real numbers and suppose that*

$$f : I \times I \to I$$

be a function $f(z_1, z_2)$ which decreases in z_1 and increases in z_2. Then for every solution, $\{x_n\}_{n=-1}^{\infty}$, of Equation (1.2) *the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution are either both monotonically increasing, both monotonically decreasing, or eventually one subsequence is monotonically increasing while the other is monotonically decreasing.*

Our work was motivated by the results of Amleh, Camouzis, and Ladas in [1, 2], where they investigated the autonomous difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \ n = 0, 1, 2, \dots, \tag{1.4}$$

where the parameters and initial conditions are nonnegative such that the denominators are always positive. Further motivation was given by Ladas in the following conjecture.

Conjecture 1.4 (G. Ladas). Let $\alpha > 0$ and

$$x_{n+1} = \frac{\alpha}{1 + \left(\prod_{i=0}^{k-1} x_{n-i}\right)}, n = 0, 1, \dots. \tag{1.5}$$

For any pair of nonnegative initial conditions x_{-1}, x_0, every solution to Eq. (1.4) converges.

Introducing periodic coefficients in rational difference equations has been studied previously, for example, see [12, 17].

Comparing the limiting behavior of difference equations with periodic coefficients to the corresponding autonomous equations where constant coefficients are the average of their periodic counterparts is of interest. A nonautonomous difference equation with periodic coefficients is said to be *advantageous* if the average of the periodic limits of the corresponding nonautonomous case is more than the limit of the autonomous equation with average coefficients. For more insight on deleterious and advantageous difference equations, see [17, 19].

For some basic results in the area of difference equations and systems, see [3–11, 15, 16, 20–29].

In this paper, we consider several special cases of Eq. (1.1), assuming nonnegative periodic coefficients. A sequence of coefficients $\{\alpha_n\}_{n=0}^{\infty}$ is said to have period p if $\alpha_n = \alpha_{n+p}$ for all n. We follow the numbering system used in [1, 2].

2 Equation #1

In this section, we consider the special case of Equation (1.1) given below.

$$x_{n+1} = \frac{\alpha_n}{1 + x_n x_{n-1}}, \ n = 0, 1, \dots. \tag{2.1}$$

The case for $\alpha_n = \alpha$ for all n, was studied by Amleh, Camouzis and Ladas in [1], where it was shown that every solution of this equation converges for $\alpha < 2$. There is was conjectured that the convergence holds for all positive values of α, which is a special case of Conjecture 1.4.

In this section we confirm Conjecture 1.4 for $k = 2$. Furthermore, we investigate the behavior of Eq. (2.1) when the coefficients have a periodic nature.

When $\alpha_n = \alpha$ for all n, then Eq. (2.1) can be written as follows:

$$x_{n+1} = \frac{\alpha}{1 + x_n x_{n-1}}, \; n = 0, 1, \ldots. \tag{2.2}$$

According to [1], Eq. (2.2) has a unique equilibrium point $\bar{x}$, which is the unique positive root of the cubic equation

$$\bar{x}^3 + \bar{x} - \alpha = 0,$$

and is locally asymptotically stable for all values of the parameter α.

Theorem 2.1. *Let $\alpha > 0$. Then the unique positive equilibrium point $\bar{x}$ of Eq.* (2.2) *is a global attractor of all positive solutions of Eq.* (2.2).

Remark 2.2. Theorem 2.1 completes the proof that Eq. (2.2) converges for all values of $\alpha \geq 0$.

Proof of Theorem 2.1. Our proof is based on the global character of the rational system

$$u_{n+1} = \frac{\alpha^2 + u_n}{v_n} \quad \text{and} \quad v_{n+1} = \frac{u_n}{v_n}, n = 0, 1, \ldots \tag{2.3}$$

with the parameter α positive and both initial conditions u_0 and v_0 positive. This system has the boundedness characterization (B,B), which means that both components of every solution of System (2.3) are always bounded. See [6, 10]. Actually, for System (2.3) it holds that both components are also bounded from below by positive numbers. In [10] it was shown that every solution of System (2.3) converges to a finite limit. Based on that proof we establish that every positive solution of Eq. (2.2) converges to $\bar{x}$ for all values of $\alpha \geq 0$.

The equilibrium points of System (2.3) are the points $(\bar{u}, \bar{v}) \in (0, \infty)^2$ that are solutions of the system

$$\bar{u} = \frac{\alpha^2 + \bar{u}}{\bar{v}} \quad \text{and} \quad \bar{v} = \frac{\bar{u}}{\bar{v}}$$

from which it follows that $\bar{u}$ is the unique positive root of the cubic equation

$$\bar{u}^3 - \bar{u}^2 - 2\alpha^2 \bar{u} - \alpha^2 = 0,$$

and $\bar{v}$ is the unique positive root of the cubic equation

$$\bar{v}^3 - \bar{v}^2 - \alpha^2 = 0.$$

Thus System (2.3) has a unique equilibrium point $(\bar{u}, \bar{v})$.

By substituting the value of v_n from the first equation of System 2.3 into the second, we see that the component u_n satisfies the second-order rational difference equation

$$u_{n+1} = \frac{(\alpha^2 + u_n)(\alpha^2 + u_{n-})}{u_n u_{n-1}}, n = 1, 2, \ldots. \tag{2.4}$$

Similarly, by substituting the value of u_n from the second equation of the system into the first, we see that the component v_n satisfies the second-order rational difference equation

$$v_{n+1} = \frac{\alpha^2 + v_n v_{n-1}}{v_n v_{n-1}}, n = 1, 2, \ldots. \tag{2.5}$$

Both Eqs. (2.4) and (2.5) have unique equilibrium points which are $\bar{u}$ and $\bar{v}$ respectively. By iterating the value of u_n in the denominator of Eq. (2.4) we have

$$u_{n+1} = (\alpha^2 + u_n)\frac{u_{n-2}}{\alpha^2 + u_{n-2}}, n = 2, 3, \ldots. \tag{2.6}$$

The function associated with Eq. (2.6) increases in all of its arguments and every point of Eq. (2.6) is an equilibrium point. Thus, Theorem 1.2 applies, implying that every solution of Eq. (2.6) has a finite limit. Therefore, the unique equilibrium point $\bar{u}$ of Eq. (2.4) is a global attractor of every solution of Eq. (2.4). Then, since $v_n = \dfrac{\alpha^2 + u_n}{u_{n+1}}$, we have that

$$v_n \to \frac{\alpha^2 + \bar{u}}{\bar{u}} = \bar{v}. \tag{2.7}$$

This shows that $\bar{v}$ is a global attractor for every solution of Eq. (2.5). The change of variables

$$v_n = \frac{\alpha}{x_n} \tag{2.8}$$

transforms Equation (2.5) into Equation (2.2) from which the result follows. □

Theorem 2.3. *Assume that* $\{\alpha_n\}_{n=0}^{\infty}$ *is a periodic sequence of prime period* p. *Then, every solution to Equation* (2.1) *is bounded.*

Proof. The case when $p = 1$ has been shown in [1]. Thus, we assume $p \geq 2$, and that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence with prime period p. Let us define $M = \max\{\alpha_0, \ldots, \alpha_{p-1}\}$ and $m = \min\{\alpha_0, \ldots, \alpha_{p-1}\}$. We can see that

$$x_{n+1} = \frac{\alpha_n}{1 + x_n x_{n-1}} \leq \alpha_n \leq M, \tag{2.9}$$

and

$$x_{n+1} = \frac{\alpha_n}{1 + x_n x_{n-1}} \geq \frac{m}{1 + M^2}. \tag{2.10}$$

Thus, the sequence is bounded. □

Convergence of Equation (2.1) with Period-2 Coefficients

We now consider the case when $\{\alpha_n\}_{n=0}^{\infty}$ is a prime period-two sequence. That is,

$$\{\alpha_n\}_{n=0}^{\infty} = \{\alpha_0, \alpha_1, \alpha_0, \alpha_1, \ldots\},$$

with $\alpha_0 \neq \alpha_1$.

For any solution of Eq. (2.1), the odd and even terms satisfy the following equations:

$$x_{2n+1} = \frac{\alpha_0}{1 + x_{2n}x_{2n-1}}, n = 0, 1, \ldots, \tag{2.11}$$

$$x_{2n+2} = \frac{\alpha_1}{1 + x_{2n+1}x_{2n}}, n = 0, 1, \ldots. \tag{2.12}$$

We can see that if one of the coefficients is zero, say α_0, then $x_{2n+1} = 0$ for all $n \geq 0$, and $x_{2n+2} = \alpha_1$ for all $n \geq 0$. The case for $\alpha_1 = 0$ and $\alpha_0 > 0$ follows similarly. In the sequel, we assume that $\alpha_0 \cdot \alpha_1 > 0$.

Let us define

$$z_{n+1} = x_{2n+1} \cdot x_{2n+2}.$$

And further, set $A = \alpha_0 \cdot \alpha_1$. Then,

$$z_{n+1} = \frac{A}{(1 + z_{n-1})(1 + z_n)}. \tag{2.13}$$

Lemma 2.4. *Every solution of Eq.* (2.13) *converges to a finite limit.*

Proof. The change of variables $z_n = \dfrac{\sqrt{A}}{y_n} - 1$ transforms Eq. (2.13) into

$$y_{n+1} = \frac{\sqrt{A}}{1 + y_n y_{n-1}}. \tag{2.14}$$

Therefore, by Theorem 2.1, the result follows. □

Theorem 2.5. *When* $\{\alpha_n\}_{n=0}^{\infty}$ *is a periodic sequence with prime period two, every solution of Eq.* (2.1) *converges to a prime period-two sequence.*

Proof. It follows from Lemma 2.4 that $\lim_{n\to\infty} x_n x_{n-1} = L$, where L is the unique positive equilibrium of Eq. (2.13). Thus, the subsequences of odd and even terms of every solution of Eq. (2.1) converge to $\dfrac{\alpha_0}{1+L}$ and $\dfrac{\alpha_1}{1+L}$ respectively. □

Theorem 2.6. *When* $\{\alpha_n\}_{n=0}^{\infty}$ *is a prime period-two sequence, Eq.* (2.1) *is advantageous.*

Proof. Let α_0, α_1 be given, and let $a = \dfrac{\alpha_0 + \alpha_1}{2}$.

We consider the equation

$$y_{n+1} = \frac{a}{1 + y_n y_{n-1}}, \quad n = 0, 1, \ldots. \tag{2.15}$$

According to Theorem 2.5, every solution to Eq. (2.15) converges to $\bar{y}$, the unique positive solution to $\bar{y}^3 + \bar{y} - a = 0$. Let us define the following function, which is strictly increasing on $(0, \infty)$.

$$f(y) = y^3 + y - a.$$

The unique positive equilibrium point, $\bar{z}$, of Eq. (2.13) is the unique positive root of the cubic equation

$$\bar{z}^3 + 2\bar{z}^2 + \bar{z} - \alpha_0\alpha_1 = 0.$$

From Theorem 2.5, we have that the subsequence $\{x_{2n+1}\}_{n=0}^{\infty}$ converges to $\dfrac{\alpha_0}{1+\bar{z}}$ and the subsequence $\{x_{2n}\}_{n=0}^{\infty}$ converges to $\dfrac{\alpha_1}{1+\bar{z}}$. We define

$$l = \frac{\frac{\alpha_0}{1+\bar{z}} + \frac{\alpha_1}{1+\bar{z}}}{2} = \frac{a}{1+\bar{z}}.$$

We claim that $l > \bar{y}$. It suffices to show that $f(l) > 0$. We have

$$\begin{aligned} f(l) &= \frac{a^3}{(1+\bar{z})^3} + \frac{a}{1+\bar{z}} - a \\ &= \frac{a(\alpha_0 - \alpha_1)^2}{4(1+\bar{z})^3} > 0. \end{aligned}$$

This completes the proof. □

3 Equation #2

In this section, we consider the special case of Eq. (1.1) given below.

$$x_{n+1} = \frac{\alpha_n}{(1+x_n)x_{n-1}}, \ n = 0, 1, \ldots \tag{3.1}$$

When $\alpha_n = \alpha$ for all $n \geq 0$, it was shown in [1] that Eq. (3.1) possess the invariant

$$x_{n-1} + x_n + x_{n-1}x_n + \alpha\left(\frac{1}{x_{n-1}} + \frac{1}{x_n}\right) = \text{constant}, \quad \text{for all } n \geq 0, \tag{3.2}$$

from which it follows that every positive solution is bounded from above and from below by positive constants.

Amazingly, when $\{\alpha_n\}_{n=0}^{\infty}$ is a prime period-two sequence, Eq. (3.1) also possesses an invariant, which is as follows:

$$x_{n-1} + x_n + x_{n-1}x_n + \frac{\alpha_n}{x_{n-1}} + \frac{\alpha_{n+1}}{x_n} = \text{constant}, \quad \text{for all } n \geq 0. \tag{3.3}$$

The proof of the following theorem is a consequence of (3.3).

Theorem 3.1. *Assume* $\{\alpha_n\}_{n=0}^{\infty}$ *is a prime period-two sequence. Then, every positive solution of Equation* (3.1) *is bounded from above and below by positive constants.*

4 Equation #3

In this section, we consider the special case of Eq. (1.1) given below.

$$x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1 + x_n x_{n-1}}, \; n = 0, 1, \ldots . \tag{4.1}$$

The case for $\beta_n = \beta$ was studied in [1], where it was established that every positive solution to Eq. (4.1) converges to a finite limit.

In the nonautonomous case, we have established the following results.

Theorem 4.1. *Let $\{\beta_n\}_{n=0}^{\infty}$ be a periodic sequence with period $p \geq 1$. Then every solution to Eq.* (4.1) *is bounded.*

Proof. The case when $p = 1$ has been shown in [1]. Thus, we assume $p \geq 2$ and that $\{\beta_n\}_{n=0}^{\infty}$ is a periodic sequence with prime period p. Let $b = \max\{\beta_0, \ldots, \beta_{p-1}\}$. We can see that

$$x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1 + x_n x_{n-1}} \leq \beta_n \leq b. \tag{4.2}$$

The proof is complete. □

Convergence of (4.1) with Period-two Coefficients

We now consider the case when $\{\beta_n\}_{n=0}^{\infty}$ is a prime period-two sequence. That is,

$$\{\beta_n\}_{n=0}^{\infty} = \{\beta_0, \beta_1, \beta_0, \beta_1, \ldots\}$$

with $\beta_0 \neq \beta_1$. For any solution of Eq. (4.1) the odd and even terms satisfy the following equations:

$$x_{2n+1} = \frac{\beta_0 x_{2n} x_{2n-1}}{1 + x_{2n} x_{2n-1}}, \; n = 0, 1, \ldots, \tag{4.3}$$

$$x_{2n+2} = \frac{\beta_1 x_{2n+1} x_{2n}}{1 + x_{2n+1} x_{2n}}, \; n = 0, 1, \ldots . \tag{4.4}$$

Clearly, if one of β_0 or β_1 is equal to zero, then $x_n = 0$ for all $n \geq 1$. In the sequel, we consider $\beta_0 \cdot \beta_1 > 0$.

Let us define

$$z_{n+1} = x_{2n+1} \cdot x_{2n+2},$$

and furthermore, set $B = \beta_0 \cdot \beta_1$. Then,

$$z_{n+1} = \frac{B z_n z_{n-1}}{(1 + z_n)(1 + z_{n-1})}, \; n = 0, 1, \ldots . \tag{4.5}$$

Clearly,

$$0 \leq z_{n+1} = \frac{B z_n z_{n-1}}{(1 + z_n)(1 + z_{n-1})} \leq B, \text{ for all } n \geq 0. \tag{4.6}$$

Lemma 4.2. *Every solution of Eq.* (4.5) *converges to a finite limit.*

Proof. The function $f(x,y) = \dfrac{Bxy}{(1+x)(1+y)}$ associated with Eq. (4.5) is increasing in both variables. Therefore, by Theorem 1.1, the subsequences $\{z_{2n}\}_{n=0}^{\infty}$ and $\{z_{2n+1}\}_{n=0}^{\infty}$ of every solution are eventually monotonic. Since $\{z_n\}_{n=0}^{\infty}$ is bounded, they must converge to a finite limit. Furthermore, Eq. (4.5) does not have any prime period-two solutions, thus both subsequences must converge to the same equilibrium of Eq. (4.5). □

For Eq. (4.5), $\bar{z} = 0$ is always an equilibrium solution. Furthermore, Eq. (4.5) can have at most two additional equilibria, which are given by

$$\bar{z} = \frac{(B-2) \pm \sqrt{B(B-4)}}{2}. \tag{4.7}$$

In view of Eq. (4.7), if $B = 4$, then we have one additional equilibrium point, $\bar{z} = 1$. If $B > 4$ then we have two additional equilibrium points, say $\bar{z}_1 < \bar{z}_2$. Finally, when $B < 4$, $\bar{z} = 0$ is the only equilibrium point.

Lemma 4.3. *When $B < 4$, the equilibrium point $\bar{z} = 0$ of Eq.* (4.5) *is globally asymptotically stable.*

Proof. The proof follows directly from Lemma 4.2 and the fact that when $B < 4$, $\bar{z} = 0$ is the unique equilibrium point of Eq. (4.5). □

Lemma 4.4. *When $B = 4$, in addition to the zero equilibrium, we also have a unique positive equilibrium $\bar{z} = 1$. Furthermore, $\bar{z} = 0$ is locally asymptotically stable and $\bar{z} = 1$ is non-hyperbolic. Furthermore, we have,*

(i) if for some $N > 0$, $z_{N-1}, z_N \in [0,1)$, then

$$z_n \in [0,1) \text{ for all } n \geq N, \tag{4.8}$$

and

$$\lim_{n\to\infty} z_n = 0.$$

(ii) if for some $N > 0$, $z_{N-1}, z_N \in [1,\infty)$, then

$$z_n \in [1,\infty) \text{ for all } n \geq N, \tag{4.9}$$

and

$$\lim_{n\to\infty} z_n = 1.$$

(iii) if the subsequences $\{z_{2n+1}\}$ and $\{z_{2n}\}$ lie one in the interval $[0,1]$ and the other in the interval $[1,\infty)$, then

$$\lim_{n\to\infty} z_n = 1.$$

Lemma 4.5. *When $B > 4$, in addition to the zero equilibrium we have exactly two positive equilibria, namely*

$$\bar{z} = \frac{(B-2) \pm \sqrt{B(B-4)}}{2},$$

where $\bar{z}_1$ is the smaller root and $\bar{z}_2$ is the larger. Furthermore, we have,

(i) if for some $N > 0$, $z_{N-1}, z_N \in [0, \bar{z}_1)$, then

$$z_n \in [0, \bar{z}_1) \textit{ for all } n \geq N, \tag{4.10}$$

and

$$\lim_{n\to\infty} z_n = 0.$$

(ii) if for some $N > 0$, $z_{N-1}, z_N \in (\bar{z}_1, \bar{z}_2]$, then

$$z_n \in (\bar{z}_1, \bar{z}_2] \textit{ for all } n \geq N, \tag{4.11}$$

and

$$\lim_{n\to\infty} z_n = \bar{z}_2.$$

(iii) if for some $N > 0$, $z_{N-1}, z_N \in [\bar{z}_2, \infty)$, then

$$z_n \in [\bar{z}_2, \infty) \textit{ for all } n \geq N, \tag{4.12}$$

and

$$\lim_{n\to\infty} z_n = \bar{z}_2.$$

(iv) if the subsequences $\{z_{2n+1}\}$ and $\{z_{2n}\}$ eventually lie one in the interval $[0, \bar{z}_1]$ and one in the interval $[\bar{z}_1, \bar{z}_2)$, then the $\lim_{n\to\infty} z_n = \bar{z}_1$. And if the subsequences lie one in the interval $(\bar{z}_1, \bar{z}_2]$ and one in $[\bar{z}_2, \infty)$ then the $\lim_{n\to\infty} z_n = \bar{z}_2$.

The proof of Theorem 4.6 follows from the lemmas above.

Theorem 4.6. *Let $\{\beta_n\}_{n=0}^{\infty}$ be a periodic sequence with prime period-two. Then for $B = \beta_0 \cdot \beta_1 < 4$, every solution of Eq.* (4.1) *converges to* 0*. For $B \geq 4$, every solution of Eq.* (4.1) *converges to a period-two solution.*

5 Equation #5

In this section, we consider the special case of Eq. (1.1) given below.

$$x_{n+1} = \frac{\gamma_n x_{n-1}}{1 + x_n x_{n-1}}, \; n = 0, 1, \ldots \tag{5.1}$$

For the case when $\gamma_n = \gamma$ for all n, it was shown in [1] that every positive solution to Eq. (5.1) is bounded.

In the sequel, we consider the case when $\{\gamma_n\}_{n=0}^{\infty}$ is a positive, prime period-two sequence. That is,

$$\{\gamma_n\}_{n=0}^{\infty} = \{\gamma_0, \gamma_1, \ldots\},$$

with $\gamma_0 \neq \gamma_1$, and $\gamma_0 \cdot \gamma_1 > 0$. In this case, we have the following main result.

Theorem 5.1. *If $\{\gamma_n\}_{n=0}^{\infty}$ is a periodic sequence with prime period-two, then there exists unbounded solutions of Eq.* (5.1).

For any solution of Eq. (5.1), the odd and even terms satisfy the following equations:

$$x_{2n+1} = \frac{\gamma_0 x_{2n-1}}{1 + x_{2n} x_{2n-1}}, \; n = 0, 1, \ldots, \tag{5.2}$$

$$x_{2n+2} = \frac{\gamma_1 x_{2n}}{1 + x_{2n+1} x_{2n}}, \; n = 0, 1, \ldots. \tag{5.3}$$

Let us define

$$z_{n+1} = x_{2n+1} x_{2n+2}$$

from which it follows that

$$z_{n+1} = \frac{\gamma_0 \gamma_1 z_{n-1}}{(1 + z_n)(1 + z_{n-1})} \; n = 0, 1, \ldots. \tag{5.4}$$

Lemma 5.2. *When*

$$\gamma_0 \cdot \gamma_1 \leq 1,$$

zero is a globally asymptotically stable equilibrium of Eq. (5.4).

Proof. Consider the monotone character of the function

$$f(x, y) = \frac{\gamma_0 \gamma_1 y}{(1 + x)(1 + y)},$$

which is decreasing in x and increasing in y. Therefore, Theorem 1.3 applies,and because Eq. (5.4) has no period-two solutions when $\gamma_0 \cdot \gamma_1 < 1$, it follows that every solution of Eq. (5.4) converges to a finite limit. Furthermore, the equilibrium points, $\bar{z}$, of Eq. (5.4) satisfy the following equation:

$$\bar{z}(1 + 2\bar{z} + \bar{z}^2) = \gamma_0 \gamma_1 \bar{z}, \tag{5.5}$$

which has the unique solution $\bar{z} = 0$ when $\gamma_0 \gamma_1 < 1$. Since $\bar{z} = 0$ is locally asymptotically stable, the result follows. □

With Lemma 5.2 in place, we proceed to prove Theorem 5.1.

Proof of Theorem 5.1. Choose initial conditions x_{-1} and x_0 and the parameters γ_0 and γ_1 such that the following conditions are satisfied:

$$x_{-1} < \gamma_0 < 1 < \gamma_1 < x_0, \tag{5.6}$$

$$\gamma_0 \cdot \gamma_1 = 1. \tag{5.7}$$

We see that

$$x_1 = \frac{\gamma_0 x_{-1}}{1 + x_0 x_{-1}} < \gamma_0 x_{-1}. \tag{5.8}$$

Thus, by induction, we see that the subsequence of odd terms $\{x_{2n+1}\}$ is decreasing, and converges to zero.

Furthermore, by Lemma 5.2 we know that there exists some $N > 0$ such that for all $n \geq N$, $x_n \cdot x_{n-1} < \gamma_1 - 1$. Thus,

$$\begin{aligned} x_{2N+1}x_{2N} &< \gamma_1 - 1 & (5.9)\\ \gamma_1 &> 1 + x_{2N+1}x_{2N} & (5.10)\\ \frac{\gamma_1}{1 + x_{2N+1}x_{2N}} &> 1, & (5.11) \end{aligned}$$

and so,

$$x_{2N+2} = \frac{\gamma_1 x_{2N}}{1 + x_{2N+1}x_{2N}} = \left(\frac{\gamma_1}{1 + x_{2N+1}x_{2N}}\right) x_{2N}. \tag{5.12}$$

By induction, it follows that $\lim_{n\to\infty} x_{2n} = \infty$. □

Theorem 5.3. *If*

$$\gamma_0, \gamma_1 \in [0, 1),$$

then every positive solution of Eq. (5.1) *converges to zero.*

Proof. We know that Eq. (5.1) is bounded from below by zero. Further,

$$x_{2n+1} = \frac{\gamma_0 x_{2n-1}}{1 + x_{2n}x_{2n-1}} < \gamma_0 x_{2n-1}, \tag{5.13}$$

and

$$x_{2n+2} = \frac{\gamma_1 x_{2n}}{1 + x_{2n+1}x_{2n}} < \gamma_1 x_{2n}. \tag{5.14}$$

Thus, by induction, both of the sequences of even and odd terms of Eq. (5.1) converge to zero, and the result follows. □

6 Equation #15

In this section we consider the special case of Eq. (1.1) given below.

$$x_{n+1} = \frac{\alpha_n + x_{n-1}}{(1 + B_n x_n)x_{n-1}}, \ n = 0, 1, \ldots. \tag{6.1}$$

In the autonomous case, that is when $\alpha_n = \alpha$ and $B_n = B$ for all $n \geq 0$, Amleh, Camouzis, and Ladas, in [1], showed that every solution of Eq. 6.1 is bounded. Interestingly enough, periodicity destroys boundedness for this equation, as the following theorem shows.

Theorem 6.1. *Assume that* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{B_n\}_{n=0}^{\infty}$ *are period-three sequences such that*

$$\alpha_{3n} = 0, \ \alpha_{3n+1} = 1, \ \alpha_{3n+2} = 2, \ \textit{for all } n \geq 0,$$

and

$$B_{3n} = 1, \ B_{3n+1} = 2, \ B_{3n+2} = 1, \ \textit{for all } n \geq 0.$$

Then

$$x_{3n} \to \infty, \ x_{3n+1} \to 0, \ \textit{and } x_{3n+2} \to 1.$$

Proof. We see that

$$x_{3n+1} = \frac{1}{1 + x_{3n}}, n = 0, 1, \ldots, \tag{6.2}$$

$$x_{3n+2} = \frac{1 + x_{3n}}{(1 + 2x_{3n+1})x_{3n}}, n = 0, 1, \ldots, \tag{6.3}$$

$$x_{3n+3} = \frac{2 + x_{3n+1}}{(1 + x_{3n+2})x_{3n+1}}, n = 0, 1, \ldots. \tag{6.4}$$

It suffices to show that $\lim_{n\to\infty} x_{3n+3} = \infty$. By substituting the values of x_{3n+1} and x_{3n+2} from Eqs. (6.2) and (6.3) into Eq. (6.4), we see that Eq. (6.4) can be written as

$$x_{3n+3} = \left(\frac{1 + 9x_{3n} + 2(x_{3n})^2}{1 + 5x_{3n} + 2(x_{3n})^2} \right) x_{3n}. \tag{6.5}$$

In view of Eq. (6.5), we see that $x_{3n+3} > x_{3n}$ for all $n \geq 0$. Observe that the function

$$f(u) = \frac{9 + 9u + 2u^2}{1 + 5u + 2u^2} > 1$$

for all $u \geq 0$. Thus Eq. (6.5) has zero as the unique equilibrium point, from which the result follows. □

References

[1] A. M. Amleh, E. Camouzis, and G. Ladas. On the dynamics of a rational difference equation. I. *Int. J. Difference Equ.*, 3(1):1–35, 2008.

[2] A. M. Amleh, E. Camouzis, and G. Ladas. On the dynamics of a rational difference equation. II. *Int. J. Difference Equ.*, 3(2):195–225, 2008.

[3] A. M. Amleh, E. Camouzis, G. Ladas, and M. A. Radin. Patterns of boundedness of a rational system in the plane. *J. Difference Equ. Appl.*, 16(10):1197–1236, 2010.

[4] A. M. Brett, E. Camouzis, G. Ladas, and C. D. Lynd. On the boundedness character of a rational system. *J. Numer. Math. Stoch.*, 1(1):1–10, 2009.

[5] E. Camouzis, E. Drymonis, and G. Ladas. On the global character of the system $x_{n+1} = \frac{a}{x_n + y_n}$ and $y_{n+1} = \frac{y_n}{Bx_n + y_n}$. *Comm. Appl. Nonlinear Anal.*, 16(2):51–64, 2009.

[6] E. Camouzis, E. Drymonis, and G. Ladas. Patterns of boundedness of the rational system $x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{A_1 + C_1 y_n}$ and $y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}$. *Fasc. Math.*, (44):9–18, 2010.

[7] E. Camouzis, E. Drymonis, and G. Ladas. Patterns of boundedness of the rational system $x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{A_1 + B_1 x_n + C_1 y_n}$ and $y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}$. *Comm. Appl. Nonlinear Anal.*, 18(1):1–23, 2011.

[8] E. Camouzis, E. Drymonis, G. Ladas, and W. Tikjha. Patterns of boundedness of the rational system $x_{n+1} = \frac{\alpha_1}{A_1 + B_1 x_n + C_1 y_n}$ and $y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}$. *J. Difference Equ. Appl.*, 2011.

[9] E. Camouzis, A. Gilbert, M. Heissan, and G. Ladas. On the boundedness character of the system $x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{x_n}$ and $y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + x_n + y_n}$. *Commun. Math. Anal.*, 7(2):41–50, 2009.

[10] E. Camouzis, C.M. Kent, G. Ladas, and C.D. Lynd. On the global character of the solutions of the system $x_{n+1} = \frac{\alpha_1 + y_n}{x_n}$ and $y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}$. *J. Difference Equ. Appl.*, 2011.

[11] E. Camouzis, M. R. S. Kulenović, G. Ladas, and O. Merino. Rational systems in the plane. *J. Difference Equ. Appl.*, 15(3):303–323, 2009.

[12] E. Camouzis and G. Ladas. When does periodicity destroy boundedness in rational equations? *J. Difference Equ. Appl.*, 12(9):961–979, 2006.

[13] E. Camouzis and G. Ladas. Global convergence in difference equations. *Comm. Appl. Nonlinear Anal.*, 14(4):1–16, 2007.

[14] E. Camouzis and G. Ladas. *Dynamics of third-order rational difference equations with open problems and conjectures*, volume 5 of *Advances in Discrete Mathematics and Applications*. Chapman & Hall/CRC, Boca Raton, FL, 2008.

[15] E. Camouzis and G. Ladas. Global results on rational systems in the plane, part 1. *J. Difference Equ. Appl.*, 16(8):975–1013, 2010.

[16] E. Camouzis, G. Ladas, and L. Wu. On the global character of the system $x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{x_n}$ and $y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{B_2 x_n + C_2 y_n}$. *Int. J. Pure Appl. Math.*, 53(1):21–36, 2009.

[17] J. M. Cushing and S. Henson. Global dynamics of some periodically forced, monotone difference equations. *J. Differ. Equations Appl.*, 7(6):859–872, 2001. On the occasion of the 60th birthday of Calvin Ahlbrandt.

[18] H. El-Metwally, E. A. Grove, G. Ladas, and H. D. Voulov. On the global attractivity and the periodic character of some difference equations. *J. Differ. Equations Appl.*, 7(6):837–850, 2001. On the occasion of the 60th birthday of Calvin Ahlbrandt.

[19] S. Elaydi and R. Sacker. Periodic difference equations, population biology and the Cushing-Henson conjectures. *Math. Biosci.*, 201(1-2):195–207, 2006.

[20] M. Garić-Demirović, M. R. S. Kulenović, and M. Nurkanović. Global behavior of four competitive rational systems of difference equations in the plane. *Discrete Dyn. Nat. Soc.*, pages Art. ID 153058, 34, 2009.

[21] E. A. Grove and G. Ladas. *Periodicities in nonlinear difference equations*, volume 4 of *Advances in Discrete Mathematics and Applications*. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[22] S. Kalabušić, M. R. S. Kulenović, and E. Pilav. Global dynamics of a competitive system of rational difference equations in the plane. *Adv. Difference Equ.*, pages Art. ID 132802, 30, 2009.

[23] V. L. Kocić and G. Ladas. *Global behavior of nonlinear difference equations of higher order with applications*, volume 256 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1993.

[24] M. R. S. Kulenović and G. Ladas. *Dynamics of second order rational difference equations*. Chapman & Hall/CRC, Boca Raton, FL, 2002. With open problems and conjectures.

[25] M. R. S. Kulenović and O. Merino. Competitive-exclusion versus competitive-coexistence for systems in the plane. *Discrete Contin. Dyn. Syst. Ser. B*, 6(5):1141–1156, 2006.

[26] M. R. S. Kulenović and M. Nurkanović. Asymptotic behavior of a two dimensional linear fractional system of difference equations. *Rad. Mat.*, 11(1):59–78, 2002. Dedicated to the memory of Prof. Dr. Naza Tanović-Miller.

[27] M. R. S. Kulenović and M. Nurkanović. Asymptotic behavior of a system of linear fractional difference equations. *J. Inequal. Appl.*, (2):127–143, 2005.

[28] M. R. S. Kulenović and M. Nurkanović. Asymptotic behavior of a competitive system of linear fractional difference equations. *Adv. Difference Equ.*, pages Art. ID 19756, 13, 2006.

[29] G. Ladas. Open problems on the boundedness of some difference equations. *J. Differ. Equations Appl.*, 1(4):413–419, 1995.