International Journal of Difference Equations
ISSN 0973-6069, Volume 7, Number 1, pp. 1–18 (2012)
http://campus.mst.edu/ijde

Boundedness of Solutions of a Rational System

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Abstract

The main result in this paper is that every solution, $\{x_n, y_n\}_{n=0}^{\infty}$, of the system

$$x_{n+1} = \frac{x_n}{y_n} \text{ and } y_{n+1} = x_n + \gamma y_n, \quad n = 0, 1, \ldots,$$

with positive initial conditions x_0 and y_0, is bounded when $\gamma \in (0, 1)$. Our proof will be based on certain properties of convergence of double sequences. This result confirms the longstanding conjecture, posed in [32], that every positive solution of the difference equation

$$x_{n+1} = \frac{x_n^2}{x_{n-1}(\gamma + x_n)}, \quad n = 1, 2, \ldots$$

is bounded. In the Appendix, we also present the boundedness characterizations of all special cases of a rational system, which contains 98 special cases each with positive parameters.

AMS Subject Classifications: 39A10.
Keywords: Boundedness, boundedness characterization, rational systems, double sequences.

1 Introduction

The main result in this paper is that every solution of the system,

$$x_{n+1} = \frac{x_n}{y_n} \text{ and } y_{n+1} = x_n + \gamma y_n, \quad n = 0, 1, \ldots, \tag{1.1}$$

Received March 12, 2012; Accepted August 15, 2012
Communicated by Martin Bohner

with positive initial conditions, is bounded when $\gamma \in (0,1)$. The component, $\{x_n\}_{n=0}^{\infty}$, of every solution of this system, satisfies the second-order difference equation

$$x_{n+1} = \frac{x_n^2}{x_{n-1}(\gamma + x_n)}, \quad n = 1, 2, \ldots,$$

for which it has been conjectured for a long time that every positive solution is bounded when $\gamma \in (0,1)$. See [32].

The dynamics of System (1.1) in terms of boundedness, as it will be shown throughout this paper, are equivalent with the dynamics of the system,

$$x_{n+1} = \frac{x_n y_n}{\gamma + x_n} \text{ and } y_{n+1} = \frac{y_n}{\gamma + x_n}, \quad n = 0, 1, \ldots, \tag{1.2}$$

when $\gamma \in (0,1]$ and the initial conditions are positive. Also, in Section 3, we will prove that the two maps that correspond to the two systems are topologically conjugate. System (1.2), up to an appropriate change of variables, is a special case of the system

$$\left.\begin{aligned} x_{n+1} &= y_n - \frac{y_{n+1}}{F} \\ y_{n+1} &= F y_n \left(1 + \frac{a x_n}{k}\right)^{-k} \end{aligned}\right\}, \quad n = 0, 1, \ldots, \tag{1.3}$$

with $k = 1$. System (1.3) is the so-called **May's Host Parasitoid Model.** See [34]. For some local stability results of System (1.2), see [33].

By replacing x_n with y_n and y_n with x_n, in System (1.1), we obtain the "symmetric" system

$$x_{n+1} = \gamma x_n + y_n \text{ and } y_{n+1} = \frac{y_n}{x_n}, \quad n = 0, 1, \ldots . \tag{1.4}$$

System (1.4) is a special case of the rational system,

$$\left.\begin{aligned} x_{n+1} &= \alpha_1 + \beta_1 x_n + \gamma_1 y_n \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots . \tag{1.5}$$

If we assume that $\beta_1, \gamma_1 > 0$ and the remaining 7 parameters are nonnegative, System (1.5) contains 98 special cases each with positive parameters. More precisely, it contains 49 special cases in which the first equation of the system is:

$$x_{n+1} = \beta_1 x_n + \gamma_1 y_n, \quad n = 0, 1, \ldots,$$

with β_1 and γ_1 positive. In the numbering system introduced in [13], this equation is numbered $\#25$. The second equation of the system varies among the 49 "second" equations listed in the Appendix. So, for example, System (25,6), which is one of the 49 systems for which the first equation is Equation $\#25$, is:

$$(25,6): \quad x_{n+1} = \beta_1 x_n + \gamma_1 y_n \text{ and } y_{n+1} = \frac{y_n}{x_n}, \quad n = 0, 1, \dots . \tag{1.6}$$

It so appears that System (1.4), which is the "symmetric" system of (1.1), is a special case of (1.6). However, we will show in Section 2 of this paper that the dynamics of these two systems are equivalent.

In the remaining 49 special cases, the first equation of the system is:

$$x_{n+1} = \alpha_1 + \beta_1 x_n + \gamma_1 y_n, \quad n = 0, 1, \dots,$$

with α_1, β_1 and γ_1 positive, which in the numbering system introduced in [13], is numbered $\#40$.

In the Appendix of this paper, we present the boundedness characterizations of all 98 special cases of System (1.5). Each one of them has exactly one of the following three boundedness characterizations:

(U,B), **(B,U)**, or **(U,U)**.

A special case of System (1.5) has the boundedness characterization **(U,B)** means: Given an arbitrary set with positive values which correspond to the parameters and the initial conditions of the special case, the second component of the corresponding solution is bounded and also there exists at least one solution for which the first component is unbounded. For the definitions of the boundedness characterizations **(U,B)** and **(U,U)** see the Appendix.

Example 1.1. Consider the system

$$(25,12): \quad x_{n+1} = \beta_1 x_n + \gamma_1 y_n \text{ and } y_{n+1} = \frac{1}{B_2 x_n + C_2 y_n}, \quad n = 0, 1, \dots .$$

Given an arbitrary set of positive values that correspond to β_1, γ_1, B_2, C_2, x_0, and y_0, the sequence $\{y_n\}_{n=0}^{\infty}$, which corresponds to the solution, $\{x_n, y_n\}_{n=0}^{\infty}$, is bounded, because from

$$x_{n+1} y_{n+1} = \frac{\beta_1 x_n + \gamma_1 y_n}{B_2 x_n + C_2 y_n} > \frac{\min\{\beta_1, \gamma_1\}}{\max\{B_2, C_2\}}, \quad \text{for all } n \geq 0,$$

the two sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ cannot both go to zero along the same subsequence. On the other hand, when $\beta_1 = \gamma_1 = B_2 = C_2 = 2$ and $x'_0 = y'_0 = 1$, the values of the sequence $\{x'_n\}_{n=0}^{\infty}$, of the solution $\{x'_n, y'_n\}_{n=0}^{\infty}$, satisfy the inequality

$$x'_n \geq 2^n, \quad \text{for all } n \geq 0,$$

and so, $\{x'_n\}_{n=0}^{\infty}$ is unbounded. Therefore, the boundedness characterization of System (25,12) is **(U,B)**.

In Section 2, we establish that the boundedness characterization of System (1.1) is **(B,U)** and that the boundedness characterization of the special case (25,6) of System (1.5) is **(U,B)**. For the proofs of the boundedness characterizations of the remaining 97 special cases of System (1.5), see [15]. For some results on difference equations and systems of difference equations and applications, see [1–22, 24, 25, 27–34].

The following will be useful in the sequel. A sequence of the form $\{\phi(k,n)\}_{k,n=1}^{\infty}$, where for each pair of positive integers (k,n), $\phi(k,n) > 0$, is called a double sequence of positive real numbers. The double limit of the sequence, $\{\phi(k,n)\}_{k,n=1}^{\infty}$, exists and is equal with $M \in [0,\infty)$, if for every $\epsilon > 0$, there exists $N(\epsilon)$ such that

$$|\phi(k,n) - M| < \epsilon, \quad \text{for all} \quad k, n \geq N.$$

Symbolically,

$$\lim_{k,n\to\infty} \phi(k,n) = M.$$

The two limits

$$\lim_{k\to\infty}\lim_{n\to\infty} \phi(k,n) \quad \text{and} \quad \lim_{n\to\infty}\lim_{k\to\infty} \phi(k,n)$$

are called iterated limits. For the proof of the next theorem, which gives sufficient conditions for the existence of the double limit of a double sequence, see [26, p. 80].

Theorem 1.2. *If the double sequence of positive real numbers, $\{\phi(k,n)\}_{k,n=1}^{\infty}$, is increasing in n, for each k, and if there exists a sequence of positive real numbers, $\{w_k\}_{k=1}^{\infty}$, such that*

$$\lim_{n\to\infty} \phi(k,n) \leq w_k, \quad \textit{for all} \quad k,$$

with

$$\lim_{k\to\infty} w_k = \lim_{n\to\infty}\lim_{k\to\infty} \phi(k,n) \in [0,\infty),$$

then the double limit exists and

$$\lim_{k,n\to\infty} \phi(k,n) = \lim_{k\to\infty}\lim_{n\to\infty} \phi(k,n) = \lim_{n\to\infty}\lim_{k\to\infty} \phi(k,n).$$

2 The Proof of Boundedness

In this section we establish that the boundedness characterization of the system,

$$x_{n+1} = \frac{x_n}{y_n} \quad \text{and} \quad y_{n+1} = x_n + \gamma y_n, \quad n = 0, 1, \ldots, \tag{2.1}$$

where γ is positive and the initial conditions x_0 and y_0 are positive real numbers is **(B,U)**.

Clearly, the dynamics of System (2.1) are equivalent with the dynamics of the "symmetric" system

$$x_{n+1} = \gamma x_n + y_n \quad \text{and} \quad y_{n+1} = \frac{y_n}{x_n}, \quad n = 0, 1, \ldots . \tag{2.2}$$

When the values of the parameters are positive, the dynamics of System (2.2) are equivalent with the dynamics of System $(25, 6)$, that is the system

$$(25,6):\qquad x_{n+1} = \beta_1 x_n + \gamma_1 y_n \ \text{ and } \ y_{n+1} = \frac{y_n}{x_n}, \quad n = 0, 1, \ldots . \tag{2.3}$$

To see this observe that, by setting $\beta_1 = \gamma$ and given a solution $\{x_n, y_n\}_{n=0}^{\infty}$, of System (2.3), the sequence $\{x_n, \gamma_1 y_n\}_{n=0}^{\infty}$ satisfies (2.2) and also given a solution $\{x_n, y_n\}_{n=0}^{\infty}$ of System (2.2), the sequence $\{x_n, \dfrac{y_n}{\gamma_1}\}_{n=0}^{\infty}$ satisfies (2.3). In past articles we would have referred to (2.2), as: the "normalized form" of (2.3). The conclusion of this analysis is: The dynamics of System (2.1) are equivalent with the dynamics of System (25,6) and provided that the boundedness characterization of System (2.1) is **(B,U)**, the boundedness characterization of System (25,6) is **(U,B)**.

Now, when

$$\gamma > 1,$$

the component $\{y_n\}_{n=0}^{\infty}$ of the solution $\{x_n, y_n\}_{n=0}^{\infty}$ of System (2.1), increases to infinity and the component $\{x_n\}_{n=0}^{\infty}$ decreases to zero. This establishes the **U** characterization of the component $\{y_n\}_{n=0}^{\infty}$. When $\gamma = 1$, every solution $\{x_n, y_n\}_{n=0}^{\infty}$ of System (2.1), with positive initial conditions x_0 and y_0, satisfies the identity

$$x_n + y_n + \frac{x_n}{y_n} + \frac{1}{y_n} = x_0 + y_0 + \frac{x_0}{y_0} + \frac{1}{y_0}, \quad \text{for all } \ n \geq 0.$$

From this together with,

$$y_{n+1} = x_n + y_n > y_n, \quad \text{for all } \ n \geq 0,$$

it follows that, $\{y_n\}_{n=0}^{\infty}$ increases to a positive and finite limit, which depends on the initial conditions x_0 and y_0, while the component $\{x_n\}_{n=0}^{\infty}$ goes to zero, and so, the solution is bounded. For the remaining of the sequel we assume that

$$0 < \gamma < 1.$$

Remark 2.1. Note that the question on the boundedness characterization **(B,U)** is not on the letter **U** but on the letter **B**, that is, we only need to establish that the component $\{x_n\}_{n=0}^{\infty}$ is bounded when $\gamma \in (0, 1)$. The boundedness behavior of the component $\{y_n\}_{n=0}^{\infty}$, when $\gamma \in (0, 1)$, with respect to the boundedness characterization, is immaterial. That is, even if the sequence $\{y_n\}_{n=0}^{\infty}$ is bounded for all $\gamma \in (0, 1)$, which we prove to be the case, the second letter of the boundedness characterization remains **U**.

In view of the remark above and in order to prove that the boundedness characterization of System (2.1) is **(B,U)**, it suffices to establish the statement of the next theorem.

Theorem 2.2. *Assume that* $0 < \gamma < 1$*. Then every solution* $\{x_n, y_n\}_{n=0}^{\infty}$ *of System* (2.1)*, with positive initial conditions* x_0 *and* y_0 *is bounded.*

The following lemmas will be useful in the proof of Theorem 2.2.

Lemma 2.3. *Given a solution* $\{x_n, y_n\}_{n=0}^{\infty}$ *of System* (2.1)*, with positive initial conditions* x_0 *and* y_0*, it holds:*

$$\frac{y_n}{\gamma^n} \uparrow \infty. \tag{2.4}$$

Proof. Clearly, from the second equation of System (2.1), we have

$$\frac{y_{n+1}}{\gamma^{n+1}} = \frac{x_n}{\gamma^{n+1}} + \frac{y_n}{\gamma^n}, \quad \text{for all} \ \ n \geq 0,$$

and so, the sequence $\{\frac{y_n}{\gamma^n}\}_{n=0}^{\infty}$, is strictly increasing. Assume, for the sake of contradiction, that

$$\lim_{n\to\infty} \frac{y_n}{\gamma^n} = L \in (0, \infty).$$

Then, for every $0 < \epsilon < 1$, there exists N sufficiently large, such that

$$y_n < (L + \epsilon)\gamma^n, \quad \text{for all} \ \ n \geq N.$$

Clearly,

$$\lim_{n\to\infty} y_n = 0,$$

and so, for $0 < \epsilon_1 < \epsilon$, there exists N_1 such that,

$$y_n < \epsilon_1 \ \ \text{and} \ \ x_{n+1} = \frac{x_n}{y_n} > \frac{x_n}{\epsilon_1}, \quad \text{for all} \ \ n \geq N_1.$$

Thus,

$$\lim_{n\to\infty} x_n = \infty.$$

On the other hand,

$$0 = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} (x_n + \gamma y_n) = \lim_{n\to\infty} x_n = \infty,$$

is a contradiction. The proof of the lemma is complete. □

Lemma 2.4. *Given a solution* $\{x_n, y_n\}_{n=0}^{\infty}$ *of System* (2.1)*, with positive initial conditions* x_0 *and* y_0*, the sequence* $\{x_n, w_n\}_{n=0}^{\infty}$*, with*

$$w_n = \frac{\gamma + x_n}{y_n}, \quad n = 0, 1, \ldots,$$

satisfies the system

$$x_{n+1} = \frac{x_n w_n}{\gamma + x_n} \ \ \textit{and} \ \ w_{n+1} = \frac{w_n}{\gamma + x_n}, \quad n = 0, 1, \ldots \ . \tag{2.5}$$

Also,

$$w_n = \frac{1}{y_{n-1}} \ \ \textit{and} \ \ w_{n-1} = x_n + \gamma w_n, \ \ \textit{for all} \ \ n \geq 1.$$

Proof. The proof is based on straightforward calculations and the details are omitted. □

Lemma 2.5. *Given a solution* $\{x_n, y_n\}_{n=0}^{\infty}$ *of System* (2.1), *with positive initial conditions* x_0 *and* y_0, *it holds:*

$$x_{n+1} = \frac{1}{y_{n-1}} \cdot \frac{x_n}{\gamma + x_n}, \quad \textit{for all} \quad n \geq 1. \tag{2.6}$$

Proof. The proof is based on straightforward calculations and the details are omitted. □

Lemma 2.6. *Given a solution* $\{x_n, y_n\}_{n=0}^{\infty}$ *of System* (2.1), *with positive initial conditions* x_0 *and* y_0, *it holds:*

$$\sum_{k=0}^{\infty} \gamma^k x_{k+n+1} = \frac{\gamma + x_n}{y_n} = w_n, \quad \textit{for all} \quad n \geq 0. \tag{2.7}$$

Proof. First we establish the result for $n = 0$, that is,

$$\sum_{n=0}^{\infty} \gamma^k x_{k+1} = \frac{\gamma + x_0}{y_0} = w_0. \tag{2.8}$$

In view of Lemma 2.4, we have that for all $l \geq 0$,

$$\frac{\gamma + x_0}{y_0} = w_0 = x_1 + \gamma w_1 = x_1 + \gamma x_2 + \gamma^2 w_2 = \ldots = \sum_{k=0}^{l} \gamma^k x_{k+1} + \gamma^{l+1} w_{l+1}.$$

From Lemma 2.3 and from Lemma 2.4,

$$\gamma^{l+1} w_{l+1} = \frac{\gamma^{l+1}}{y_l} \downarrow 0, \quad \text{as} \quad l \uparrow \infty,$$

and so, the result follows. For an arbitrary $n > 0$, set

$$x_0' = x_n, \quad y_0' = y_n, \quad \text{and} \quad w_0' = w_n.$$

Then, for all $l \geq 0$,

$$w_n = w_0' = \sum_{k=0}^{l} \gamma^k x_{k+1}' + \gamma^{l+1} w_{l+1}' = \sum_{k=0}^{l} \gamma^k x_{k+n+1} + \gamma^{l+1} w_{l+1}'.$$

From this together with,

$$\gamma^{l+1} w_{l+1}' = \frac{\gamma^{l+1}}{y_l'} \to 0,$$

we have

$$\sum_{k=0}^{\infty} \gamma^k x_{k+n+1} = \lim_{l \to \infty} \sum_{k=0}^{l} \gamma^k x_{k+n+1} = w_n.$$

The proof of the lemma is complete. □

Next, we present a lemma which is essential for the proof of our main theorem, but is also interesting on its own right.

Lemma 2.7. *Let* $\{x_n, y_n\}_{n=0}^{\infty}$ *be a solution of* (2.1)*, with positive initial conditions* x_0 *and* y_0*. Assume that for an infinite sequence of indices* $\{k_i\}_{i=1}^{\infty}$*,* $\{x_{k_i}\}_{i=1}^{\infty}$ *and* $\{y_{k_i}\}_{i=1}^{\infty}$ *converge to finite limits and that*

$$\lim_{i\to\infty} \frac{x_{k_i}+\gamma}{y_{k_i}} = \lim_{i\to\infty} w_{k_i} = M \in (0,\infty).$$

Then, the double limit and the two iterated limits of the double sequence

$$\phi(k_i, n) = \sum_{k=0}^{n} \gamma^k x_{k+k_i+1}, \quad i = 1, 2, \ldots, \quad n = 0, 1, \ldots,$$

are all equal. More precisely,

$$\lim_{i,n\to\infty} \phi(k_i, n) = \lim_{i\to\infty}\lim_{n\to\infty} \phi(k_i, n) = \lim_{n\to\infty}\lim_{i\to\infty} \phi(k_i, n) = M.$$

Furthermore,

$$\lim_{i\to\infty} x_{k_i} \in (0,\infty).$$

Proof. For each $i \geq 1$, Lemma 2.6 implies that,

$$\lim_{n\to\infty} \phi(k_i, n) = \sum_{k=0}^{\infty} \gamma^k x_{k_i+k+1} = w_{k_i},$$

and so,

$$\lim_{i\to\infty}\lim_{n\to\infty} \phi(k_i, n) = \lim_{i\to\infty} w_{k_i} = M \in (0,\infty).$$

In view of the hypothesis, set

$$\lim_{i\to\infty} x_{k_i} = l_0 \in [0,\infty) \ \text{ and } \ \lim_{i\to\infty} y_{k_i} = m_0 \in (0,\infty).$$

Then,

$$\lim_{i\to\infty} x_{k_i+1} = \frac{l_0}{m_0} = l_1 \in [0,\infty) \text{ and } \lim_{i\to\infty} y_{k_i+1} = l_0 + \gamma m_0 = m_1 \in (0,\infty).$$

By induction, it can be shown that, for each $n \geq 0$,

$$\lim_{i\to\infty} x_{k_i+n+1} = \frac{l_0}{m_0\cdots m_n} = l_{n+1} \in [0,\infty) \ \text{ and } \ \lim_{i\to\infty} y_{k_i+n+1} = m_{n+1} \in (0,\infty). \tag{2.9}$$

For each $n \geq 0$, set

$$L_n = \lim_{i\to\infty} \phi(k_i, n) = l_1 + \ldots + \gamma^n l_{n+1}. \tag{2.10}$$

Then,

$$L_{n+1} = \lim_{i\to\infty} \phi(k_i, n+1) \geq \lim_{i\to\infty} \phi(k_i, n) = L_n$$

and since, for each given $n \geq 0$, L_n is bounded from above by $\sup_{i\geq 1} w_{k_i}$ plus a constant, we have

$$\lim_{n\to\infty} L_n = L \in [0, \infty).$$

Furthermore, it can be shown inductively that for a fixed n, $\phi(k_i, n)$ is a continuous function of x_{k_i}, w_{k_i}, and x_{k_i+1}, and its domain is $[0,\infty) \times [0,\infty) \times [0,\infty)$. To see this observe for example that

$$x_{k_i+2} = \frac{x_{k_i+1} w_{k_i+1}}{\gamma + x_{k_i+1}} = \frac{x_{k_i+1}}{\gamma + x_{k_i+1}} \frac{w_{k_i}}{\gamma + x_{k_i}}.$$

In view of the fact, that for every $\epsilon > 0$ the values of x_{k_i}, w_{k_i}, x_{k_i+1} are eventually within the intervals $(0, l_0 + \epsilon)$, $(0, M + \epsilon)$, and $(0, l_1 + \epsilon)$, respectively, $\phi(k_i, n)$ is uniformly continuous. In addition, there exists i_0 such that

$$|x_{k_i} - x_{k_j}| < \epsilon,\ |w_{k_j} - M| < \epsilon,\ |w_{k_i} - w_{k_j}| < \epsilon,\ |x_{k_i+1} - x_{k_j+1}| < \epsilon, \text{ for all } i, j \geq i_0.$$

Thus, for a given n and for every $\epsilon > 0$,

$$\phi(k_j, n) - \epsilon < \phi(k_i, n) < \phi(k_j, n) + \epsilon, \quad \text{for all } i \geq i_0.$$

In addition, there exists N sufficiently large such that for every given $n \geq N$, there exists $j_1 \geq i_0$ such that

$$w_{k_{j_1}} - \epsilon < \phi(k_{j_1}, n) < w_{k_{j_1}},$$

and so

$$w_{k_{j_1}} - 2\epsilon < \phi(k_i, n) < w_{k_{j_1}} + \epsilon, \quad \text{for all } i \geq i_0.$$

Furthermore,

$$M - \epsilon < w_{k_{j_1}} < M + \epsilon$$

and so, as $i \to \infty$, we have

$$M - 3\epsilon < L_n < M + 2\epsilon, \quad \text{for all } n \geq N$$

from which it follows that

$$\lim_{n\to\infty} L_n = M.$$

Thus, the conditions of Theorem 1.3 are satisfied and the proof follows. Finally, one may easily see that

$$\lim_{n\to\infty} L_n = M \in (0,\infty) \Rightarrow \lim_{i\to\infty} x_{k_i} \in (0, \infty).$$

This completes the proof. □

Now we present the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $\{x_n, y_n\}$ be a solution of System (2.1). First we establish that the second component of the solution is bounded from below by a positive constant. Assume for the sake of contradiction that there exists a sequence of indices $\{n_i\}_{i=1}^{\infty}$ such that

$$y_{n_i+1} = x_{n_i} + \gamma y_{n_i} \to 0.$$

Clearly,

$$x_{n_i-t} \to 0 \text{ and } y_{n_i-t} \to 0, \quad \text{for all } t = 0, 1, \dots .$$

In addition, there exists a sequence of indices $\{k_i\}_{i=1}^{\infty}$ such that

$$k_i \leq n_i, \quad \text{for all } i,$$

and

$$(y_{k_i-1} \geq 1 \text{ and } y_{k_i} < 1) \text{ and } (y_t < 1, \quad \text{for all } t \in \{k_i+1, \dots, n_i\}). \tag{2.11}$$

Otherwise,

$$x_{n_i} = \frac{x_0}{\prod_{j=0}^{n_i-1} y_j} > x_0,$$

which is a contradiction. Also,

$$y_{k_i} = x_{k_i-1} + \gamma y_{k_i-1} \text{ and } y_{k_i-1} \geq 1, \quad \text{for all } i,$$

implies that

$$y_{k_i} \in [\gamma, 1), \quad \text{for all } i.$$

When $r \in \{k_i+1, \dots, n_i\}$,

$$x_r = \frac{x_{r-1}}{y_{r-1}} > x_{r-1},$$

and more precisely,

$$x_{k_i} < x_{k_i+1} < \dots < x_{n_i},$$

from which it follows that $x_{k_i} \to 0$. By utilizing,

$$y_{k_i} \in [\gamma, 1), \quad \text{for all } i,$$

we may select a further subsequence of $\{k_i\}$, still denoted as $\{k_i\}$, such that

$$y_{k_i} \to m_0 \text{ and } w_{k_i} = \frac{\gamma + x_{k_i}}{y_{k_i}} \to \frac{\gamma}{m_0} = M \in [\gamma, 1].$$

Since, the conditions of Lemma 2.7 are satisfied, we have

$$\lim_{i\to\infty} x_{k_i} > 0,$$

a contradiction. This contradiction establishes our claim that the component $\{y_n\}$ of the solution is bounded from below. In view of (2.6) and the fact that $\{y_n\}$ is bounded from below by a positive constant, we find that the component $\{x_n\}$ is bounded from above by a positive constant. Since, the component $\{x_n\}$ is bounded from above by a positive constant, the second equation of the system implies, that the component $\{y_n\}$ is also bounded from above by a positive constant. The proof of Theorem 2.2 is complete. □

3 Final Remarks

Given a solution of System (1.2), with positive initial conditions x_0 and y_0, the sequence

$$\{x_n, \frac{\gamma + x_n}{y_n}\}_{n=0}^{\infty} = \{x_n, \frac{1}{y_{n+1}}\}_{n=0}^{\infty},$$

satisfies System (2.1). On the other hand, given a solution of System (2.1), with positive initial conditions x_0 and y_0, the sequence

$$\{x_n, \frac{\gamma + x_n}{y_n}\}_{n=1}^{\infty} = \{x_n, \frac{1}{y_{n-1}}\}_{n=1}^{\infty},$$

satisfies System (1.2).

Furthermore, the dynamics of the two systems, in terms of boundedness are equivalent provided that $\gamma \in (0,1]$ and the initial conditions are positive. Actually, when $\gamma \in (0,1]$ both systems have the boundedness characterization **(B,B)**. Indeed, given a pair of positive initial conditions, (x_0, w_0) and by choosing $y_0 = \dfrac{\gamma + x_0}{w_0}$, the component $\{x_n\}_{n=0}^{\infty}$ of the solution $\{x_n, w_n\}_{n=0}^{\infty}$ of System (1.2), is also the first component of the solution $\{x_n, y_n\}_{n=0}^{\infty}$ of System (2.1) and so in view of Theorem 2.2 is bounded. From the results established in Section 2, the component $\{y_n\}_{n=0}^{\infty}$ of the solution of System (2.1) is bounded from above and from below by positive constants, and so, it follows that $\{w_n\}_{n=0}^{\infty}$, is also bounded from above and from below by positive constants. However, the boundedness characterizations of the two systems are not the same for all positive values of γ. More precisely, as it was established in Section 2 the boundedness characterization of System (2.1) is **(B,U)**. However, the boundedness characterization of System (1.2) is **(B,B)**. The difference in the two boundedness characterizations is due to the parameter range,

$$\gamma \in (1, \infty).$$

In this case, tt can be easily shown that, every solution $\{x_n, w_n\}_{n=0}^{\infty}$ of System (1.2), with positive initial conditions, converges to (0,0) and so is bounded. The conclusion of this analysis is described by the theorem.

Theorem 3.1. *1. When*

$$\gamma \in (0, 1],$$

every solution of System (1.2) *and also every solution of System* (2.1)*, with positive initial conditions, is bounded.*

2. When

$$\gamma > 1,$$

every solution of System (1.2) *converges to (0,0) and every solution of System* (2.1) *goes to* $(0, \infty)$.

Finally, we show that the dynamics of the two maps that correspond to the two systems are topologically conjugate, provided that the initial conditions are positive and that $\gamma \in (0, \infty)$.

The following is a definition of topologically conjugate maps from $(0, \infty) \times (0, \infty)$ to $(0, \infty) \times (0, \infty)$. For another definition see [23].

Definition 3.2. Let L, U be two maps, such that:

$$L, U : (0, \infty) \times (0, \infty) \to (0, \infty) \times (0, \infty).$$

Then L and U are topologically conjugate if there exists a homeomorphism T

$$T : (0, \infty) \times (0, \infty) \to (0, \infty) \times (0, \infty)$$

such that

$$T(L(x, y)) = U(T(x, y)), \quad \text{for all} \quad x, y \in (0, \infty).$$

The homeomorphism T is called topological conjugacy.

Let us establish that the two maps that correspond to the two systems (1.2) and (2.1) are topologically conjugate. The map associated with System (2.1) is:

$$L(x, y) = (\frac{x}{y}, x + \gamma y), \quad x, y \in (0, \infty)$$

and the map associated with System (1.2) is:

$$U(x, y) = (\frac{xy}{\gamma + x}, \frac{y}{\gamma + x}), \quad x, y \in (0, \infty).$$

Define,

$$T(x, y) = (x, \frac{\gamma + x}{y}), \quad x, y \in (0, \infty).$$

Then, it can be easily verified that:

1. L and U are homeomorphisms of $(0, \infty) \times (0.\infty)$ onto itself.

2.

$$L(U(x, y)) = U(L(x, y)) = (x, y), \quad \text{for all} \quad x, y \in (0, \infty),$$

that is

$$L = U^{-1}.$$

3. T is a homeomorphism of $(0,\infty)\times(0.\infty)$ onto itself and

$$T(T(x,y)) = (x,y), \quad \text{for all} \;\; x,y,\in(0,\infty),$$

that is,

$$T = T^{-1}.$$

4.

$$T(L(x,y)) = U(T(x,y)), \quad \text{for all} \;\; x,y,\in(0,\infty).$$

Thus, L and U are topologically conjugate and the map T is the topological conjugacy.

Appendix

The Boundedness Character of the Rational System:

$$\left.\begin{aligned} x_{n+1} &= \alpha_1 + \beta_1 x_n + \gamma_1 y_n \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned}\right\}, \quad n = 0,1,\dots \tag{3.1}$$

where $\beta_1,\gamma_1 > 0$, the remaining parameters are nonnegative and the initial conditions are positive real numbers.

The boundedness characterization **(B,U)**, next to a special case of System (3.1) means, that given an arbitrary set of positive values which correspond to the parameters and the initial conditions of the special case, the first component of the corresponding solution is bounded and there exists at least one solution for which the second component is unbounded.

The boundedness characterization **(U,B)**, next to a special case of System (3.1) means, that given an arbitrary set of positive values which correspond to the parameters and the initial conditions of the special case, the second component of the corresponding solution is bounded and there exists at least one solution for which the first component is unbounded.

The boundedness characterization **(U,U)**, next to a special case of System (3.1), means that there exists at least one solution for which the first component is unbounded and also at least one solution for which the second component is unbounded.

$(25-40,1):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\alpha_2$	**(U,B)**
$(25-40,2):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\alpha_2}{y_n}$	**(U,B)**
$(25-40,3):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\alpha_2}{x_n}$	**(U,B)**
$(25-40,4):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\gamma_2y_n$	**(U,U)**
$(25-40,5):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\beta_2$	**(U,B)**
$(25-40,6):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\gamma_2y_n}{x_n}$	**(U,B)**
$(25-40,7):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\beta_2x_n$	**(U,U)**
$(25-40,8):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{x_n}{y_n}$	**(U,U)**
$(25-40,9):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\gamma_2$	**(U,B)**
$(25-40,10):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\alpha_2}{A_2+y_n}$	**(U,B)**
$(25-40,11):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\alpha_2}{1+x_n}$	**(U,B)**
$(25-40,12):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\alpha_2}{x_n+y_n}$	**(U,B)**
$(25-40,13):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\gamma_2y_n}{A_2+y_n}$	**(U,B)**
$(25-40,14):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\gamma_2y_n}{1+x_n}$	**(U,B)**
$(25-40,15):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\gamma_2y_n}{x_n+y_n}$	**(U,B)**
$(25-40,16):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\beta_2x_n}{1+y_n}$	**(U,U)**
$(25-40,17):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\beta_2x_n}{1+x_n}$	**(U,B)**
$(25-40,18):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\beta_2x_n}{x_n+y_n}$	**(U,B)**
$(25-40,19):$	$x_{n+1}=\dfrac{\beta_1x_n}{y_n},\ \ y_{n+1}=\alpha_2+\gamma_2y_n$	**(U,U)**
$(25-40,20):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\alpha_2+\gamma_2y_n}{y_n}$	**(U,B)**
$(25-40,21):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\dfrac{\alpha_2+\gamma_2y_n}{x_n}$	**(U,B)**
$(25-40,22):$	$x_{n+1}=\alpha_1+\beta_1x_n+\gamma_1y_n,\ \ y_{n+1}=\alpha_2+x_n$	**(U,U)**

$(25-40,23):$	$x_{n+1}=\dfrac{\beta_1 x_n}{y_n},\ \ y_{n+1}=\dfrac{\alpha_2+x_n}{y_n}$	**(U,U)**
$(25-40,24):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\alpha_2+x_n}{x_n}$	**(U,B)**
$(25-40,25):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\beta_2 x_n+\gamma_2 y_n$	**(U,U)**
$(25-40,26):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{x_n+\gamma_2 y_n}{y_n}$	**(U,U)**
$(25-40,27):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\beta_2 x_n+\gamma_2 y_n}{x_n}$	**(U,B)**
$(25-40,28):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\alpha_2+\gamma_2 y_n}{A_2+y_n}$	**(U,B)**
$(25-40,29):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\alpha_2+\gamma_2 y_n}{1+x_n}$	**(U,B)**
$(25-40,30):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\alpha_2+\gamma_2 y_n}{x_n+y_n}$	**(U,B)**
$(25-40,31):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\alpha_2+x_n}{A_2+y_n}$	**(U,U)**
$(25-40,32):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\alpha_2+x_n}{A_2+x_n}$	**(U,B)**
$(25-40,33):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\alpha_2+\beta_2 x_n}{x_n+y_n}$	**(U,B)**
$(25-40,34):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\beta_2 x_n+\gamma_2 y_n}{A_2+y_n}$	**(U,U)**
$(25-40,35):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\beta_2 x_n+\gamma_2 y_n}{1+x_n}$	**(U,B)**
$(25-40,36):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\beta_2 x_n+\gamma_2 y_n}{x_n+y_n}$	**(U,B)**
$(25-40,37):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\alpha_2}{A_2+x_n+y_n}$	**(U,B)**
$(25-40,38):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\gamma_2 y_n}{A_2+x_n+y_n}$	**(U,B)**
$(25-40,39):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\beta_2 x_n}{A_2+x_n+y_n}$	**(U,B)**
$(25-40,40):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\alpha_2+x_n+\gamma_2 y_n$	**(U,U)**
$(25-40,41):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\alpha_2+x_n+\gamma_2 y_n}{y_n}$	**(U,U)**
$(25-40,42):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\alpha_2+x_n+\gamma_2 y_n}{x_n}$	**(U,B)**
$(25-40,43):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\alpha_2+\gamma_2 y_n}{A_2+x_n+y_n}$	**(U,B)**
$(25-40,44):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\alpha_2+x_n}{A_2+x_n+y_n}$	**(U,B)**
$(25-40,45):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,\ \ y_{n+1}=\dfrac{\beta_2 x_n+\gamma_2 y_n}{A_2+x_n+y_n}$	**(U,B)**

$(25-40,46):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,$	$y_{n+1}=\dfrac{\alpha_2+x_n+\gamma_2 y_n}{A_2+y_n}$	**(U,U)**
$(25-40,47):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,$	$y_{n+1}=\dfrac{\alpha_2+x_n+\gamma_2 y_n}{A_2+x_n}$	**(U,B)**
$(25-40,48):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,$	$y_{n+1}=\dfrac{\alpha_2+\beta_2 x_n+\gamma_2 y_n}{x_n+y_n}$	**(U,B)**
$(25-40,49):$	$x_{n+1}=\alpha_1+\beta_1 x_n+\gamma_1 y_n,$	$y_{n+1}=\dfrac{\alpha_2+x_n+\gamma_2 y_n}{A_2+x_n+y_n}$	**(U,B)**

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