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Existence and Approximation of Solutions to Dynamic Equations on Time Scales

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Abstract

In this paper, we develop the method of upper and lower solutions and the generalized quasilinearization technique for second order nonlinear 3-point dynamic equations on time scales

$$x^{\Delta\Delta}(t) = f(t, x^\sigma),\ t \in [0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T},$$
$$x(0) = 0,\ x(\sigma^2(1)) = g(x(\eta)),\ \eta \in (0,1)_{\mathbb{T}}.$$

Existence and uniqueness of solutions is established via the method of upper and lower solutions. Approximation of solution uses the generalized quasilinearization method. We show that there exists a monotone sequence of solutions of linear problems which converges uniformly and quadratically to a solution of the original problem.

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1 Introduction

Traditionally, researchers have assumed that a dynamical process is only continuous or only discrete. However, many of them contain both continuous and discrete elements simultaneously. Thus, traditional mathematical modeling techniques, such as differential or difference equations, provide a limited understanding of these types of models [17]. A simple example of this hybrid continuous-discrete behavior appears in many natural

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populations: for example, insects that lay their eggs at the end of the season just before the generation dies out, with the eggs laying dormant, hatching at the start of the next season giving rise to a new generation. See Ref. [11] for different examples of species which follow this behavior. Stefan Hilger [13] introduced the notion of time scales in 1990 in order to unify the theory of continuous and discrete calculus. An excellent resource with an extensive bibliography on time scales was produced by Martin Bohner and Allan C. Peterson [8, 9]. Recently, existence theory for positive solutions of boundary value problems (BVPs) on time scale have received much attention from many authors, see for example, [1, 4, 7, 12, 14, 20] and the references therein for two-point (BVPs) and [3, 10, 15] for three-point BVPs. However, very little work have been done on the method of upper and lower solutions and the quasilinearization technique [2, 5, 6, 17]. In [2], the method of quasilinearization is developed for some two point BVPs and in [6], some periodic boundary value problem is studied, while in [5], the method of quasilinearization is developed for 3-point BVPs with linear boundary conditions.

In this paper, we develop the method of upper and lower solutions for the existence of solution and the method of generalized quasilinearization to approximate the solution of a class of nonlinear three-point boundary value problem corresponding to dynamic equation on time scales and nonlinear boundary conditions of the type

$$\begin{aligned} &x^{\Delta\Delta}(t) = f(t, x^{\sigma}(t)),\ t \in [0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T}, \\ &x(0) = 0,\ x(\sigma^2(1)) = g(x(\eta)), \end{aligned} \tag{1.1}$$

where $\eta_i \in (0,1)_{\mathbb{T}}$ and t is from a so-called time scale $\mathbb{T}$ (which is an arbitrary closed subset of $\mathbb{R}$). We assume that $\mathbb{T}$ has a topology that it inherits from the standard topology on $\mathbb{R}$.

We approximate the solution by the method of generalized quasilinearization [2, 5, 6, 16, 18, 19] and show that under suitable conditions on f, there exists a monotone sequence of solutions of *linear* problems that converges uniformly and quadratically to a unique solution of the original nonlinear problem.

For $a < b$, points in $\mathbb{T}$, the time scale interval is defined by $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. If $\sigma(t) > t$, t is said to be right-scattered and if $\sigma(t) = t$, t is said to be right-dense. If $\rho(t) < t$, t is said to be left-scattered and if $\rho(t) = t$, t is said to be left-dense.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided it is continuous at all right-dense points of $\mathbb{T}$ and its left-sided limit exists at left-dense points of $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be ld-continuous provided it is continuous at all left-dense points of $\mathbb{T}$ and its right-sided limit exists at right-dense points of $\mathbb{T}$. Define $\mathbb{T}^{\mathrm{k}} = \mathbb{T} - \{m\}$ if $\mathbb{T}$ has a left scattered maximum at m; otherwise $\mathbb{T}^{\mathrm{k}} = \mathbb{T}$.

Definition 1.1. Define $C^2_{rd}([0, \sigma^2(1)]_{\mathbb{T}})$ to be the set of all functions $y : \mathbb{T} \to \mathbb{R}$ such that

$$C^2_{rd}([0,\sigma^2(1)]_{\mathbb{T}}) = \{y : y,\ y^{\Delta} \in C([0,\sigma^2(1)]_{\mathbb{T}}) \text{ and } y^{\Delta\Delta} \in C_{rd}([0,1]_{\mathbb{T}})\}.$$

A solution of (1.1) is a function $y \in C^2_{rd}([0,\sigma^2(1)]_{\mathbb{T}})$ which satisfies (1.1) for each $t \in [0,1]_{\mathbb{T}}$.

Let us denote

$$\begin{aligned} C_{rd}[[0,1]_{\mathbb{T}} \times \mathbb{R}] = \{y(t,x) : & y(\cdot,x) \text{ is } C_{rd}[0,1]_{\mathbb{T}} \text{ for every } x \in \mathbb{R} \text{ and } y(t,\cdot) \\ & \text{is continuous in } \mathbb{R} \text{ uniformly at } t \in [0,1]_{\mathbb{T}}\} \end{aligned}$$

and

$$\begin{aligned} C^2_{rd}([0,1]_{\mathbb{T}} \times \mathbb{R}) = \{ & y(t,x) : y(\cdot,x),\ y_x(\cdot,x),\ y_x(\cdot,x) \in C_{rd}([0,1]_{\mathbb{T}}) \text{ for every } x \in \mathbb{R} \\ & \text{and } y(t,\cdot),\ y_x(t,\cdot),\ y_{xx}(t,\cdot) \in C(\mathbb{R}) \text{ uniformly at } t \in [0,1]_{\mathbb{T}}\}. \end{aligned}$$

2 Upper and Lower Solutions Method

We write the BVP (1.1) as an equivalent perturbed Δ-integral equation

$$x(t) = \int_0^{\sigma(1)} G(t,s) f(s, x^{\sigma}(s))\Delta s + \frac{g(x(\eta))}{\sigma^2(1)} t,\ t \in [0,\sigma^2(1)]_{\mathbb{T}}, \tag{2.1}$$

where $G(t,s)$ is the Green's function corresponding to the homogeneous problem

$$y^{\Delta\Delta}(t) = 0,\ t \in [0,1]_{\mathbb{T}},\ y(0) = 0,\ y(\sigma^2(1)) = 0$$

and is given by

$$G(t,s) = -\frac{1}{\sigma^2(1)} \begin{cases} \sigma(s)(\sigma^2(1) - t),\ \sigma(s) \le t \\ t(\sigma^2(1) - \sigma(s)),\ t \le s. \end{cases}$$

Notice that $G(t,s) < 0$ on $(0,\sigma^2(1))_{\mathbb{T}} \times (0,\sigma(1))_{\mathbb{T}}$ and is rd-continuous. Define an operator $T : C_{rd}[0,\sigma^2(1)]_{\mathbb{T}} \to C_{rd}[0,\sigma^2(1)]_{\mathbb{T}}$ by

$$(Tx)(t) = \int_0^{\sigma(1)} G(t,s) f(s, x^{\sigma}(s))\Delta s + \frac{g(x(\eta))}{\sigma^2(1)} t,\ t \in [0,\sigma^2(1)]_{\mathbb{T}}. \tag{2.2}$$

By a solution of (2.1), we mean a solution of the operator equation

$$(I - T)x = 0, \text{ that is, a fixed point of } T, \tag{2.3}$$

where I is the identity map. If $f \in C_{rd}[[0,1]_{\mathbb{T}} \times \mathbb{R}]$ and is bounded on $[0,1]_{\mathbb{T}} \times \mathbb{R}$ and $g(x(\eta))$ exists and is finite, then by Arzela–Ascoli theorem T is compact and the Schauder's fixed point theorem yields a fixed point of T. We discuss the case when f is not necessarily bounded on $[0,\sigma^2(1)]_{\mathbb{T}} \times \mathbb{R}$. Firstly, recall the concept of upper and lower solutions for the BVP (1.1).

Definition 2.1. We say that $\alpha \in C^2_{rd}[0, \sigma^2(1)]_{\mathbb{T}}$ is a lower solution of the BVP (1.1), if

$$\begin{aligned}&\alpha^{\Delta\Delta}(t) \geq f(t, \alpha^\sigma(t)), \quad t \in [0,1]_{\mathbb{T}}\\ &\alpha(0) \leq 0,\ \alpha(\sigma^2(1)) \leq g(\alpha(\eta)).\end{aligned}$$

Similarly, $\beta \in C^2_{rd}[0, \sigma^2(1)]_{\mathbb{T}}$ is an upper solution of the BVP (1.1) if

$$\begin{aligned}&\beta^{\Delta\Delta}(t) \leq f(t, \beta^\sigma(t)), \quad t \in [0,1]_{\mathbb{T}}\\ &\beta(0) \geq 0,\ \beta(\sigma^2(1)) \geq g(\beta(\eta)).\end{aligned}$$

In the following theorem, we develop a comparison result that leads to a uniqueness of solution for the 3-point BVP (1.1).

Theorem 2.2 (Comparison result)**.** *Assume that α, β are lower and upper solutions of the boundary value problem* (1.1) *such that $\alpha(\eta) < \beta(\eta)$. If $f(t,x) \in C_{rd}[[0,1]_{\mathbb{T}} \times \mathbb{R}]$ and is strictly increasing in x for each $t \in [0, \sigma^2(1)]_{\mathbb{T}}$ and the boundary functional g is nondecreasing. Then $\alpha \leq \beta$ on $[0, \sigma^2(1)]_{\mathbb{T}}$.*

Proof. Define $v(t) = \alpha(t) - \beta(t)$, $t \in [0, \sigma^2(1)]_{\mathbb{T}}$. Clearly $v \in C^2_{rd}[0, \sigma^2(1)]_{\mathbb{T}}$ and the BCs together with the nondecreasing property of g imply that

$$v(0) \leq 0,\ v(\sigma^2(1)) \leq g(\alpha(\eta)) - g(\beta(\eta)) \leq 0. \tag{2.4}$$

Assume that the conclusion of the theorem is not true. Then, v has a positive maximum at some $t_0 \in [0, \sigma^2(1)]_{\mathbb{T}}$. Clearly, $t_0 \in (0, \sigma^2(1))_{\mathbb{T}}$. Moreover, the point t_0 is not simultaneously left-dense and right-scattered, see for example [20]. Hence by lemma 1 of [20] or lemma 2 of [10],

$$v^{\Delta\Delta}(\rho(t_0)) \leq 0.$$

On the other hand, using the definitions of lower and upper solutions, we obtain

$$v^{\Delta\Delta}(\rho(t_0)) = \alpha^{\Delta\Delta}(\rho(t_0)) - \beta^{\Delta\Delta}(\rho(t_0)) \geq f(\rho(t_0), \alpha^\sigma(\rho(t_0))) - f(\rho(t_0), \beta^\sigma(\rho(t_0))).$$

Since t_0 is not simultaneously left-dense and right-scattered, it is either left-scattered and right-scattered or left-dense and right-dense or left-scattered and right-dense. In either case $\sigma(\rho(t_0)) = t_0$. Using the increasing property of $f(t,x)$ in x, we obtain $v^{\Delta\Delta}(\rho(t_0)) > 0$, a contradiction. Hence $v(t) \leq 0$ on $[0, \sigma^2(1)]_{\mathbb{T}}$. □

Now we establish existence of solution of the BVP (1.1) in the presence of well ordered lower and upper solutions. We do not require f to be increasing. However, we require continuity of g on $[\alpha(\eta), \beta(\eta)]$.

Theorem 2.3. *Assume that α, β are lower and upper solutions of the BVP* (1.1) *such that $\alpha \leq \beta$ on $[0, \sigma^2(1)]_{\mathbb{T}}$. Assume that $f(t,x) \in C_{rd}[[0,1]_{\mathbb{T}} \times \mathbb{R}]$ and g is continuous and nondecreasing on $[\alpha(\eta), \beta(\eta)]$, then the BVP* (1.1) *has a solution x such that $\alpha \leq x \leq \beta$ on $[0, \sigma^2(1)]_{\mathbb{T}}$.*

Proof. Consider the modifications F of f and G of g as follows:

$$F(t,x) = \begin{cases} f(t,\beta^\sigma(t)) + \dfrac{x(t)-\beta^\sigma(t)}{1+|x(t)-\beta^\sigma(t)|}, & \text{if } x \geq \beta^\sigma(t), \\ f(t,x), & \text{if } \alpha^\sigma(t) \leq x \leq \beta^\sigma(t), \\ f(t,\alpha^\sigma(t)) + \dfrac{x-\alpha^\sigma(t)}{1+|\alpha^\sigma(t)-x|}, & \text{if } x \leq \alpha^\sigma(t). \end{cases} \tag{2.5}$$

$$G(x) = \begin{cases} g(\beta(\eta)), & \text{if } x \geq \beta(\eta), \\ g(x), & \text{if } \alpha(\eta) \leq x \leq \beta(\eta), \\ g(\alpha(\eta)), & \text{if } x \leq \alpha(\eta). \end{cases} \tag{2.6}$$

Clearly, $F \in C_{rd}[[0,1]_{\mathbb{T}} \times \mathbb{R}]$ and bounded on $[0,1]_{\mathbb{T}} \times \mathbb{R}$. Also, G is continuous and bounded on $\mathbb{R}$. Hence the modified problem

$$\begin{aligned} &x^{\Delta\Delta}(t) = F(t, x^\sigma(t)),\ t \in [0,1]_{\mathbb{T}}, \\ &x(0) = 0,\ x(\sigma^2(1)) = G(x(\eta)), \end{aligned} \tag{2.7}$$

has a solution. Notice that any solution x of (2.7) such that

$$\alpha(t) \leq x(t) \leq \beta(t),\ t \in [0,\sigma^2(1)]_{\mathbb{T}}, \tag{2.8}$$

is a solution of (1.1). We only need to show that any solution x of (2.7) in fact does satisfy (2.8). By the definition of F and G, we have

$$\begin{aligned} &F(t,\alpha^\sigma(t)) = f(t,\alpha^\sigma(t)) \leq \alpha^{\Delta\Delta}(t),\ t \in [0,1]_{\mathbb{T}} \\ &\alpha(0) \leq 0,\ G(\alpha(\eta)) = g(\alpha(\eta)) \geq \alpha(\sigma^2(1)) \text{ and} \\ &F(t,\beta^\sigma(t)) = f(t,\beta^\sigma(t)) \geq \beta^{\Delta\Delta}(t),\ t \in [0,1]_{\mathbb{T}}, \\ &\beta(0) \geq 0,\ G(\beta(\eta)) = g(\beta(\eta)) \leq \beta(\sigma^2(1)), \end{aligned}$$

which imply that α and β are lower and upper solutions of the BVP (2.7).

Now, set $v(t) = \alpha(t) - x(t)$, $t \in [0,\sigma^2(1)]_{\mathbb{T}}$, where x is a solution of (2.7). Then, $v \in C^2_{rd}[0,\sigma^2(1)]_{\mathbb{T}}$ and the BCs imply that

$$v(0) \leq 0,\ v(\sigma^2(1)) \leq g(\alpha(\eta)) - G(x(\eta)). \tag{2.9}$$

It is required to show that $v(t) \leq 0$ on $[0,\sigma^2(1)]_{\mathbb{T}}$. Assume that it is not true and

$$\max\{v(t) : t \in [0,\sigma^2(1)]_{\mathrm{T}}\} = v(t_0) > 0.$$

Clearly, $t_0 \neq 0$. Moreover, t_0 is not simultaneously left-dense and right-scattered. Consequently,

$$v^{\Delta\Delta}(\rho(t_0)) \leq 0 \text{ and } x(t_0) < \alpha(t_0) = \alpha^\sigma(\rho(t_0)).$$

On the other hand, using the definitions of lower solution and that of the modified function, we obtain

$$\begin{aligned} v^{\Delta\Delta}(\rho(t_0)) = \alpha^{\Delta\Delta}(\rho(t_0)) - x^{\Delta\Delta}(\rho(t_0)) &\geq f(\rho(t_0), \alpha^{\sigma}(\rho(t_0))) - F(\rho(t_0), x^{\sigma}(\rho(t_0))) \\ &= f(\rho(t_0), \alpha^{\sigma}(\rho(t_0))) - f(\rho(t_0), \alpha^{\sigma}(\rho(t_0))) + \frac{v^{\sigma}(\rho(t_0))}{1+|v^{\sigma}(\rho(t_0))|} > 0, \end{aligned}$$

a contradiction. Hence $v(t)$ has no positive local maximum. Thus, $t_0 = \sigma^2(1)$ that implies $v(\sigma^2(1)) > 0$. However, if $x(\eta) < \alpha(\eta)$, then $G(x(\eta)) = g(\alpha(\eta))$ and in view of (2.9), we have $v(\sigma^2(1)) \leq 0$, a contradiction. If $x(\eta) > \beta(\eta)$, then using the nondecreasing property of g, we obtain $G(x(\eta)) = g(\beta(\eta)) \geq g(\alpha(\eta))$ that again leads to $v(\sigma^2(1)) \leq 0$, a contradiction. If $\alpha(\eta) \leq x(\eta) \leq \beta(\eta)$, then using the nondecreasing property of g, we obtain $G(x(\eta)) = g(x(\eta)) \geq g(\alpha(\eta))$ that again leads to $v(\sigma^2(1)) \leq 0$, a contradiction. Hence, $t_0 \neq \sigma^2(1)$ and consequently $v(t) \leq 0$ on $[0, \sigma^2(1)]_{\mathbb{T}}$. Similarly, we can show that $x(t) \leq \beta(t)$ on $[0, \sigma^2(1)]_{\mathbb{T}}$. □

Example 2.4. For the three point BVP

$$\begin{aligned} &x^{\Delta\Delta}(t) = f(t, x^{\sigma}(t)) = x^2(\sigma(t)) - t^2, \ t \in [0,1]_{\mathbb{T}}, \\ &x(0) = 0, \ x(\sigma^2(1)) = \delta x(\eta), \end{aligned} \tag{2.10}$$

where $0 < \delta\eta < 1$.

$\alpha = 0$ and $\beta = t$, hence the BVP (2.10) has a solution x such that

$$0 \leq x(t) \leq t \text{ on } [0, \sigma^2(1)]_{\mathbb{T}}.$$

Corollary 2.5. *Under the hypotheses of Theorem 2.2, the solution of the BVP* (1.1) *is unique.*

3 Generalized Quasilinearization Technique

We develop the approximation technique and show that under suitable conditions on f and g, there exists a bounded monotone sequence of solutions of linear problems that converges uniformly and quadratically to a solution of the nonlinear original problem. If $\frac{\partial^2}{\partial x^2} f(t,x) \in C[[0,1]_{\mathbb{T}} \times \mathbb{R}]$ and is bounded on $[0, \sigma^2(1)]_{\mathbb{T}} \times [\bar{\alpha}, \bar{\beta}]$, where

$$\bar{\alpha} = \min\{\alpha(t), t \in [0, \sigma^2(1)]_{\mathbb{T}}\}, \bar{\beta} = \max\{\beta(t), t \in [0, \sigma^2(1)]_{\mathbb{T}}\},$$

there always exists a function Φ such that

$$\frac{\partial^2}{\partial x^2}[f(t,x) + \Phi(t,x)] \leq 0 \text{ on } [0,1]_{\mathbb{T}} \times [\bar{\alpha}, \bar{\beta}], \tag{3.1}$$

where $\Phi \in C^2([0,\sigma^2(1)]_{\mathbb{T}} \times \mathbb{R})$ and is such that $\dfrac{\partial^2}{\partial x^2}\Phi(t,x) \le 0$ on $[0,\sigma^2(1)]_{\mathbb{T}} \times [\bar{\alpha},\bar{\beta}]$. Define $F : [0,1]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ by $F(t,x) = f(t,x) + \Phi(t,x)$. Note that

$$F \in C^2_{rd}([0,\sigma^2(1)]_{\mathbb{T}} \times \mathbb{R}) \text{ and } \frac{\partial^2}{\partial x^2}F(t,x) \le 0 \text{ on } [0,1]_{\mathbb{T}} \times [\bar{\alpha},\bar{\beta}]. \tag{3.2}$$

Theorem 3.1. *Assume that*

(A$_1$) α, β *are lower and upper solutions of* (1.1) *such that* $\alpha \le \beta$ *on* $[0,\sigma^2(1)]_{\mathbb{T}}$.

(A$_2$) $f \in C^2_{rd}([0,1]_{\mathbb{T}} \times \mathbb{R})$ *and* f *is increasing in* x *for each* $t \in [0,1]_{\mathbb{T}}$.

(A$_3$) $g \in C^2([\alpha(\eta),\beta(\eta)])$ *and is nondecreasing. Further,*

$$\max\{g'(x) : x \in [\alpha(\eta),\beta(\eta)]\} < \frac{\sigma^2(1)}{\eta} \text{ and } g'' \ge 0 \text{ on } [\alpha(\eta),\beta(\eta)].$$

Then, there exists a monotone sequence $\{w_n\}$ *of solutions of* linear problems *converging uniformly and quadratically to a unique solution of the BVP* (1.1).

Proof. The conditions (A_1), (A_2) and (A_3) ensure the existence of a unique solution x of the BVP (1.1) such that

$$\alpha(t) \le x(t) \le \beta(t),\ t \in [0,\sigma^2(1)]_{\mathbb{T}}.$$

For $t \in [0,1]_{\mathbb{T}}$, using (3.2), we obtain

$$f(t,x) \le f(t,y) + F_x(t,y)(x-y) - [\Phi(t,x) - \Phi(t,y)], \tag{3.3}$$

where $x,\ y \in [\bar{\alpha},\bar{\beta}]$. The mean value theorem and the fact that Φ_x is non-increasing in x on $[\bar{\alpha},\bar{\beta}]$ for each $t \in [0,1]_{\mathbb{T}}$, yields

$$\Phi(t,x) - \Phi(t,y) = \Phi_x(t,c)(x-y) \ge \Phi_x(t,\bar{\beta})(x-y) \text{ for } x \ge y, \tag{3.4}$$

where $x,\ y \in [\bar{\alpha},\bar{\beta}]$ such that $y \le c \le x$. Substituting in (3.3), we have

$$f(t,x) \le f(t,y) + [F_x(t,y) - \Phi_x(t,\bar{\beta})](x-y), \text{ for } x \ge y \tag{3.5}$$

on $[0,1]_{\mathbb{T}} \times [\bar{\alpha},\bar{\beta}]$. Define $F^* : [0,1]_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}$ by

$$F^*(t,x,y) = f(t,y) + [F_x(t,y) - \Phi_x(t,\bar{\beta})](x-y). \tag{3.6}$$

We note that $F^*(t,.,.)$ is continuous for each $t \in [0,1]_{\mathbb{T}}$ and $F^*(.,x,y)$ is rd-continuous for each $(x,y) \in \mathbb{R}^2$. Moreover, F^* satisfies the following relations on $[0,1]_{\mathbb{T}} \times [\bar{\alpha},\bar{\beta}]$:

$$F^*_x(t,x,y) = F_x(t,y) - \Phi_x(t,\bar{\beta}) \ge F_x(t,y) - \Phi_x(t,y) = f_x(t,y) \ge 0 \text{ and}$$

$$\begin{cases} f(t,x) \le F^*(t,x,y), \text{for } x \ge y, \\ f(t,x) = F^*(t,x,x). \end{cases} \tag{3.7}$$

Using (A_3), we obtain

$$g(x) \ge g(y) + g'(y)(x-y), \text{ for } x,\, y \in [\alpha(\eta), \beta(\eta)]. \tag{3.8}$$

Define $G : \mathbb{R}^2 \to \mathbb{R}$ by

$$G(x,y) = g(y) + g'(y)(x-y).$$

Clearly G is continuous and bounded on $[\alpha(\eta), \beta(\eta)]$ and satisfies

$$\begin{cases} g(x) \ge G(x,y) \text{ on } [\alpha(\eta), \beta(\eta)], \\ g(x) = G(x,x). \end{cases} \tag{3.9}$$

Now, we develop the iterative scheme to approximate the solution. As an initial approximation, we choose $w_0 = \alpha$ and consider the linear problem

$$\begin{aligned} x^{\Delta\Delta}(t) &= F^*(t, x^\sigma(t), w_0^\sigma(t)),\, t \in [0,1]_{\mathbb{T}} \\ x(0) = 0,\, x(\sigma^2(1)) &= G(x(\eta), w_0(\eta)). \end{aligned} \tag{3.10}$$

Using (3.7), (3.9) and the definition of lower and upper solutions, we get

$$\begin{aligned} &F^*(t, w_0^\sigma(t), w_0^\sigma(t)) = f(t, w_0^\sigma(t)) \le w_0^{\Delta\Delta}(t),\, t \in [0,1]_{\mathbb{T}}, \\ &w_0(0) \le 0,\, G(w_0(\eta), w_0(\eta)) = g(w_0(\eta)) \ge w_0(\sigma^2(1)), \\ &F^*(t, \beta^\sigma(t), w_0^\sigma(t)) \ge f(t, \beta^\sigma(t)) \ge \beta^{\Delta\Delta}(t),\, t \in [0,1]_{\mathbb{T}}, \\ &\beta(0) \le 0,\, G(\beta(\eta), w_0(\eta)) \le g(\beta(\eta)) \le \beta(\sigma^2(1)) \end{aligned}$$

which imply that w_0 and β are lower and upper solutions of (3.10) respectively. Hence by Theorems 2.3, 2.2, there exists a unique solution $w_1 \in C^2_{rd}[0, \sigma^2(1)]_{\mathbb{T}}$ of (3.10) such that

$$w_0(t) \le w_1(t) \le \beta(t) \text{ on } [0, \sigma^2(1)]_{\mathbb{T}}.$$

Using (3.7), (3.9) and the fact that w_1 is a solution of (3.10), we obtain

$$\begin{aligned} w_1^{\Delta\Delta}(t) &= F^*(t, w_1^\sigma(t), w_0^\sigma(t)) \ge f(t, w_1^\sigma(t)),\, t \in [0,1]_{\mathbb{T}} \\ w_1(0) = 0,\, w_1(\sigma^2(1)) &= G(w_1(\eta), w_0(\eta)) \le g(w_1(\eta)), \end{aligned} \tag{3.11}$$

which implies that w_1 is a lower solution of (1.1). Similarly, in view of (A_1), (3.7), (3.9) and (3.11), we can show that w_1 and β are lower and upper solutions of

$$\begin{aligned} x^{\Delta\Delta}(t) &= F^*(t, x^\sigma(t), w_1^\sigma(t)),\, t \in [0,1]_{\mathbb{T}} \\ x(0) = 0,\, x(\sigma^2(1)) &= G(x(\eta), w_1(\eta)). \end{aligned} \tag{3.12}$$

Hence by Theorems 2.3,2.2, there exists a unique solution $w_2 \in C^2_{rd}[0, \sigma^2(1)]_{\mathrm{T}}$ of (3.12) such that

$$w_1(t) \leq w_2(t) \leq \beta(t) \text{ on } [0, \sigma^2(1)]_{\mathbb{T}}.$$

Continuing in the above fashion, we obtain a bounded monotone sequence $\{w_n\}$ of solutions of linear problems satisfying

$$w_0(t) \leq w_1(t) \leq w_2(t) \leq w_3(t) \leq ... \leq w_n(t) \leq \beta(t) \text{ on } [0, \sigma^2(1)]_{\mathbb{T}}, \tag{3.13}$$

where the element w_n of the sequence is a solution of the linear problem

$$x^{\Delta\Delta}(t) = F^*(t, x^\sigma(t), w^\sigma_{n-1}(t)),\ t \in [0,1]_{\mathbb{T}} x(0) = 0,\ x(\sigma^2(1)) = G(x(\eta), w_{n-1}(\eta)),$$

and is given by

$$\begin{aligned} w_n(t) = \int_0^{\sigma(1)} & G(t,s) F^*(s, w^\sigma_n(s), w^\sigma_{n-1}(s))\Delta s \\ & + \frac{G(w_n(\eta), w_{n-1}(\eta))}{\sigma^2(1)} t, \quad t \in [0, \sigma^2(1)]_{\mathbb{T}}. \end{aligned} \tag{3.14}$$

By standard arguments as in [6], the sequence converges to some function w uniformly. Passing to the limit as $n \to \infty$, we obtain

$$\begin{aligned} w(t) &= \int_0^{\sigma(1)} G(t,s) F^*(s, w^\sigma(s), w^\sigma(s))\Delta s + \frac{G(w(\eta), w(\eta))}{\sigma^2(1)} t \\ &= \int_0^{\sigma(1)} G(t,s) f(s, w^\sigma(s))\Delta s + \frac{g(w(\eta))}{\sigma^2(1)} t,\ t \in [0, \sigma^2(1)]_{\mathbb{T}}, \end{aligned}$$

that is, w is a solution of (1.1).

Now, we show that the convergence is quadratic. Set $v_{n+1}(t) = x(t) - w_{n+1}(t),\ t \in [0, \sigma^2(1)]_{\mathbb{T}}$, where x is a solution of (1.1). Then, $v_{n+1}(t) \geq 0$ on $[0, \sigma^2(1)]_{\mathbb{T}}$ and the boundary conditions imply that $v_{n+1}(0) = 0$ and

$$v_{n+1}(\sigma^2(1)) = g(x(\eta)) - G(w_n(\eta), w_{n-1}(\eta)) \geq g(x(\eta)) - g(w_n(\eta)) = g'(\xi) v_n(\eta),$$

where $w_n(\eta) < \xi < x(\eta)$. Now, for $t \in [0,1]_{\mathbb{T}}$, using the definitions of F and F^*, we obtain

$$\begin{aligned} v_n^{\Delta\Delta}(t) &= f(t, x^\sigma(t)) - F^*(s, w^\sigma_n(t), w^\sigma_{n-1}(t)) = [F(t, x^\sigma(t)) - \Phi(t, x^\sigma(t))] \\ &\quad - [f(t, w^\sigma_{n-1}(t)) + (F_x(t, w^\sigma_{n-1}(t)) - \Phi_x(t, \bar{\beta}))(w^\sigma_n(t) - w^\sigma_{n-1}(t))] \\ &= [F(t, x^\sigma(t)) - F(t, w^\sigma_{n-1}(t)) - F_x(t, w^\sigma_{n-1}(t))(w^\sigma_n(t) - w^\sigma_{n-1}(t))] \\ &\quad - [\Phi(t, x^\sigma(t)) - \Phi(t, w^\sigma_{n-1}(t)) - \Phi_x(t, \bar{\beta}))(w^\sigma_n(t) - w^\sigma_{n-1}(t))]. \end{aligned} \tag{3.15}$$

Using the mean value theorem repeatedly, and the fact that $\Phi_{xx} \le 0$ on $[0,1]_{\mathbb{T}} \times [\bar{\alpha}, \bar{\beta}]$, we obtain $\Phi(t, x^\sigma(t)) - \Phi(t, w^\sigma_{n-1}(t)) \le \Phi_x(t, w^\sigma_{n-1}(t))(x^\sigma(t) - w^\sigma_{n-1}(t))$ and

$$\begin{aligned}
&F(t, x^\sigma(t)) - F(t, w^\sigma_{n-1}(t)) - F_x(t, w^\sigma_{n-1}(t))(w^\sigma_n(t) - w^\sigma_{n-1}(t)) \\
&= F_x(t, w^\sigma_{n-1}(t))(x^\sigma(t) - w^\sigma_{n-1}(t)) + \frac{F_{xx}(t,\xi_1)}{2}(x^\sigma(t) - w^\sigma_{n-1}(t))^2 \\
&- F_x(t, w^\sigma_{n-1}(t))(w^\sigma_n(t) - w^\sigma_{n-1}(t)) \\
&= F_x(t, w^\sigma_{n-1}(t))(x^\sigma(t) - w^\sigma_n(t)) + \frac{F_{xx}(t,\xi_1)}{2}(x^\sigma(t) - w^\sigma_{n-1}(t))^2 \\
&\ge F_x(t, w^\sigma_{n-1}(t))(x^\sigma(t) - w^\sigma_n(t)) - d\|v_{n-1}\|^2,
\end{aligned}$$

where $w^\sigma_{n-1}(t) \le \xi_1 \le x^\sigma(t)$, $d = \max\{\frac{|F_{xx}(t,x)|}{2} : (t,x) \in [0,1]_{\mathbb{T}} \times [\bar{\alpha}, \bar{\beta}]\}$ and $\|v\| = \max\{v(t) : t \in [0, \sigma^2(1)]_{\mathbb{T}}\}$. Hence the equation (3.15) can be rewritten as

$$\begin{aligned}
v_n^{\Delta\Delta}(t) &\ge F_x(t, w^\sigma_{n-1}(t))(x^\sigma(t) - w^\sigma_n(t)) - d\|v_{n-1}\|^2 \\
&- \Phi_x(t, w^\sigma_{n-1}(t))(x^\sigma(t) - w^\sigma_{n-1}(t)) + \Phi_x(t, \bar{\beta}))(w^\sigma_n(t) - w^\sigma_{n-1}(t)) \\
&= f_x(t, w^\sigma_{n-1}(t))(x^\sigma(t) - w^\sigma_n(t)) - d\|v_{n-1}\|^2 + [\Phi_x(t,\bar{\beta}) - \Phi_x(t, w^\sigma_{n-1}(t))]\times \\
&(w^\sigma_n(t) - w^\sigma_{n-1}(t)) = f_x(t, w^\sigma_{n-1})(x^\sigma(t) - w^\sigma_n(t)) - d\|v_{n-1}\|^2 + \\
&\Phi_{xx}(t,\xi_2)(\bar{\beta} - w^\sigma_{n-1}(t))(w^\sigma_n(t) - w^\sigma_{n-1}(t)) \ge f_x(t, w^\sigma_{n-1})(x^\sigma(t) - w^\sigma_n(t)) \\
&- d\|v_{n-1}\|^2 + \Phi_{xx}(t,\xi_2)(\bar{\beta} - w^\sigma_{n-1}(t))(x^\sigma_n(t) - w^\sigma_{n-1}(t)) \\
&\ge -d\|v_{n-1}\|^2 - d_1|\bar{\beta} - w^\sigma_{n-1}(t)||x^\sigma(t) - w^\sigma_{n-1}(t)|,\ t \in [0,1]_{\mathbb{T}}
\end{aligned} \tag{3.16}$$

where $w^\sigma_{n-1}(t) \le \xi_2 \le w^\sigma_n(t)$, $d_1 = \max\{|\Phi_{xx}| : (t,x) \in [0,1]_{\mathbb{T}} \times [\bar{\alpha}, \bar{\beta}]\}$ and we used the fact that $f_x \ge 0$ on $[0,1]_{\mathbb{T}} \times [\bar{\alpha}, \bar{\beta}]$. Choose $r > 1$ such that

$$|\beta^\sigma(t) - w^\sigma_{n-1}(t)| \le r|x^\sigma(t) - w^\sigma_{n-1}(t)| \text{ on } [0,1]_{\mathbb{T}},$$

we obtain

$$v_n^{\Delta\Delta}(t) \ge -d_2\|v_{n-1}\|^2,\ t \in [0,1]_{\mathbb{T}}, \tag{3.17}$$

where $d_2 = d + rd_1$.

By comparison result, $v_n(t) \le z(t),\ t \in [0, \sigma^2(1)]_{\mathbb{T}}$, where $z(t)$ is a unique solution of the linear BVP

$$\begin{aligned}
z^{\Delta\Delta}(t) &= -d_2\|v_{n-1}\|^2,\ t \in [0,1]_{\mathbb{T}}, \\
z(0) = 0,\ z(\sigma^2(1)) &= g'(\xi)v_n(\eta),
\end{aligned} \tag{3.18}$$

and is given by

$$z(t) = d_2 \int_0^{\sigma(1)} |G(t,s)|\|v_{n-1}\|^2 \Delta s + \frac{g'(\xi)v_n(\eta)}{\sigma^2(1)} t \le d_3\|v_{n-1}\|^2 + \frac{g'(\xi)v_n(\eta)}{\sigma^2(1)} t, \tag{3.19}$$

where $d_3 = d_2 \max\{\int_0^{\sigma(1)} |G(t,s)|\Delta s : t \in [0,1]_{\mathbb{T}}\}$.

At $t = \eta$, using (A_3), we obtain

$$v_n(\eta) \leq d_3\|v_{n-1}\|^2 + \frac{g'(\xi)v_n(\eta)}{\sigma^2(1)}\eta \leq d_3\|v_{n-1}\|^2 + \frac{\theta v_n(\eta)}{\sigma^2(1)}\eta,$$

where $\theta = \max\{g'(x) : x \in [\alpha(\eta), \beta(\eta)]$. Solving for $v_n(\eta)$, we have

$$v_n(\eta) \leq \frac{\sigma^2(1)d_3}{\sigma^2(1) - \theta\eta}\|v_{n-1}\|^2.$$

Substituting in (3.19) and taking the maximum over $[0,1]_{\mathbb{T}}$, we obtain

$$\|v_n\| \leq d_4\|v_{n-1}\|^2,$$

where $d_4 = d_3\frac{\sigma^2(1) + \theta(1-\eta)}{\sigma^2(1) - \theta\eta}$, which shows the quadratic convergence. □

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