International Journal of Difference Equations
ISSN 0973-6069, Volume 6, Number 2, pp. 113–125 (2011)
http://campus.mst.edu/ijde

Periodic Solutions of Linearizable Difference Equations

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Abstract

Consider the nonlinear difference equation

$$x_{n+l} = f(x_n, \ldots, x_{n-k}), \quad k, n = 0, 1, \ldots .$$

with real initial conditions. We investigate this equation when it has the linearization of the form

$$x_{n+l} = \sum_{i=1-l}^{k} g_i(x_{n+l-1}, \ldots, x_{n-k})x_{n-i}, \quad n = 0, 1, \ldots, \; l = 1, 2, \ldots .$$

We use the generalized identity to study the problem of existence and nonexistence of periodic solutions of the considered equation.

We illustrate our results by numerous examples.

AMS Subject Classifications: Identity, linear, periodic, solution.
Keywords: 39A10, 39A11.

1 Introduction

We investigate the difference equation

$$x_{n+1} = f(x_n, \ldots, x_{n-k}), \quad k, n = 0, 1, \ldots, \tag{1.1}$$

Received August 19, 2011; Accepted September 11, 2011
Communicated by Martin Bohner

where the initial conditions are real numbers.

Our first objective is to bring (1.1) into the linearized form

$$x_{n+l} = \sum_{i=1-l}^{k} g_i x_{n-i}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where $l \in \{1, 2, \ldots\}$ and the functions $g_i : \mathbb{R} \to \mathbb{R}$.

There are many ways to obtain a linearization of (1.1) such as the following "forced linearization". Assume that for $j = 0, \ldots, k$ the functions $c_j : \mathbb{R} \to \mathbb{R}$ are such that $\sum_{j=0}^{k} c_j \neq 0$. Then

$$x_{n+l} = \frac{c_0 f(x_n, \ldots, x_{n-k})}{x_n \sum_{j=0}^{k} c_j} x_n + \ldots + \frac{c_k f(x_n, \ldots, x_{n-k})}{x_{n-k} \sum_{j=0}^{k} c_j} x_{n-k}, \quad n = 0, 1, \ldots . \tag{1.3}$$

Define the functions

$$g_i = \frac{c_i f(x_n, \ldots, x_{n-k})}{x_{n-i} \sum_{j=0}^{k} c_j}, \quad i = 0, \ldots, k; n = 0, 1, \ldots$$

Then (1.3) can be written as (1.2). If the linearization of (1.1) is not of the form (1.2), then we can try iterates of (1.1) such as the first iterate

$$x_{n+2} = f(f(x_n, \ldots, x_{n-k}), x_n, \ldots ., \ldots, x_{n-k+1}), \quad n = 0, 1, \ldots .$$

From (1.2) we obtain a generalized identity so that we can study the relationship between the functions g_i and the behavior of the solutions. Identities are often used to study the behavior of the solutions of the difference equation. Usually they compare two terms of the solution with each other or one term of the solution with a specific constant, such as the equilibrium or term of periodic solution. Such identities are building blocks of semicycle analysis, method of invariants and rate of convergence. See [1–3, 5–8, 10–15, 18, 19]. In this paper we will generalize the identity by comparing terms of the solution with an arbitrary constant. For example consider the linear equation

$$x_{n+1} = Ax_n + Bx_{n-k}, \quad n = 0, 1, \ldots, \tag{1.4}$$

where the real valued constants $A, B > 0$. Let K be any real number. Then (1.4) has the generalized identity

$$x_{n+1} - (A + B)K = A(x_n - K) + B(x_{n-k} - K), \quad n = 0, 1, \ldots,$$

Choosing K appropriately we can derive useful identities that will reveal local or global behavior of solutions of (1.4). Similar holds for general nonlinear equation as is illustrated by Examples 2.3–3.7.

We can obtain the generalized identity for (1.1) by applying the following theorem to the linearization (1.2) when the summation of the functions g_i is constant for all n. See [9].

Theorem 1.1. *Let $K \in \mathbb{R}$ and let $l \in \{1, 2, \ldots\}$. Suppose that on some interval $I \subset \mathbb{R}$,* (1.1) *has the linearization* (1.2)*. Then* (1.1) *has the generalized identity*

$$x_{n+l} - aK = \sum_{i=1-l}^{k} g_i(x_{n-i} - K), \quad n = 0, 1, \ldots \tag{1.5}$$

if and only if

$$a = \sum_{i=1-l}^{k} g_i, \quad n = 0, 1, \ldots.$$

A more direct method of obtaining a generalized identity for (1.1), where $\sum_{i=1-l}^{k} g_i$ is a constant for $n \geq 0$, which the authors have used in [9], consists of algebraic manipulation of

$$x_{n+1} - K = f(x_n, \ldots, x_{n-k}) - K, \quad k, n = 0, 1, \ldots\ , \tag{1.6}$$

or by manipulation of the iterates of (1.6) such as the first iterate

$$x_{n+2} - K = f(f(x_n, \ldots, x_{n-k}), \ldots, x_{n+1-k}) - K, \quad n = 0, 1, \ldots\ . \tag{1.7}$$

2 Generalized Identities and Periodicity

In this section we find some properties of the linearization (1.2) that allows us to determine which periodic solutions of (1.1) are feasible.

Theorem 2.1. *Let $l \in \{1, 2, \ldots\}$. Suppose that on some interval $I \subset \mathbb{R}$,* (1.1) *has the linearization* (1.2)*, where the functions g_i are nonnegative for $n \geq 0$ and $\sum_{i=1-l}^{k} g_i =$ $1,\ n = 0, 1, \ldots$. Assume that* (1.1) *has a periodic solution of minimal period p with a unique minimum or a unique maximum. Then, for all $i \in \{1-l, \ldots, k\}$, if $g_i > 0$ for all $n \geq 0$, then p divides $i + l$. Furthermore, $2 \leq p \leq k + l$. In particular, if $k + l = 2$, then* (1.1) *may possess only period-two solution.*

Proof. Since (1.1) has the linearization it has also the generalized identity (1.5) with $a = 1$. Assume that (1.1) has periodic solution of prime period p. Then $x_{n+l} = x_{n+l-mp}$ for $n = 0, 1, \ldots;\ m = 1, 2, \ldots$. Choose N sufficiently large such that for some $m = 1, 2, \ldots\ N \geq mp - k$ and x_{N+l-mp} is either the unique minimum or the unique maximum. Take $K = x_{N+l-mp}$ in (1.5) to obtain

$$0 = x_{N+l} - x_{N+l-mp} = \sum_{i=1-l}^{k} g_i(x_{N-i} - x_{N+l-mp}). \tag{2.1}$$

Thus if $i \neq mp - l$, then $x_{N-i} - x_{N+l-mp} \neq 0$. Hence in view of (2.1) if $i \neq mp - l$ then $g_i = 0$. Thus for $i \in \{1-l, \dots, k\}$ such that $g_i > 0$ we have that $i = mp - l \leq k$ for $m = 1, 2, \dots$. Therefore for all $i \in \{1-l, \dots, k\}$ if $g_i > 0$ for all $n \geq 0$, then p divides $i + l$. Since $k \geq i$, then $k + l \geq \dfrac{k+l}{m} \geq p$ for $m = 1, 2, \dots$. Therefore $2 \leq p \leq k + l$. □

If we want to determine which minimal period p solutions of (1.1) with a unique minimum or a unique maximum are possible, then we can iterate (1.1) until $l \geq p - k$ and see if the hypothesis of Theorem 2.1 are satisfied.

Remark 2.2. An immediate consequence of Theorem 2.1 is that if there are i, m such that $i + l$ and $m + l$ are relatively prime and $g_i > 0, g_m > 0$, then (1.1) does not possess periodic solutions.

The following examples illustrate Theorem 2.1, which can be used also to prove the nonexistence of periodic solutions.

Example 2.3. The difference equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_{n-1}^2 + Cx_{n-3}^2}{Ax_n + Bx_{n-1} + Cx_{n-3}}, \quad n = 0, 1, \dots \tag{2.2}$$

with all the parameters and initial conditions positive real numbers has the linearization

$$x_{n+1} = \frac{Ax_n}{Ax_n + Bx_{n-1} + Cx_{n-3}} x_n + \frac{Bx_{n-1}}{Ax_n + Bx_{n-1} + Cx_{n-3}} x_{n-1}$$
$$+ \frac{Cx_{n-3}}{Ax_n + Bx_{n-1} + Cx_{n-3}} x_{n-3}, \; n = 0, 1, \dots \, .$$

Observe that

$$g_0 + g_1 + g_3 = 1, \quad n = 0, 1, \dots$$

and $k + l = 3 + 1 = 4$. Thus by Theorem 2.1 all periodic solutions are of period p which is a factor of $2, 3$ and 4, which proves that (2.2) does not have any periodic solution.

Example 2.4. The difference equation

$$x_{n+1} = \frac{A + Bx_{n-2}}{x_n x_{n-1} x_{n-2}}, \quad n = 0, 1, \dots, \tag{2.3}$$

where A, B and the initial conditions are all positive real numbers has the linearization

$$x_{n+2} = \frac{Bx_{n-2}}{A + Bx_{n-2}} x_{n-1} + \frac{A}{A + Bx_{n-2}} x_{n-2} = g_1 x_{n-1} + g_2 x_{n-2}, \quad n = 0, 1, \dots \, .$$

Observe that

$$g_1 + g_2 = 1, \quad n = 0, 1, \dots$$

and $k+l = 2+2 = 4$. Thus by Theorem 2.1 all periodic solutions are of period p which is a factor of $2, 3$ and 4. Thus, (2.3) does not have any periodic solution.

Theorem 2.1 can be also applied to the following $k+1$-th order generalization of (2.3) of the form:

$$x_{n+1} = \frac{A + Bx_{n-k}}{x_n x_{n-1} \dots x_{n-k}}, \quad n = 0, 1, \dots; k = 2, 3, \dots, \tag{2.4}$$

where A, B and the initial conditions are all positive real numbers. This equation can be embedded into the $k+2$-th order linearization

$$x_{n+2} = \frac{Bx_{n-k}}{A + Bx_{n-k}} x_{n-k+1} + \frac{A}{A + Bx_{n-k}} x_{n-k} = g_k x_{n-k} + g_{k-1} x_{n-k+1},$$

for $n = 0, 1, \dots$, with

$$g_k + g_{k-1} = 1, \quad n = 0, 1, \dots .$$

Thus by Theorem 2.1 the only minimal period p solutions (2.4) may possess are for p which is a factor of $4, 5, \dots, k+2$ and so $p = 1$. Thus, (2.4) does not have any periodic solution. In fact, more can be concluded. In view of [9, Theorem 5], we can conclude that every positive solution of (2.4) converges to the unique positive equilibrium of that equation.

Theorem 2.5. *Let $i \in \{1-l, \dots, k\}$ and let g_i be real functions.*

(a) *Assume that $g_k = 1, g_i = 0, i = 1-l, \dots, k-1$ for $n = 0, 1, \dots$. Then* (1.1) *has the linearization* (1.2) *if and only if every solution of* (1.1) *is periodic with period $l+k$.*

(b) *Suppose that* (1.1) *has the linearization* (1.2) *with $g_j = 1$ for $j \in \{2-l, \dots, k-1\}, j \neq 1-l$ and $g_i = 0$ for all $i \in \{1-l, \dots, k\}, i \neq j$. Then every solution of* (1.1) *is eventually periodic with period $l+j$.*

Proof. (a) Equation (1.1) has the linearization (1.2) if and only if

$$x_{n+l} = \sum_{i=1-l}^{k} g_i x_{n-i} = g_k x_{n-k} = x_{n-k}, \quad n = 0, 1, \dots$$

which is true if and only if $x_{n+l} = x_{n-k}$, that is if and only if every solution of (1.1) is periodic with period $l+k$.

(b) By (1.5) we get

$$x_{n+l} = \sum_{i=1-l}^{k} g_i x_{n-i} = g_j x_{n-j} = x_{n-j}, \quad n = 0, 1, \dots$$

Since we can not determine the initial conditions $x_{-j-1}, \dots, x_{-k}$, then we can only say that every solution of (1.1) is eventually periodic with period $l+j$. □

Example 2.6. The following equations illustrate Theorem 2.5. Equations

$$x_{n+1} = \frac{1}{x_n}, \quad x_{n+1} = -x_n, \quad n = 0, 1, \ldots$$

both have linearization

$$x_{n+1} = x_{n-1}, \quad n = 0, 1, \ldots$$

which shows that all solutions of these equations are periodic of (not necessarily minimal) period two. Furthermore, the special case of well-known Lyness' equation, see [13, 15]:

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \quad n = 0, 1, \ldots ; x_{-1}, x_0 > 0$$

as well as its $\max$ analogue

$$x_{n+1} = \frac{\max\{1, x_n\}}{x_{n-1}}, \quad n = 0, 1, \ldots ; x_{-1}, x_0 > 0$$

both have linearization

$$x_{n+1} = x_{n-4}, \quad n = 0, 1, \ldots$$

which shows that all solutions of Lyness' equation are periodic of (not necessarily minimal) period five.

The following result describes the oscillatory character of solutions of linearizable equations.

Theorem 2.7. *Suppose that on some interval $I \subset \mathbb{R}$,* (1.1) *has the linearization* (1.2)*, where the functions g_i are nonnegative for $n \geq 0$ and $\sum_{i=1-l}^{k} g_i = 1$. Then every solution of* (1.1) *strictly oscillates about some $L \in I$.*

Proof. Let $\gamma = l + k$ and $K \in \mathbb{R}$. Define for $N = 0, 1, \ldots$

$$M_N = \max\{x_{\gamma N+l-1}, \ldots, x_{\gamma N-k}\}, \quad m_N = \min\{x_{\gamma N+l-1}, \ldots, x_{\gamma N-k}\}, \ .$$

Since $\sum_{i=1-l}^{k} g_i = 1$, then (1.1) has the generalized identity (1.5) with $a = 1$. If for some N, $x_{\gamma N+l-1} = \ldots = x_{\gamma N-k}$, then $m_N = M_N$. By using (1.5) with $a = 1$ and $K = m_N$ we get that $x_{\gamma N+l+j} = m_N = M_N,\ j = 0, 1, \ldots$ and so the solution of (1.1) is constant solution. So for every $N \geq 0$ at least one term of $x_{\gamma N+l-1}, \ldots, x_{\gamma N-k}$ must be strictly greater than m_N. Hence $m_N < M_N$ for $N = 0, 1, \ldots$. Then there exists $L \in I$ such that for every $N \geq 0$ one term of $x_{\gamma N+l-1}, \ldots, x_{\gamma N-k}$ is above L and one term is below L. □

3 Nonexistence of Periodic Solutions

In this section we present some results on nonexistence of periodic solutions. The results presented here are surprisingly efficient as one can see from the examples included and are different from discrete versions of classical nonexistence results of Bendixson and Dulac, see [16, 17] and the similar results which hold for equations on time scales, see [4]. Although some of these results follow from the theorems in [9] they also extend them, for example when $\lim_{n\to\infty} \sum_{i=1-l}^{k} g_i = 1$.

Theorem 3.1. *Let* $l \in \{1, 2, \ldots\}$. *Suppose* (1.1) *has the linearization* (1.2)*, where the real functions* g_i *are such that* $g_{1-l} > 0$ *for all* $n = 0, 1, \ldots$. *If for all* $i \in \{2-l, \ldots, k\}$ $g_i \geq 0$ *and* $\sum_{i=1-l}^{k} g_i \leq 1$, *then* (1.1) *does not have a periodic solution.*

Proof. Assume that (1.1) has a periodic solution of period p. Let $a_n = \sum_{i=1-l}^{k} g_i$ for $n = 0, 1, \ldots$ and $K \in \mathbb{R}$. Then (1.1) has the generalized identity

$$x_{n+1} - a_n K = g_{1-l}(x_{n-1+l} - K) + \sum_{i=2-l}^{k} g_i(x_{n-i} - K), \quad n = 0, 1, \ldots \tag{3.1}$$

First consider the case where the maximum term of the period p solution is strictly positive. Choose n such that x_{n+l} is the maximum term of the period p solution and $x_{n+l} \neq x_{n+l-1}$. By using (3.1) with $K = x_{n+l+p}$ we get

$$x_{n+1} - a_n x_{n+l+p} = g_{1-l}(x_{n-1+l} - x_{n+l+p}) + \sum_{i=2-l}^{k} g_i(x_{n-i} - x_{n+l+p}) < 0.$$

Hence $x_{n+l} < a_n x_{n+l} \leq x_{n+l}$, which is a contradiction.

Next consider the case where the maximum term of the period p solution is nonpositive. Then the minimum term is strictly negative. Choose n such that x_{n+l} is the minimum term of the period p solution and $x_{n+l} \neq x_{n+l-1}$. By using (3.1) with $K = x_{n+l+p}$ we get

$$x_{n+1} - a_n x_{n+l+p} = g_{1-l}(x_{n-1+l} - x_{n+l+p}) + \sum_{i=2-l}^{k} g_i(x_{n-i} - x_{n+l+p}) > 0.$$

Hence $x_{n+l} > a_n x_{n+l} \geq x_{n+l}$, which is a contradiction. □

Example 3.2. Consider the difference equation

$$x_{n+1} = \frac{a(x_n^3 + x_n) + bx_{n-1}^3}{a(x_n^2 + 1) + bx_{n-1}^2}, \quad n = 0, 1, \ldots, \tag{3.2}$$

where $a, b > 0$. Equation (3.2) can be rewritten as

$$x_{n+1} = \frac{a(x_n^2 + 1)}{a(x_n^2 + 1) + bx_{n-1}^2} x_n + \frac{bx_{n-1}^2}{a(x_n^2 + 1) + bx_{n-1}^2} x_{n-1} = g_0 x_n + g_1 x_{n-1}, \tag{3.3}$$

for $n = 0, 1, \ldots$, where $g_0 + g_1 = 1, g_0 > 0$ and so by Theorem 3.1 does not possess periodic solutions of any period.

This example easily generalizes to more general equation of the form

$$x_{n+1} = \frac{a(x_n^{2k+1} + x_n) + bx_{n-s}^{2r+1}}{a(x_n^{2k} + 1) + bx_{n-s}^{2r}}, \quad n = 0, 1, \ldots,$$

where $a, b > 0, k, r, s \in \{1, 2, \ldots\}$.

Example 3.3. In view of Theorem 3.1, the difference equation

$$x_{n+1} = \sum_{k=0}^{N} D_k(x_n, \ldots, x_{n-k}) x_{n-k}, \quad n = 0, 1, \ldots, \tag{3.4}$$

where the functions D_k are given by

$$D_k(x_n, \ldots, x_{n-k}) = \begin{cases} a_k & \text{if } x_k \text{ is rational,} \\ b_k & \text{if } x_k \text{ is irrational,} \end{cases}$$

where $a_k, b_k \geq 0, k = 0, 1, \ldots, N$ satisfy $\sum_{k=0}^{N} \max\{a_k, b_k\} \leq 1$, has no periodic solutions of any period.

Now, it is clear that the functions D_i can even be random variables and so our result applies to random difference equations.

Theorem 3.4. *Let* $l \in \{1, 2, \ldots\}$. *Suppose that* (1.1) *has the linearization* (1.2), *where for all* n *either*

$$g_{1-l} > 0, g_i \geq 0 \quad \text{and} \quad 1 \leq \sum_{i=1-l}^{k} g_i \tag{3.5}$$

or

$$g_{1-l} < 0, g_i \leq 0 \quad \text{and} \quad -1 \leq \sum_{i=1-l}^{k} g_i. \tag{3.6}$$

Then (1.1) *does not have a periodic solution whose terms are all nonpositive or all nonnegative.*

Proof. Let $a_n = \sum_{i=1-l}^{k} g_i$ for $n = 0, 1, \ldots$ and $K \in \mathbb{R}$. Then (1.1) has the generalized identity (3.1).

First, assume (3.5). First, assume that all terms of a periodic solution are nonpositive. Choose n such that x_{n+l} is the maximum term of periodic solution of period p and $x_{n+l} \neq x_{n+l-1}$. By using (3.1) with $K = x_{n+l+p}$ we get for $n = 0, 1, \ldots$

$$x_{n+1}(1 - a_n) = x_{n+1} - a_n x_{n+l+p}$$

$$= g_{1-l}(x_{n-1+l} - x_{n+l+p}) + \sum_{i=2-l}^{k} g_i(x_{n-i} - x_{n+l+p}) < 0,$$

which, in view of $x_{n+1} \leq 0$ is a contradiction.

Second, assume that all terms of a periodic solution are nonnegative. Choose n such that x_{n+l} is the minimum term of periodic solution of period p and $x_{n+l} \neq x_{n+l-1}$. By using (3.1) with $K = x_{n+l+p}$ we get for $n = 0, 1, \ldots$

$$x_{n+1}(1 - a_n) = x_{n+1} - a_n x_{n+l+p}$$

$$= g_{1-l}(x_{n-1+l} - x_{n+l+p}) + \sum_{i=2-l}^{k} g_i(x_{n-i} - x_{n+l+p}) > 0,$$

which, in view of $x_{n+1} \geq 0$ is a contradiction.

Next, assume (3.6). First, assume that all terms of a periodic solution are nonpositive. Choose n such that x_{n+l} is the maximum term of periodic solution of period p and $x_{n+l} \neq x_{n+l-1}$. By using (3.1) with $K = x_{n+l+p}$ we get for $n = 0, 1, \ldots$

$$x_{n+1}(1 + a_n) = x_{n+1} + a_n x_{n+l+p}$$

$$= g_{1-l}(x_{n-1+l} - x_{n+l+p}) + \sum_{i=2-l}^{k} g_i(x_{n-i} - x_{n+l+p}) > 0,$$

which, in view of $x_{n+1} \leq 0$ is a contradiction.

Second, assume that all terms of a periodic solution are nonnegative. Choose n such that x_{n+l} is the minimum term of periodic solution of period p and $x_{n+l} \neq x_{n+l-1}$. By using (3.1) with $K = x_{n+l+p}$ we get for $n = 0, 1, \ldots$

$$x_{n+1}(1 + a_n) = x_{n+1} + a_n x_{n+l+p}$$

$$= g_{1-l}(x_{n-1+l} - x_{n+l+p}) + \sum_{i=2-l}^{k} g_i(x_{n-i} - x_{n+l+p}) < 0,$$

which, in view of $x_{n+1} \geq 0$ is a contradiction. □

Example 3.5. The difference equation

$$y_{n+1} = \frac{\alpha + Ay_{n-1}}{A + By_n}, \quad n = 0, 1, \dots, \tag{3.7}$$

where α, A and B are positive constants and initial conditions are nonnegative, was considered in [7, 13], where we have given global dynamics. Equation (3.7) possesses the linearization

$$y_{n+2} = y_n + \frac{A^2 + ABy_n}{A^2 + ABy_n + \alpha B + ABy_{n-1}} y_{n+1} - \frac{A^2 + ABy_n}{A^2 + ABy_n + \alpha B + ABy_{n-1}} y_{n-1}, \tag{3.8}$$

for $n = 0, 1, \dots$ and the obvious generalized identity

$$y_{n+2} - K = y_n - K + \frac{A^2 + ABy_n}{A^2 + ABy_n + \alpha B + ABy_{n-1}} (y_{n+1} - K) - \frac{A^2 + ABy_n}{A^2 + ABy_n + \alpha B + ABy_{n-1}} (y_{n-1} - K), \tag{3.9}$$

where $n = 0, 1, \dots$ and K is an arbitrary number. Here

$$g_{-1} = -g_1 = \frac{A^2 + ABy_n}{A^2 + ABy_n + \alpha B + ABy_{n-1}}, \quad g_0 = 1$$

and so $g_{-1} + g_0 + g_1 = 1$. Choosing $K = y_n$ in (3.9) or rearranging terms in (3.8) we obtain

$$y_{n+2} - y_n = \frac{A^2 + ABy_n}{A^2 + ABy_n + \alpha B + ABy_{n-1}} (y_{n+1} - y_{n-1}), \quad n = 0, 1, \dots, \tag{3.10}$$

showing that both sequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are monotonic. Thus every bounded solution converges to period-two solution. Indeed, one can show that (3.7) has infinitely many period-two solutions of the form $\Phi, \Psi, \Phi, \Psi, \dots$, where $B\Phi\Psi = \alpha$ and that all solutions are bounded. Consequently, every solution converges to period-two solution which follows immediately from the identity (3.10).

Example 3.6. The difference equation

$$y_{n+1} = \frac{ay_{n-1}^2 + 1}{y_{n-1}} + \frac{by_n^2 + 1}{y_n}, \quad n = 0, 1, \dots, \tag{3.11}$$

where a and b are positive constants which satisfy $a + b > 1$ was considered in [9], where we have shown that if $a+b > 1$, then every solution with positive (resp. negative) initial conditions is asymptotic to ∞ (resp. $-\infty$). Here we consider the generalization of (3.11):

$$y_{n+1} = \sum_{k=0}^{M} \frac{A_k y_{n-k}^2 + 1}{y_{n-k}}, \quad n = 0, 1, \dots, \tag{3.12}$$

where $A_0 > 0, A_k \geq 0, k = 1, \ldots, M$ and $\sum_{k=0}^{M} A_k \geq 1$. In this case Theorem 3.4 implies that (3.12) has no periodic solution with all positive initial conditions. Indeed, (3.12) has the following linearization:

$$y_{n+1} = \sum_{k=0}^{M} \frac{A_k y_{n-k}^2 + 1}{y_{n-k}^2} y_{n-k} = \sum_{k=0}^{M} g_k y_{n-k}, \quad n = 0, 1, \ldots .$$

Clearly

$$\sum_{k=0}^{M} g_k = \sum_{k=0}^{M} A_k + \sum_{k=0}^{M} \frac{1}{y_{n-k}^2} > \sum_{k=0}^{M} A_k \geq 1.$$

Example 3.7. The difference equation

$$y_{n+1} = \frac{a y_{n-1}^2 - 1}{y_{n-1}} + \frac{b y_n^2 - 1}{y_n}, \quad n = 0, 1, \ldots, \tag{3.13}$$

where $a < 0$ and $b \leq 0$ has the linearization

$$y_{n+1} = \frac{a y_{n-1}^2 - 1}{y_{n-1}^2} y_{n-1} + \frac{b y_n^2 - 1}{y_n^2} y_n, \quad n = 0, 1, \ldots ,$$

which implies

$$g_1 + g_2 = a + b - \frac{1}{y_{n-1}^2} - \frac{1}{y_n^2} < a + b < 0,$$

for every solution $\{y_n\}$ which satisfies $y_{-1} y_0 > 0$. In this case Theorem 3.4 implies that (3.13) has no periodic solution of any period.

References

[1] A. M. Amleh, E. A. Grove, G. Ladas and D. A. Georgiou, On the recursive sequence $x_{n+1} = \alpha + x_{n-1}/x_n$, *J. Math. Anal. Appl.* 233 (1999), no. 2, 790–798.

[2] A. M. Amleh, E. Camouzis, and G. Ladas, On the dynamics of a rational difference equation. I. *Int. J. Difference Equ.* 3 (2008), no. 1, 1–35.

[3] A. M. Amleh, E. Camouzis, and G. Ladas, On the dynamics of a rational difference equation. II. *Int. J. Difference Equ.* 3 (2008), no. 2, 195–225.

[4] M. Bohner and V. Lakshmikantham, Formulas of Bendixson and Alekseev for difference equations. *Bull. London Math. Soc.* 36 (2004), no. 1, 65–71.

[5] S. Elaydi, An introduction to difference equations. Third edition. Undergraduate Texts in Mathematics. Springer, New York, 2005. 539 pp.

[6] J. Feuer, E. Janowski, and G. Ladas, Invariants for some rational recursive sequences with periodic coefficients, *J. Difference Equ. Appl.* 2(1996), 167–174.

[7] C. Gibbons, M. R. S. Kulenović and G. Ladas, On the Recursive Sequence $x_{n+1} = (a + bx_{n-1})/(g + x_n)$, *Math. Sci. Res. Hot-Line* 4(2) (2000), 1–11.

[8] C. Gibbons, M. R. S. Kulenović, G. Ladas and H. D. Voulov, On the trichotomy character of $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + x_n)$, *J. Difference Equ. Appl.* 8 (2002), 75–92.

[9] E. Janowski and M. R. S. Kulenović, Attractivity and global stability for linearizable difference equations, *Comput. Math. Appl.*, 57(2009), 1592–1607.

[10] S. Kalabušić and M. R. S. Kulenović, Rate of convergence of solutions of rational difference equation of second order, *Adv. Difference Equ.*, 1 (2004), 121–140.

[11] V. L. Kocic and G. Ladas, "Global Behavior of Nonlinear Difference Equations of Higher Order with Applications", Kluwer Academic Publishers, Dordrecht, 1993.

[12] M. R. S. Kulenović, Invariants and related Liapunov functions for difference equations, *Appl. Math. Lett.* 13(2000), 1–8.

[13] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations, with Open Problems and Conjectures*, Chapman& Hall/CRC Press, 2001.

[14] M. R. S. Kulenović, G. Ladas, and C. Overdeep , On the dynamics of $x_{n+1} = p_n + \frac{x_{n-1}}{x_n}$ with a period-two coefficient, *J. Difference Equ. Appl.*, 10(2004), 905–914.

[15] M. R. S. Kulenović and O. Merino, "Discrete Dynamical Systems and Difference Equations with Mathematica," Chapman and Hall/CRC, Boca Raton, London, 2002.

[16] C. C. McCluskey and J. S. Muldowney, Bendixson-Dulac criteria for difference equations. *J. Dynam. Differential Equations* 10 (1998), no. 4, 567–575

[17] C. C. McCluskey and J. S. Muldowney, Stability implications of Bendixson conditions for difference equations. *Proceedings of the Sixth International Conference on Difference Equations*, 181187, CRC, Boca Raton, FL, 2004.

[18] T. Nesemann, Invariants and Liapunov functions for nonautonomous systems. Advances in difference equations, III. *Comput. Math. Appl.* 42 (2001), no. 3-5, 385–392.

[19] H. Sedaghat, " Nonlinear Difference Equations, Theory with applications to social science models",. Mathematical Modelling: Theory and Applications, 15. Kluwer Academic Publishers, Dordrecht, 2003.