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Right Nabla Discrete Fractional Calculus

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Abstract

Here we define a Caputo like right discrete nabla fractional difference and we produce a right discrete nabla fractional Taylor formula for the first time. We estimate the remainder. Then we derive related right discrete nabla fractional Ostrowski, Poincaré and Sobolev type inequalities.

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1 Introduction and Background

Here we work on the time scale $\mathbb{T} = a + \mathbb{Z}$, where $a \in \mathbb{R}$. We consider functions $f : (a + \mathbb{Z}) \to \mathbb{R}$. If a function f is defined on a subset of $a + \mathbb{Z}$, then one can extend it to all of $a + \mathbb{Z}$, by assigning zero values to f on the complement, with respect to $a + \mathbb{Z}$, of that subset. For $t \in \mathbb{R}$, the rising factorial is defined as

$$t^{\bar{n}} = t(t+1)(t+2)\dots(t+n-1),\ n \in \mathbb{N}$$

and $t^{\bar{0}} = 1$. For $\alpha \in \mathbb{R}$, define in general

$$t^{\bar{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}$$

where $t \in \mathbb{R} - \{\dots, -2, -1, 0\}$, $0^{\bar{\alpha}} = 0$, where Γ is the gamma function;

$$\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt, \nu > 0.$$

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Note that

$$\nabla(t^{\bar{\alpha}}) = \alpha t^{\overline{\alpha-1}}$$

where $\nabla y(t) = y(t) - y(t-1)$ $(= y^{\nabla}(t)$, the time scale nabla discrete derivative if $t \in (a+\mathbb{Z}))$. Also $\nabla y(t)$ is called the backward difference. We further define the falling factorial

$$t^{(n)} = t(t-1)\ldots(t-n+1), n \in \mathbb{N},$$

and in general

$$t^{\bar{\alpha}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}.$$

Notice that

$$t^{\bar{\alpha}} = (t+\alpha-1)^{(\alpha)} \tag{1.1}$$

From time scales theory [4, p. 333] we know that

$$\int_{a^*}^{b^*} f(t)\nabla t = \sum_{t=a^*+1}^{b^*} f(t),\ \ a^* < b^* \tag{1.2}$$

where $a^*, b^* \in (a+\mathbb{Z})$. Also for $f : (a+\mathbb{Z}) \to \mathbb{R}$ we define the higher order backward difference

$$\nabla^k f(t) = \sum_{l=0}^{k} (-1)^l \binom{k}{l} f(t-l) = f^{\nabla^k}(t),\ \ k \in \mathbb{N},$$

the kth order time scale nabla derivative of f. If f runs on $[a^*, b^*] \cap (a+\mathbb{Z})$, then f^{∇^k} runs on $[a^*+k, b^*] \cap (a+\mathbb{Z})$. For a general time scale $\mathbb{T}$, see [2,4], we define

$$\hat{h}_k : \mathbb{T}^2 \to \mathbb{R}, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

$$\hat{h}_0(t,s) = 1,\ \ s,t \in \mathbb{T},$$

$$\hat{h}_{k+1}(t,s) = \int_s^t \hat{h}_k(\tau,s)\nabla\tau,\ s,t \in \mathbb{T}. \tag{1.3}$$

We have that the nabla derivative

$$\hat{h}_k^{\nabla}(t,s) = \hat{h}_{k-1}(t,s),\ \ k \in \mathbb{N},\ \ t \in \mathbb{T}_k,$$

(for $\mathbb{T}_k$, $\mathbb{T}^k$ see [2,4]) and $\hat{h}_1(t,s) = t-s$, $s,t \in \mathbb{T}$. Notice here that

$$(a+\mathbb{Z}) = (a+\mathbb{Z})_k = (a+\mathbb{Z})^k. \tag{1.4}$$

Lemma 1.1. *We have that*

$$\hat{h}_k(t,s) = \frac{(t-s)^{\bar{k}}}{k!},\ \ s,t \in (a+\mathbb{Z}). \tag{1.5}$$

Lemma 1.2. *It holds on $a+\mathbb{Z}$ that*

$$\left\{\frac{(t-s)^{\overline{k+1}}}{(k+1)!}\right\}^{\nabla_t} = \frac{(t-s)^{\overline{k}}}{k!}, \quad k \in \mathbb{N}_0, \tag{1.6}$$

∇_t means nabla derivative with respect to t.

Proof. We have

$$\begin{aligned}\left(\frac{(t-s)^{\overline{k+1}}}{(k+1)!}\right)^{\nabla_t} &= \frac{1}{(k+1)!}\left\{(t-s)^{\overline{k+1}} - (t-s-1)^{\overline{k+1}}\right\} \\ &= \frac{1}{(k+1)!}\{(t-s)(t-s+1)(t-s+2)\dots(t-s+k) \\ &\quad -(t-s-1)(t-s)(t-s+1)\dots(t-s+k-1)\} \\ &= \frac{1}{(k+1)!}\{(t-s)(t-s+1)\dots(t-s+k-1)\} \\ &\quad \{(t-s+k)-(t-s-1)\} \\ &= \frac{1}{k!}\{(t-s)(t-s+1)\dots(t-s+k-1)\} \\ &= \frac{(t-s)^{\overline{k}}}{k!}.\end{aligned}$$

The proof is complete. □

Proof of Lemma 1.1. We see that

$$\hat{h}_0(t,s) = \frac{(t-s)^{\overline{0}}}{0!} = 1, \;\; s,t \in (a+\mathbb{Z}).$$

Assume (1.5) is correct. Then

$$\begin{aligned}\hat{h}_{k+1}(t,s) &= \int_s^t \hat{h}_k(\tau,s)\nabla\tau = \int_s^t \frac{(\tau-s)^{\overline{k}}}{k!}\nabla\tau \\ &= \frac{(t-s)^{\overline{k+1}}}{(k+1)!}.\end{aligned}$$

This completes the proof. □

Again here we consider $f:(a+\mathbb{Z}) \to \mathbb{R}$ and let $m \in \mathbb{N}$. By the nabla time scales Taylor formula, see [2] and Lemma 1.1, applied to $(a+\mathbb{Z})$, we obtain the Taylor formula

$$f(t) = \sum_{k=0}^{m-1} \nabla^k f(s)\frac{(t-s)^{\overline{k}}}{k!} + \int_s^t \frac{(t-\tau+1)^{\overline{m-1}}}{(m-1)!}\nabla^m f(\tau)\nabla\tau, \tag{1.7}$$

for all $t, s \in (a+\mathbb{Z})$. Here $\nabla^0 f = f$. We specialize (1.7) for $t \in [a^*, b^*] \cap (a+\mathbb{Z})$ and $s = b^* \in (a+\mathbb{Z})$, $a^* \in (a+\mathbb{Z})$, where $a^* < b^*$. Hence we have

$$f(t) = \sum_{k=0}^{m-1} \nabla^k f(b^*) \frac{(t-b^*)^{\overline{k}}}{k!} + \int_{b^*}^{t} \frac{(t-\tau+1)^{\overline{m-1}}}{(m-1)!} \nabla^m f(\tau) \nabla\tau. \tag{1.8}$$

That is,

$$f(t) = \sum_{k=0}^{m-1} \nabla^k f(b^*) \frac{(t-b^*)^{\overline{k}}}{k!} - \int_{t}^{b^*} \frac{(t-\tau+1)^{\overline{m-1}}}{(m-1)!} \nabla^m f(\tau) \nabla\tau. \tag{1.9}$$

Using (1.2), we can write (1.9) as

$$f(t) = \sum_{k=0}^{m-1} \nabla^k f(b^*) \frac{(t-b^*)^{\overline{k}}}{k!} - \frac{1}{(m-1)!} \sum_{s=t+1}^{b^*} (t-s+1)^{\overline{m-1}} \nabla^m f(s). \tag{1.10}$$

We call the remainder of (1.10) as

$$R^*(t) := -\frac{1}{(m-1)!} \sum_{s=t+1}^{b^*} (t-s+1)^{\overline{m-1}} \nabla^m f(s), \tag{1.11}$$

for all $t \in [a^*, b^*) \cap (a+\mathbb{Z})$.

We need the following auxiliary result.

Lemma 1.3. *We have*

$$(t-s+1)^{\overline{m-1}} = (-1)^{m-1} (s-t-m+1)^{\overline{m-1}}. \tag{1.12}$$

Proof. Notice that

$$\begin{aligned}
(t-s+1)^{\overline{m-1}} &= (t-s+1)(t-s+1+1)(t-s+1+2)\ldots \\
&\quad (t-s+1+m-1-1) \\
&= (-(s-t)+1)(-(s-t)+2)(-(s-t)+3)\ldots \\
&\quad (-(s-t)+m-1) \\
&= (-1)^{m-1}(s-t-1)(s-t-2)(s-t-3)\ldots \\
&\quad (s-t-m+1) \\
&= (-1)^{m-1}(s-t-m+1)\ldots \\
&\quad (s-t-3)(s-t-2)(s-t-1) \\
&= (-1)^{m-1}(s-t-(m-1))^{\overline{m-1}} \\
&= (-1)^{m-1}(s-t-m+1)^{\overline{m-1}}.
\end{aligned}$$

Indeed we have

$$\begin{aligned}
&(t-s-(m-1))^{\overline{m-1}} \\
&= (s-t-m+1)(s-t-m+1+1)(s-t-m+1+2)\dots \\
&\quad (s-t-m+1+m-2) \\
&= (s-t-m+1)(s-t-m+2)(s-t-m+3)\dots \\
&\quad (s-t-3)(s-t-2)(s-t-1).
\end{aligned}$$

The proof is complete. □

Because of (1.12), we obtain

$$\begin{aligned}
R^*(t) &= \frac{(-1)^m}{(m-1)!}\sum_{s=t+1}^{b^*}(s-t-m+1)^{\overline{m-1}}\nabla^m f(s) \qquad (1.13)\\
&= \frac{(-1)^m}{(m-1)!}\sum_{s=t+1}^{b^*}(s-m-t+1)^{\overline{m-1}}\nabla^m f(s),
\end{aligned}$$

for all $t \in [a^*, b^*] \cap (a+\mathbb{Z})$. Here $s \geq t+1$, that is $s-t \geq 1$, and $b^* = (t+1)\mathrm{mod}(1)$.

Remark 1.4. We notice that

$$\begin{aligned}
&(\lambda-m+1)^{\overline{m-1}} = (\lambda+1-m)^{\overline{m-1}} \\
&= (\lambda+1-m)(\lambda+1-m+1)(\lambda+1-m+2)(\lambda+1-m+3)\dots \\
&\quad (\lambda+1-m+m-3)(\lambda+1-m+m-2) \\
&= (\lambda-(m-1))(\lambda-(m-2))(\lambda-(m-3))(\lambda-(m-4))\dots \\
&\quad (\lambda-2)(\lambda-1),
\end{aligned}$$

for any $\lambda \in \mathbb{N}$. So, when $\lambda = 1, 2, \dots, m-1$, we get that

$$(\lambda-m+1)^{\overline{m-1}} = 0.$$

Therefore

$$\sum_{s=t+1}^{t+m-1}(s-m-t+1)^{\overline{m-1}}\nabla^m f(s) = 0, \qquad (1.14)$$

since $s-t = 1, 2, \dots, m-1$. In conclusion, we get that

$$R^*(t) = \frac{(-1)^m}{(m-1)!}\sum_{s=t+m}^{b^*}(s-(m+t)+1)^{\overline{m-1}}\nabla^m f(s), \qquad (1.15)$$

which requires $t \leq b^* - m$. Since

$$(s-m-t+1)^{\overline{m-1}} = (s-m-t+1+m-2)^{(m-1)} \qquad (1.16)$$

$$= (s-t-1)^{(m-1)},$$

we can rewrite (1.15) as

$$R^*(t) = \frac{(-1)^m}{(m-1)!} \sum_{s=t+m}^{b^*} (s-t-1)^{(m-1)} \nabla^m f(s), \tag{1.17}$$

where $t \in [a^*, b^* - m] \cap (a + \mathbb{Z})$, $m \in \mathbb{N}$. Here we restricted f on $[a^*, b^*] \cap (a + \mathbb{Z})$, which implies that $\nabla^m f$ is restricted on $[a^* + m, b^*] \cap (a + \mathbb{Z})$.

We need the following definition.

Definition 1.5 (See [3]). If $\nu > 0$, the right fractional sum is given by

$$(\Delta_{b^*}^{-\nu} f)(t) := \frac{1}{\Gamma(\nu)} \sum_{s=t+\nu}^{b^*} (s-t-1)^{(\nu-1)} f(s), \tag{1.18}$$

$$(\Delta_{b^*}^{0} f)(t) := f(t),$$

where f is restricted on $[a^*, b^*] \cap (a + \mathbb{Z})$. Here $(\Delta_{b^*}^{-\nu} f)$ is defined on

$$\{a^* - \nu, a^* - \nu + 1, a^* - \nu + 2, \ldots, b^* - \nu\}.$$

Here one can take $a^* = -\infty$.

We also need the following result.

Theorem 1.6 (See [5]). *Let $\mu, \nu \geq 0$. Then*

$$(\Delta_{b^*-\nu}^{-\mu} \Delta_{b^*}^{-\nu} f)(t) = (\Delta_{b^*}^{-(\mu+\nu)} f)(t), \tag{1.19}$$

where $t \in \{a^* - (\mu + \nu), a^* - (\mu + \nu) + 1, \ldots, b^* - (\mu + \nu)\}$.

We have now established the following theorem.

Theorem 1.7. *We have*

$$R^*(t) = (-1)^m (\Delta_{b^*}^{-m} (\nabla^m f))(t), \tag{1.20}$$

where $t \in \{a^*, a^* + 1, \ldots, b^* - m\}$, $a^* \leq b^* - m$, when f is restricted on $[a^*, b^*] \cap (a + \mathbb{Z})$, $a^*, b^* \in (a + \mathbb{Z})$.

2 Main Results

We give the following definition.

Definition 2.1. Let $\mu > 0$, $m - 1 < \mu \leq m$, $m \in \mathbb{N}$, $m = \lceil \mu \rceil$ (ceiling of number), $\nu := m - \mu$, that is $\mu + \nu = m$.

The μ-th order nabla right fractional difference (Caputo way) is given by

$$\begin{aligned}(\nabla_{b^*-}^{\mu} f)(t) \quad : \quad &= (-1)^m (\Delta_{b^*}^{-\nu} (\nabla^m f))(t) \\ &= \frac{(-1)^m}{\Gamma(\nu)} \sum_{s=t+\nu}^{b^*} (s - t - 1)^{(\nu-1)} (\nabla^m f)(s),\end{aligned} \tag{2.1}$$

where $t \leq b^* - \nu$, $b^* \in (a + \mathbb{Z})$, $t \in (a - \nu + \mathbb{Z})$, $a \in \mathbb{R}$. If $\mu = m \in \mathbb{N}$, then

$$(\nabla_{b^*-}^{\mu} f)(t) = (-1)^m (\nabla^m f)(t).$$

Remark 2.2. If we restrict $f : (a + \mathbb{Z}) \to \mathbb{R}$ on $[a^* - m, b^*]$, where $a^*, b^* \in (a + \mathbb{Z})$, $a^* < b^*$, $m \in \mathbb{N}$, then $\nabla^m f$ is defined on $[a^*, b^*] \cap (a + \mathbb{Z})$, and then $\nabla_{b^*-}^{\mu} f$ is defined on $\{a^* - \nu, a^* - \nu + 1, \ldots, b^* - \nu\}$.

We present the following results.

Theorem 2.3. *Let* $a^*, b^* \in (a + \mathbb{Z})$, $a^* \leq b^* - m$, *and* $t \in \{a^*, a^* + 1, \ldots, b^* - m\}$. *Then*

$$R^*(t) = \left(\Delta_{b^*-\nu}^{-\mu} \left(\nabla_{b^*-}^{\mu} f\right)\right)(t), \tag{2.2}$$

where f *is restricted on* $[a^*, b^*] \cap (a + \mathbb{Z})$, *and* $\mu > 0$, $m - 1 < \mu \leq m$, $m \in \mathbb{N}$.

Proof. Let $m - 1 < \mu < m$. With the help of (1.19) and (1.20), we see that

$$\begin{aligned}\left(\Delta_{b^*-\nu}^{-\mu} \left(\nabla_{b^*-}^{\mu} f\right)\right)(t) &= (-1)^m \left(\Delta_{b^*-\nu}^{-\mu} \Delta_{b^*}^{-\nu} (\nabla^m f)\right)(t) \\ &= (-1)^m \left(\Delta_{b^*}^{-(\mu+\nu)} (\nabla^m f)\right)(t) \\ &= (-1)^m \left(\Delta_{b^*}^{-m} (\nabla^m f)\right)(t) \\ &= R^*(t).\end{aligned}$$

If $\mu = m$, then (2.2) is trivial. □

We have proved the following nabla right discrete fractional Taylor formula.

Theorem 2.4. *Let* $a \in \mathbb{R}$, $f : (a + \mathbb{Z}) \to \mathbb{R}$. *Let* $\mu > 0$, $m - 1 < \mu \leq m$, $(m = \lceil \mu \rceil)$, $m \in \mathbb{N}$, $\nu = m - \mu$. *Let* $a^*, b^* \in (a + \mathbb{Z}) : a^* \leq b^* - m$. *Here* a^* *could be* $-\infty$. *Let* $t \in \{a^*, a^* + 1, \ldots, b^* - m\}$. *Then*

$$f(t) = \sum_{k=0}^{m-1} \nabla^k f(b^*) \frac{(t - b^*)^{\overline{k}}}{k!} + R^*(t), \tag{2.3}$$

where the remainder

$$R^*(t) = \left(\Delta_{b^*-\nu}^{-\mu}\left(\nabla_{b^*-}^{\mu} f\right)\right)(t) = \frac{1}{\Gamma(\mu)} \sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{(\mu-1)} (\nabla_{b^*-}^{\mu} f)(s). \tag{2.4}$$

Remark 2.5. Using (1.1) we get that

$$(s-t-1)^{(\mu-1)} = (s-t-\mu+1)^{\overline{\mu-1}}. \tag{2.5}$$

Therefore by (2.4) we also derive

$$R^*(t) = \frac{1}{\Gamma(\mu)} \sum_{s=t+\mu}^{b^*-\nu} (s-t-\mu+1)^{\overline{\mu-1}} (\nabla_{b^*-}^{\mu} f)(s). \tag{2.6}$$

Corollary 2.6. *Suppose the hypotheses of Theorem 2.4 hold. Assume further that*

$$\nabla^k f(b^*) = 0, \text{ for } k = 0, 1, \ldots, m-1.$$

Then

$$f(t) = \frac{1}{\Gamma(\mu)} \sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{(\mu-1)} (\nabla_{b^*-}^{\mu} f)(s). \tag{2.7}$$

We need the following result.

Proposition 2.7. *We have*

$$\sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{(\mu-1)} = \frac{(b^*-\nu-t)^{(\mu)}}{\mu} > 0. \tag{2.8}$$

Proof. We proved in [1, p. 579] that

$$\frac{\Gamma(x+1)}{\Gamma(x-k+1)} = \frac{1}{(k+1)}\left(\frac{\Gamma(x+2)}{\Gamma(x-k+1)} - \frac{\Gamma(x+1)}{\Gamma(x-k)}\right),$$

$x > k$, $x, k \in \mathbb{R}$; $k > -1$, $x > -1$. We have that

$$\sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{(\mu-1)} = \sum_{s=t+\mu}^{b^*-\nu} \frac{\Gamma(s-t)}{\Gamma(s-t-\mu+1)} = \sum_{s=t+\mu+1}^{b^*-\nu} \frac{\Gamma(s-t)}{\Gamma(s-t-\mu+1)} + \Gamma(\mu).$$

Here $\mu > 0$ and $t+\mu+1 \leq s \leq b^* - \nu$. See that $x := s-t-1 \geq \mu$, $k := \mu - 1 > -1$, $x > k$, and $x - k = s - t - \mu$. Therefore

$$\begin{aligned}
\frac{\Gamma(s-t)}{\Gamma(s-t-\mu+1)} &= \frac{\Gamma(x+1)}{\Gamma(x+1-k)} = \frac{\Gamma(x+1)}{\Gamma(x-k+1)} \\
&= \frac{1}{\mu}\left(\frac{\Gamma(x+2)}{\Gamma(x-k+1)} - \frac{\Gamma(x+1)}{\Gamma(x-k)}\right) \\
&= \frac{1}{\mu}\left(\frac{\Gamma(s-t+1)}{\Gamma(s-t-\mu+1)} - \frac{\Gamma(s-t)}{\Gamma(s-t-\mu)}\right).
\end{aligned}$$

Consequently we have

$$\begin{aligned}
\sum_{s=t+\mu+1}^{b^*-\nu} \frac{\Gamma(s-t)}{\Gamma(s-t-\mu+1)} &= \frac{1}{\mu}\sum_{s=t+\mu+1}^{b^*-\nu}\left(\frac{\Gamma(s-t+1)}{\Gamma(s-t+1-\mu)} - \frac{\Gamma(s-t)}{\Gamma(s-t-\mu)}\right) \\
(s-t &\geq \mu+1) \text{ (telescoping sum)} \\
&= \frac{1}{\mu}\Big[(\Gamma(\mu+2)-\Gamma(\mu+1)) + \\
&\left(\frac{\Gamma(\mu+3)}{2} - \Gamma(\mu+2)\right) + \\
&\left(\frac{\Gamma(\mu+4)}{3!} - \frac{\Gamma(\mu+3)}{2!}\right) + \\
&\left(\frac{\Gamma(\mu+5)}{4!} - \frac{\Gamma(\mu+4)}{3!}\right) + \ldots \\
(\text{so for } s &= b^* - \nu = t+\mu+1+\lambda,\ \lambda \in \mathbb{N}) \\
&\ldots + \frac{\Gamma(\mu+1+\lambda+1)}{\Gamma(2+\lambda)} - \frac{\Gamma(\mu+1+\lambda)}{\Gamma(1+\lambda)}\Big] \\
&= \frac{\Gamma(\mu+\lambda+2)}{\mu\Gamma(2+\lambda)} - \frac{\Gamma(\mu+1)}{\mu} \\
(\text{By } b^*-\nu &= t+\mu+1+\lambda, \text{ then } b^*-\nu+1 = t+\mu+2+\lambda \text{ and} \\
b^*-\nu+1-t &= \mu+2+\lambda, \text{ that is } \lambda+2 = b^*-\nu+1-t-\mu. \\
\text{Also } \frac{\Gamma(\mu+1)}{\mu} &= \Gamma(\mu).) \\
&= \frac{\Gamma(b^*-\nu-t+1)}{\mu\Gamma(b^*-\nu-t+1-\mu)} - \Gamma(\mu) \\
&= \frac{(b^*-\nu-t)^{(\mu)}}{\mu} - \Gamma(\mu).
\end{aligned}$$

So we have proved

$$\sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{\overline{(\mu-1)}} = \frac{(b^*-\nu-t)^{(\mu)}}{\mu}.$$

The last is positive because

$$b^* - \nu - t - \mu - 1 \geq 0$$

and

$$b^* - \nu - t - \mu + 1 \geq 2,$$

also $b^* - \nu - t + 1 \geq \mu + 2 > 0$, and $b^* - \nu - t > 0$. □

We give the following estimate.

Theorem 2.8. *In the assumptions of Theorem 2.4 we have that*

$$\left| f(t) - \sum_{k=0}^{m-1} \nabla^k f(b^*) \frac{(t-b^*)^{\overline{k}}}{k!} \right| = \tag{2.9}$$
$$|R^*(t)| \leq \frac{(b^* - \nu - t)^{(\mu)}}{\Gamma(\mu+1)} \max_{s \in \{t+\mu,\dots,b^*-\nu\}} \left| \nabla^{\mu}_{b^*-} f(s) \right|.$$

Proof. By (2.4) and (2.8) we get

$$\begin{aligned} |R^*(t)| &\leq \frac{1}{\Gamma(\mu)} \left| \sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{(\mu-1)} \right| \max_{s \in \{t+\mu,\dots,b^*-\nu\}} \left| \nabla^{\mu}_{b^*-} f(s) \right| \\ &= \frac{1}{\Gamma(\mu)} \frac{(b^* - \nu - t)^{(\mu)}}{\mu} \max_{s \in \{t+\mu,\dots,b^*-\nu\}} \left| \nabla^{\mu}_{b^*-} f(s) \right| \\ &= \frac{(b^* - \nu - t)^{(\mu)}}{\Gamma(\mu+1)} \max_{s \in \{t+\mu,\dots,b^*-\nu\}} \left| \nabla^{\mu}_{b^*-} f(s) \right|. \end{aligned}$$

□

We need the following auxiliary result.

Lemma 2.9 (See [1, p. 580]). *Let* $a > \nu$, $a, \nu > -1$, $a, \nu \in \mathbb{R}$, $a \leq b$. *Then*

$$\sum_{r=a}^{b} r^{(\nu)} = \left(\frac{(b+1)^{(\nu+1)} - a^{(\nu+1)}}{\nu+1} \right). \tag{2.10}$$

We give a related Ostrowski inequality.

Theorem 2.10. *Let* $a \in \mathbb{R}$, $f : (a + \mathbb{Z}) \to \mathbb{R}$. *Let* $\mu > 0, m - 1 < \mu \leq m$, $m \in \mathbb{N}$, $\nu = m - \mu$. *Let* $a^*, b^* \in (a + \mathbb{Z}) : b^* - a^* \geq m + 1$. *Assume that* $\nabla^k f(b^*) = 0$, $k = 1, \dots, m-1$. *Then*

$$\left| \frac{1}{(b^* - m - a^* + 1)} \sum_{t=a^*}^{b^*-m} f(t) - f(b^*) \right| \tag{2.11}$$
$$\leq \frac{(b^* - \nu - a^* + 1)^{(\mu+1)}}{\Gamma(\mu+2)(b^* - m - a^* + 1)} \left(\max_{s \in \{a^*+\mu,\dots,b^*-\nu\}} \left| \nabla^{\mu}_{b^*-} f(s) \right| \right).$$

Proof. Using (2.3) and (2.4), since $\nabla^k f(b^*) = 0$, $k = 1, \ldots, m-1$, we get

$$f(t) - f(b^*) = R^*(t).$$

Then we observe that

$$\begin{aligned}
E \quad : \quad &= \left| \frac{1}{(b^* - m - a^* + 1)} \sum_{t=a^*}^{b^*-m} f(t) - f(b^*) \right| \\
&= \frac{1}{(b^* - m - a^* + 1)} \left| \sum_{t=a^*}^{b^*-m} (f(t) - f(b^*)) \right| \\
&\leq \frac{1}{(b^* - m - a^* + 1)} \left(\sum_{t=a^*}^{b^*-m} |f(t) - f(b^*)| \right) \\
&= \frac{1}{(b^* - m - a^* + 1)} \sum_{t=a^*}^{b^*-m} |R^*(t)| \\
&\quad \text{(by (2.9))} \\
&\leq \frac{1}{(b^* - m - a^* + 1)} \left(\sum_{t=a^*}^{b^*-m} (b^* - \nu - t)^{(\mu)} \right) \frac{\max_{s \in \{a^*+\mu, \ldots, b^*-\nu\}} \left| \nabla_{b^*-}^{\mu} f(s) \right|}{\Gamma(\mu+1)} \\
&= \ : (*).
\end{aligned}$$

Call $\tau := b^* - \nu - t$. Here $a^* \leq t \leq b^* - m$, and $a^* \geq -t \geq m - b^*$, which implies that $\mu \leq \tau \leq b^* - \nu - a^*$. Also $\mu^{(\mu)} = \Gamma(\mu + 1)$.

We calculate

$$\begin{aligned}
\sum_{t=a^*}^{b^*-m} (b^* - \nu - t)^{(\mu)} &= \sum_{\tau=\mu}^{b^*-\nu-a^*} \tau^{(\mu)} \\
&= \mu^{(\mu)} + \sum_{\tau=\mu+1}^{b^*-\nu-a^*} \tau^{(\mu)} \\
&= \Gamma(\mu+1) + \sum_{\tau=\mu+1}^{b^*-\nu-a^*} \tau^{(\mu)} \\
(\text{notice } b^* - a^* &\geq m + 1 \text{ by assumption, hence } \mu + 1 \leq b^* - \nu - a^*) \\
&\quad \text{(by (2.10))} \\
&= \Gamma(\mu+1) + \left(\frac{(b^* - \nu - a^* + 1)^{(\mu+1)} - (\mu+1)^{(\mu+1)}}{\mu+1} \right) \\
&= \Gamma(\mu+1) + \left(\frac{(b^* - \nu - a^* + 1)^{(\mu+1)}}{\mu+1} - \frac{\Gamma(\mu+2)}{\mu+1} \right) \\
&= \Gamma(\mu+1) + \frac{(b^* - \nu - a^* + 1)^{(\mu+1)}}{\mu+1} - \Gamma(\mu+1)
\end{aligned}$$

$$= \quad \frac{(b^* - \nu - a^* + 1)^{(\mu+1)}}{\mu + 1}.$$

So we have proved that

$$\sum_{t=a^*}^{b^*-m} (b^* - \nu - t)^{(\mu)} = \frac{(b^* - \nu - a^* + 1)^{(\mu+1)}}{\mu + 1}.$$

Therefore

$$(*) \quad = \quad \frac{1}{(b^* - m - a^* + 1)} \left(\frac{(b^* - \nu - a^* + 1)^{(\mu+1)}}{\mu + 1} \right) \frac{\max_{s \in \{a^*+\mu, \ldots, b^*-\nu\}} \left| \nabla^{\mu}_{b^*-} f(s) \right|}{\Gamma(\mu + 1)}.$$

That is,

$$E \leq \frac{(b^* - \nu - a^* + 1)^{(\mu+1)} \max_{s \in \{a^*+\mu, \ldots, b^*-\nu\}} \left| \nabla^{\mu}_{b^*-} f(s) \right|}{\Gamma(\mu + 2)(b^* - m - a^* + 1)},$$

which establishes the theorem. □

A related Poincaré type inequality follows.

Theorem 2.11. *Suppose the hypotheses of Theorem 2.4 hold. Assume*

$$\nabla^k f(b^*) = 0, \; k = 0, 1, \ldots, m - 1.$$

Let $p, q > 1$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\sum_{t=a^*}^{b^*-m} |f(t)|^q \quad \leq \quad \frac{1}{(\Gamma(\mu))^q} \left(\sum_{t=a^*}^{b^*-m} \left(\left(\sum_{s=t+\mu}^{b^*-\nu} (s - t - 1)^{p(\mu-1)} \right)^{\frac{q}{p}} \right) \right) \left(\sum_{s=a^*+\mu}^{b^*-\nu} \left| \nabla^{\mu}_{b^*-} f(s) \right|^q \right). \tag{2.12}$$

Proof. Notice here $s - t \geq \mu > 0$ and $s - t - \mu \geq 0$ and $s - t - \mu + 1 > 0$, so that

$$(s - t - 1)^{(\mu-1)} = \frac{\Gamma(s - t)}{\Gamma(s - t - \mu + 1)} > 0.$$

By (2.7) we get

$$\begin{aligned}
|f(t)| &\leq \frac{1}{\Gamma(\mu)} \sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{(\mu-1)} \left|(\nabla_{b^*-}^{\mu} f)(s)\right| \\
&\leq \frac{1}{\Gamma(\mu)} \left(\sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{p(\mu-1)} \right)^{\frac{1}{p}} \left(\sum_{s=t+\mu}^{b^*-\nu} \left|(\nabla_{b^*-}^{\mu} f)(s)\right|^q \right)^{\frac{1}{q}} \\
&\leq \frac{1}{\Gamma(\mu)} \left(\sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{p(\mu-1)} \right)^{\frac{1}{p}} \left(\sum_{s=a^*+\mu}^{b^*-\nu} \left|(\nabla_{b^*-}^{\mu} f)(s)\right|^q \right)^{\frac{1}{q}}.
\end{aligned}$$

Therefore we found

$$|f(t)| \leq \frac{1}{\Gamma(\mu)} \left(\sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{p(\mu-1)} \right)^{\frac{1}{p}} \left(\sum_{s=a^*+\mu}^{b^*-\nu} \left|(\nabla_{b^*-}^{\mu} f)(s)\right|^q \right)^{\frac{1}{q}}. \tag{2.13}$$

Hence

$$|f(t)|^q \leq \frac{1}{(\Gamma(\mu))^q} \left(\sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{p(\mu-1)} \right)^{\frac{q}{p}} \left(\sum_{s=a^*+\mu}^{b^*-\nu} \left|(\nabla_{b^*-}^{\mu} f)(s)\right|^q \right). \tag{2.14}$$

Applying $\sum_{t=a^*}^{b^*-m}$ to both sides of (2.14) we derive (2.12). □

We finish with a related Sobolev type inequality.

Theorem 2.12. *Suppose the hypotheses of Theorem 2.11 hold. Let $r \geq 1$. Then*

$$\begin{aligned}
\left(\sum_{t=a^*}^{b^*-m} |f(t)|^r \right)^{\frac{1}{r}} \leq\ & \frac{1}{\Gamma(\mu)} \\
& \left(\sum_{t=a^*}^{b^*-m} \left(\left(\sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{p(\mu-1)} \right)^{\frac{r}{p}} \right) \right)^{\frac{1}{r}} \\
& \left(\sum_{s=a^*+\mu}^{b^*-\nu} \left|\nabla_{b^*-}^{\mu} f(s)\right|^q \right)^{\frac{1}{q}}.
\end{aligned} \tag{2.15}$$

Proof. Raising (2.13) to the power of r, we obtain

$$|f(t)|^r \leq \frac{1}{(\Gamma(\mu))^r} \left(\sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{p(\mu-1)} \right)^{\frac{r}{p}} \left(\sum_{s=a^*+\mu}^{b^*-\nu} \left|(\nabla_{b^*-}^{\mu} f)(s)\right|^q \right)^{\frac{r}{q}}.$$

Hence we have

$$\sum_{t=a^*}^{b^*-m} |f(t)|^r \leq \frac{1}{(\Gamma(\mu))^r} \left(\sum_{t=a^*}^{b^*-m} \left(\sum_{s=t+\mu}^{b^*-\nu} (s-t-1)^{p(\mu-1)} \right)^{\frac{r}{p}} \right) \left(\sum_{s=a^*+\mu}^{b^*-\nu} \left| (\nabla_{b^*-}^{\mu} f)(s) \right|^q \right)^{\frac{r}{q}}. \tag{2.16}$$

Raising (2.16) to the power of $\frac{1}{r}$, we derive (2.15). □

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