International Journal of Difference Equations
ISSN 0973-6069, Volume 5, Number 2, pp. 266–271 (2010)
http://campus.mst.edu/ijde

Asymptotic Behavior for Nonoscillatory Solutions of Nonlinear Delay Difference Equations

Yu-Ping Zhao and **Xi-Lan Liu**
Qinghai Nationalities University
Department of Mathematics and Statistics
Xining, Qinghai 810007, China
zhyp1975@163.com, doclanliu2002@yahoo.com.cn

Abstract

This paper is concerned with the difference equation with several delays in the form of $\Delta\left(y_n + \sum_{i=0}^{L} p_i y_{n-k_i} - \sum_{j=0}^{M} r_j y_{n-\rho_j}\right) + f(n, y_{n-\iota}) = 0$ where $f(n,u)$ is continuous. We study the asymptotic behavior for nonoscillatory solution of the equation, which improves some known results.

AMS Subject Classifications: 39A11.
Keywords: Asymptotic behavior, nonlinear difference equation, oscillation.

1 Introduction and Preliminaries

In this work, we consider the nonlinear delay difference equation

$$\Delta\left(y_n + \sum_{i=0}^{L} p_i y_{n-k_i} - \sum_{j=0}^{M} r_j y_{n-\rho_j}\right) + f(n, y_{n-\iota}) = 0 \tag{1.1}$$

with the initial conditions

$$y_n = \phi_n, \text{ for } n \in \{-s, \dots, 0\}, \tag{1.2}$$

where $p_i \geq 0$, $k_i \in Z^+(1)$ for $i = 0, 1, 2, \dots, L$, $r_j \geq 0$, $\rho_j \in Z^+(1)$ for $j = 0, 1, 2, \dots, M$, $\iota \in Z^+(0)$, $k = \max_{0\leq i\leq L}\{k_i\}$, $\rho = \max_{0\leq j\leq M}\{\rho_j\}$, $s = \max\{k, \rho, \iota\}$ for

Received April 17, 2010; Accepted August 6, 2010
Communicated by Wan-Tong Li

$n \in Z^+(0)$, and the continuous function $f(n,u) : N \times R \to R$ is increasing in u and n and satisfies $f(n,u)u > 0$ for $u \neq 0$.

Recently, many papers concerning asymptotic behavior for nonoscillatory solutions of difference equations were published, see [1–5]. In [2], the authors investigated the asymptotic behavior of linear difference equations of the form

$$\Delta\left(y_n + \sum_{i=1}^{L} p_i y_{n-k_i} - \sum_{j=1}^{M} r_j y_{n-\rho_j}\right) + q_n y_{n-\iota} = 0. \tag{1.3}$$

Our goal of the present paper is to consider the equation (1.1) which is nonlinear and can be reduced to (1.3) by letting $f(n,u) = q_n u$, and hence the results of this paper are more general. We obtain the asymptotic behavior for nonoscillatory solutions of the equation (1.1) under some hypotheses.

Throughout this paper, we shall use the following conditions:

(H_1) $L > 0$ and $M = 0$;

(H_2) $L = 0, M > 0$ and $0 < \sum_{j=0}^{M} r_j \leq 1$;

(H_3) $L > 0, M > 0$ and $0 < \sum_{j=0}^{M} r_j \leq 1$;

(H_4) $L = 0, M > 0$ and $\sum_{j=0}^{M} r_j > 1$;

(H_5) $L > 0,\ M > 0$ and $\sum_{j=0}^{M} r_j > 1$.

2 Main Results

Theorem 2.1. *If one of the conditions* (H_1), (H_2), *and* (H_3) *is satisfied, then every nonoscillatory solution of the problem* (1.1) *and* (1.2) *tends to* 0 *as* $n \to \infty$. *If* (H_4) *is satisfied, then every nonoscillatory solution of the problem* (1.1) *and* (1.2) *tends to* ∞ *or* $-\infty$ *as* $n \to \infty$.

Proof. Without loss of the generality, we let $\{y_n\}$ be an eventually positive solution of the problem (1.1) and (1.2). Then there exists $n_1 \in Z^+(1)$ such that $y_n > 0$ for $n \in Z^+(n_1)$. It follows that y_{n-k_i}, $y_{n-\rho_j}$ and $y_{n-\iota} > 0$ for $n \in Z^+(n_2)$, $i = 0, 1, 2, \ldots, L$ and $j = 0, 1, 2, \ldots, M$, where $n_2 = n_1 + s$. Let

$$z_n = y_n + \sum_{i=0}^{L} p_i y_{n-k_i} - \sum_{j=0}^{M} r_j y_{n-\rho_j} \text{ for } n \in Z^+(0). \tag{2.1}$$

First, we consider condition (H_1), i.e., $L > 0$ and $M = 0$. From (1.1) and (2.1), we have

$$\Delta z_n = -f(n, y_{n-\iota}) < 0 \text{ and } z_n > 0 \text{ for } n \in Z^+(n_2). \tag{2.2}$$

In what follows, we shall show that $\lim_{n\to\infty} y_n = 0$. If it is not true, then there exists an infinite subsequence $\{n^{(i)}\} \subset \{n\}$ such that $\lim_{i\to\infty} y_{n^{(i)}-\iota} = b > 0$. So, we can take a sequence of subsets $N_i \subset Z^+(0)$ such that $n^{(i)} \in N_i$ for $i \in Z^+(0)$. We choose $i_1 \in Z^+(1)$ such that $n \in N_i$ and $y_{n-\iota} > \frac{b}{2}$, for $i \in Z^+(i_1)$. Thus,

$$f(n, y_{n-\iota}) > f\left(n, \frac{b}{2}\right) \text{ for } n \in N_i \text{ and } i \in Z^+(i_1). \tag{2.3}$$

For any $n \in Z^+(n_2)$, there exists $i_2 \in Z^+(i_1)$ such that $\sup N_{i_2} \le n \le \sup N_{i_2+1}$. It follows from (2.2) and (2.3) that

$$z_{n+1} - z_{\inf N_{i_2}} \le z_{\sup N_{i_2}+1} - z_{\inf N_{i_2}} \le -\sum_{n\in N_{i_2}} f\left(n, \frac{b}{2}\right).$$

which implies that $z_n \to -\infty$ as $n \to \infty$. This is a contradiction. Hence, $\lim_{n\to\infty} y_n = 0$.

Next, we consider condition (H_2), i.e., $L = 0, M > 0$ and $0 < \sum_{j=0}^{M} r_j \le 1$. From (1.1) and (2.1), it is easy to see that (2.2) holds. If it is not true, then $z_n \le 0$ for $n \in Z^+(n_2)$, and there exist $n_3 \in Z^+(n_2)$ and $\beta \ge 0$ such that $z_n \le -\beta$ for $n \in Z^+(n_3)$. That is,

$$y_n \le -\beta + \sum_{j=0}^{M} r_j y_{n-\rho_j}. \tag{2.4}$$

In this situation, we consider the following two cases.

Case I. $\{y_n\}$ is unbounded, i.e., $\limsup_{n\to\infty} y_n = \infty$. Thus, there exists a subsequence $\{n_s\}_{s=1}^{\infty} \subset \{0, 1, \ldots\}$ such that $n_s \ge n_2 + \rho$ and $n_s \to \infty$ as $s \to \infty$ and $y_{n_s} = \max\{y_n\}$, where $\rho = \max_{0\le j\le M} \rho_j$. In view of (2.4), we find that

$$y_{n_s} \le -\beta + \sum_{j=0}^{M} r_j y_{n_s-\rho_j} \le -\beta + \sum_{j=0}^{M} r_j y_{n_s},$$

which is a contradiction.

Case II. $\{y_n\}$ is bounded, i.e., $\limsup_{n\to\infty} y_n = b < \infty$. Choose a subsequence $\{n_s^*\}_{s=1}^{\infty} \subset \{0, 1, ...\}$ with $n_s^* \to \infty$ and $y_{n_s^*} \to b$ as $s \to \infty$. Let $y_{r_s} = \max_{0\le i\le\rho} \{y_{n_s^*-i}\}$.

Then $r_s \to \infty$ as $s \to \infty$ and $\limsup_{s\to\infty} y_{r_s} \le b$. Then by (2.4), we get

$$y_{n_s^*} \le -\beta + \sum_{j=0}^{M} r_j y_{n_s^*-\rho_j} \le -\beta + \sum_{j=0}^{M} r_j y_{n_{r_s}}.$$

Taking the superior limit as $s \to \infty$, we get

$$b \le -\beta + \sum_{j=0}^{M} r_j (\limsup_{s\to\infty} y_{r_s}) \le -\beta + \sum_{j=0}^{M} r_j b,$$

which is also a contradiction. The remainder of the proof is similar to the first part, and hence we omit it here.

We consider condition (H$_3$), i.e., $L > 0$, $M > 0$ and $0 < \sum_{j=0}^{M} r_j \le 1$. From (1.1) and (2.1), we can find that (2.2) holds. If it is not the case, then $z_n \le 0$ for $n \in Z^+(n_2)$. So there exist $n_3 \in Z^+(n_2)$ and $\beta \ge 0$ such that $z_n \le -\beta$ for $n \in Z^+(n_3)$, that is,

$$y_n \le -\beta - \sum_{i=0}^{L} p_i y_{n-k_i} + \sum_{j=0}^{M} r_j y_{n-\rho_j} \le -\beta + \sum_{j=0}^{M} r_j y_{n-\rho_j}$$

The proof of the following is the same as in the second part, and hence we omit it.

Finally, we consider condition (H$_4$), i.e., $L = 0, M > 0$ and $\sum_{j=0}^{M} r_j > 1$. Without loss of generality, we let y_n be an eventually positive solution of the problem (1.1) and (1.2). We first show that it is impossible that $\lim_{n\to\infty} y_n = 0$. If it is not the case, then $\lim_{n\to\infty} z_n = 0$ by (2.1). But $\Delta z_n = -f(n, y_{n-\iota}) < 0$ for $n \in Z^+(n_2)$. So $z_n \ge 0$ for $n \in Z^+(n_2)$. Thus, we have $y_n - \sum_{j=0}^{M} r_j y_{n-\rho_j} \ge 0$ for $n \in Z^+(n_2)$. We may let $y_{n_i} = \min_{0\le s\le M}\{y_{n-\rho_s}\}$. So we have $y_n \ge \sum_{j=0}^{M} r_j y_{n_i}$. Because of $\sum_{j=0}^{M} r_j > 1$, we have $y_n \to \infty$ as $n \to \infty$. This is a contradiction.

In the following, we shall show that $\lim_{n\to\infty} y_n = \infty$. If it is not the case, then there exists a sequence of points $\{n^{(i)}\} \subset Z^+(0)$ such that $0 < \lim_{i\to\infty} y_{n^{(i)}-\iota} = b < \infty$. We choose $n_3 \in Z^+(n_2)$ and $i_1 \in Z^+(1)$ such that $n^{(i)} \in Z^+(n_3)$ and $y_{n^{(i)}-\iota} > \frac{b}{2}$ for $i \in Z^+(i_1)$. As in the proof of the first part, we have $\lim_{n\to\infty} z_n = -\infty$. Now, let $y_{n-k} = \max_{0\le j\le M}\{y_{n-\rho_j}\}$. Dividing both sides of (2.1) by y_{n-k}, we have

$$\frac{z_n}{y_{n-k}} = \frac{y_n}{y_{n-k}} - \frac{\sum_{j=0}^{M} r_j y_{n-\rho_j}}{y_{n-k}} \ge -\sum_{j=0}^{M} r_j \text{ for } n \in Z^+(n_2). \tag{2.5}$$

Taking $\{n^{(i)}\}, n^{(i)} \in Z^+(n_3)$ for $i \in Z^+(i_2)$, we obtain that $\lim\limits_{i\to\infty} \dfrac{z_{n^{(i)}}}{y_{n^{(i)}-k}} = -\infty$. This contradicts to (2.5). The proof is complete. $\square$

Theorem 2.2. *Assume that $f(n,u) \geq q_n u^\alpha$ for all $u > 0$, where $\alpha \in (0,1)$ is a ratio of odd positive integer and $q_n > 0$. Suppose that for any sequence of subsets $N_i \subset Z^+(0)$ such that $\sum\limits_{n\in N_i} q_n = \infty$. Then for every nonoscillatory solution $\{y_n\}$ of the problem* (1.1) *and* (1.2) *with* (H_5)*, either $\liminf\limits_{n\to\infty} y_n = 0$ or $\limsup\limits_{n\to\infty} y_n = \infty$.*

Proof. Without loss of generality, we let $\{y_n\}$ be an eventually positive solution of the problem (1.1) and (1.2). So, there exists $n_1 \in Z^+(1)$ such that $y_n > 0$ for $n \in Z^+(n_1)$. It follows that y_{n-k_i}, $y_{n-\rho_j}$ and $y_{n-\iota} > 0$, for $n \in Z^+(n_2)$, $i = 0,1,2,\ldots,L$ and $j = 0,1,2,\ldots,M$, where $n_2 = n_1 + s$. Let

$$z_n = y_n + \sum_{i=0}^{L} p_i y_{n-k_i} - \sum_{j=0}^{M} r_j y_{n-\rho_j} \text{ for } n \in Z^+(0). \tag{2.6}$$

From (1.1) and (2.6), we have

$$\Delta z_n = -f(n, y_{n-\iota}) \leq -q_n y_{n-\iota}^\alpha < 0 \text{ for } n \in Z^+(n_2). \tag{2.7}$$

Therefore there exist two possibilities: (i) $z_n > 0$ eventually or (ii) $z_n \leq 0$ eventually.

If (i) holds, as in the proof of Theorem 2.1, we can prove $\lim\limits_{n\to\infty} y_n = 0$; if (ii) holds, then $0 > \lim\limits_{n\to\infty} z_n = l \geq -\infty$.

First, we assume that $-\infty < l < 0$. So, we can take a sequence of subsets $N_i \subset Z^+(0)$ such that $n^{(i)} \in N_i$ for $i \in Z^+(0)$. For any $n \in Z^+(n_2)$, there exists $i_2 \in Z^+(i_1)$ such that $\sup N_{i_2} \leq n \leq \sup N_{i_2+1}$. It follows from (2.7)

$$z_{n+1} - z_{\inf N_{i_2}} \leq z_{\sup N_{i_2}+1} - z_{\inf N_{i_2}} \leq -\sum_{n\in N_{i_2}} q_n y_{n-\iota}^\alpha.$$

$$z_{n+1} - z_{\inf N_{i_2}} \leq -\sum_{n\in N_{i_2}} q_n y_{n-\iota}^\alpha,$$

which implies that

$$\sum_{n\in N_{i_2}} q_n y_{n-\iota}^\alpha < \infty.$$

For $\sum\limits_{n\in N_{i_2}} q_n = \infty$, we have $\liminf\limits_{n\to\infty} y_n = 0$.

Next, we assume $l = -\infty$. Then by (2.6), we have $\limsup\limits_{n\to\infty} y_n = \infty$. The proof is complete. $\square$

Acknowledgement

The authors sincerely express their thanks for the referees' helpful suggestions. This work was supported by the NNSF of China (10961020), the Key Project of Chinese Ministry of Education (208154), the Chunhui Project of Chinese Ministry of Education (Z2009-1-81007).

References

[1] R.P. Agarwal, Difference Equations and Inequalities, Theory, Methods, and Applications, 2nd ed., Monographs and Textbooks in Pure and Applied Mathematics, Vol. 228, Marcel Dekker, New York 2000.

[2] M.R. Xu, B. Shi and X.Y. Zeng, Asymptotic behavior for non-oscillatory solutions of difference equations with several delays in the neutral term, J. Comput. Appl. Math. 27 (2008), 33–45.

[3] B. Shi, Z.C. Wang and J.S. Yu., Asymptotic constancy of solutions of linear parabolic Volterra difference equations, J. Comput. Math. Appl. 32 (1996), 65–77.

[4] J.W. Luo and Bainov.D.D, Oscillation and asymptotic behavior of second-order neutral difference equations with maxima, J. Comput. Appl. Math. 131 (2001) 333–341.

[5] J. Morchalo, On convergence of solutions of a second order nonlinear difference equations, J. Appl. Math. Comput. 12 (2003), 59–66.