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Oscillation in Nonlinear Neutral Difference Equations with Positive and Negative Coefficients

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Abstract

In this paper, we study the oscillatory and asymptotic behaviour of solutions of a class of nonlinear second order difference equations of the form

$$\Delta^2[y(n) + p(n)y(n-m)] + q(n)G(y(n-\sigma)) - r(n)H(y(n-\tau)) = 0$$

under various ranges of $p(n)$. The nonlinear functions G and H could be linear, sublinear or superlinear.

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1 Introduction

Functional difference equations are gaining interest because they are the discrete analogue of differential equations. They also have physical applications as evident by [9] and [14]. Another reason can be attributed to the development of "time scales" and time scale calculus, where we find the unification of continuous and discrete aspects (see

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e.g., [13]). Keeping this fact in view, an attempt is made here to study the second order nonlinear neutral delay difference equation with positive and negative coefficients of the form

$$\Delta^2[y(n) + p(n)y(n-m)] + q(n)G(y(n-\sigma)) - r(n)H(y(n-\tau)) = 0, \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta y(n) = y(n+1) - y(n), p, q$ and r are real-valued functions such that $q(n) > 0, r(n) \geq 0$ for all $n \in N(0) = \{0, 1, 2, 3, 4 \ldots\}$ and $G, H \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing functions satisfying $xG(x) > 0, xH(x) > 0$ for $x \neq 0$, and m, σ, τ are positive integers.

Oscillation of first and second order neutral delay differential / difference equations with either positive or negative coefficients of the terms involving delay is well developed (see e.g., [1,2,4,11]). But neutral equations with positive and negative coefficients are of special interest. Some authors have studied the oscillatory behaviour of solutions of first order neutral delay equations with positive and negative coefficients. In this direction, we refer the readers to [2, 3, 6, 10, 12]. However, we find a very few works about the oscillatory behaviour of solutions of second order neutral equations (see e.g., [5, 7, 8]). In this work, our purpose is to establish some oscillation criteria for a class of nonlinear difference equations of the form (1.1) for various ranges of $p(n)$.

In this work, it has been observed that the nonlinear equation (1.1) immediately converts into a first order difference inequality which further initiates the analysis that has been incorporated here. With this, our object is achieved by using the assumption

(H_0) $\sum_{n=0}^{\infty} nr(n) < \infty.$

By a solution of equation (1.1), we understand a real-valued function $y(n)$ defined on $N(-\rho) = \{-\rho, -\rho+1, \ldots\}$ which satisfies (1.1) for $n \geq 0$ where $\rho = \max\{m, \tau, \sigma\}$. If

$$y(n) = A_n, n = -\rho, -\rho+1, \ldots, 0 \tag{1.2}$$

are given, then (1.1) admits a unique solution satisfying the initial condition (1.2). A solution $y(n)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. Eq. (1.1) or its solution space is said to be oscillatory if all solutions are oscillatory. In other words, a solution $y(n)$ of (1.1) is said to be oscillatory, if for every integer $N > 0$, there exists an $n \geq N$ such that $y(n)y(n+1) \leq 0$.

2 Oscillation Results

In this section, sufficient conditions are obtained for the oscillation of solutions of Eq. (1.1). To facilitate the discussion, we begin with the assumption (H_0).

Theorem 2.1. *Let* $0 \leq p(n) \leq a < \infty, \sigma + 1 \geq m$. *Suppose that*

(H_1) *there exists* $\lambda > 0$ *such that* $G(u) + G(v) \geq \lambda G(u + v), u \in \mathbb{R}, v \in \mathbb{R}$,

(H_2) $G(u)G(v) \geq G(uv)$ *for* $u > 0, v > 0$,

(H_3) $G(-u) = -G(u)$ *for* $u \in \mathbb{R}$,

(H_4) $\sum_{n=m}^{\infty} Q(n) = \infty, Q(n) = \min\{q(n), q(n-m)\}, n \geq m$,

(H_5) G *is sublinear and* $\int_0^{\pm c} \frac{du}{G(u)} < \infty, c > 0$

hold. Then every solution of (1.1) *either oscillates or tends to zero as* $n \to \infty$.

Proof. On the contrary, let $y(n)$ be a nonoscillatory solution of (1.1). Then there exists $n_0 > 0$ such that $y(n) > 0$ or < 0 for $n \geq n_0 > \rho$. Assume that $y(n) > 0$ for $n \geq n_0$. Define

$$k(n) = \sum_{s=n}^{\infty} (s - n + 1)r(s)H(y(s - \tau)).$$

Clearly, $\Delta k(n) < 0$ and $\Delta^2 k(n) > 0$ for large n. It is easy to verify that (H_0) implies

$$\sum_{s=n}^{\infty} (s - n + 1)r(s) < \infty$$

and hence $\lim_{n\to\infty} k(n)$ exists. Setting

$$z(n) = y(n) + p(n)y(n - m), \qquad w(n) = z(n) - k(n) \tag{2.1}$$

for $n \geq n_1 > n_0 + 2\rho$, it follows from Eq. (1.1) that

$$\Delta^2 w(n) + q(n)G(y(n - \sigma)) = 0, \tag{2.2}$$

that is,

$$\Delta^2 w(n) = -q(n)G(y(n - \sigma)) \leq 0, \neq 0.$$

Hence $\Delta w(n)$ is nonincreasing on $[n_1, \infty)$. If $\Delta w(n) < 0$ for $n \geq n_2 > n_1$, then (2.2) yields

$$\Delta w(n + 1) + q(n)G(y(n - \sigma)) = \Delta w(n),$$

that is,

$$\Delta z(n + 1) + q(n)G(y(n - \sigma)) = \Delta k(n + 1) + \Delta w(n) < 0$$

for $n \geq n_2$. Consequently, for $n \geq n_2$

$$\begin{aligned}\Delta z(n+1) + q(n)G(y(n-\sigma)) + G(a)\Delta z(n-m+1) \\ + G(a)q(n-m)G(y(n-m-\sigma)) < 0.\end{aligned}$$

Using (H_1) and (H_2), the last inequality becomes

$$\Delta z(n+1) + G(a)\Delta z(n-m+1) + \lambda Q(n)G(z(n-\sigma)) < 0, \tag{2.3}$$

where $0 < z(n) < y(n) + ay(n-m)$ for $n \geq n_2$. Since $\Delta w(n) < 0$, we have $\Delta z(n) < 0$ for $n \geq n_3 > n_2$. Choose u and v such that $z(n+2) < u < z(n+1)$ and $z(n-m+2) < v < z(n-m+1)$ for $n \geq n_3$. Since $n - \sigma < n+1$, inequality (2.3) becomes

$$\lambda Q(n) + \frac{\Delta z(n+1)}{G(z(n-\sigma))} + G(a)\frac{\Delta z(n-m+1)}{G(z(n-\sigma))} < 0,$$

that is,

$$\lambda Q(n) + \int_{z(n+1)}^{z(n+2)} \frac{du}{G(u)} + G(a)\int_{z(n-m+1)}^{z(n-m+2)} \frac{dv}{G(v)} < 0.$$

Thus for $n \geq n_3$,

$$\lambda \sum_{s=n_3}^{n} Q(s) + \sum_{s=n_3}^{n} \int_{z(s+1)}^{z(s+2)} \frac{du}{G(u)} + G(a)\sum_{s=n_3}^{n} \int_{z(s-m+1)}^{z(s-m+2)} \frac{dv}{G(v)} < 0.$$

As $\lim_{n\to\infty} z(n)$ exists and due to (H_5), the last inequality implies that

$$\sum_{s=n_3}^{n} Q(s) < \infty,$$

a contradiction to (H_4). Hence $\Delta w(n) > 0$ for $n \geq n_2 > n_1$. Then two cases arise viz. $w(n) > 0$ or $w(n) < 0$. If the former holds, then there exists a constant $\alpha > 0$ and $n^* > 0$ such that $w(n) > \alpha$ for $n \geq n^*$, that is, $z(n) > k(n) + \alpha > \alpha$. Let $n_3 = \max\{n_2, n^*\}$. Then for $n \geq n_3$ and with repeated application of (2.2), we obtain

$$\begin{aligned}\Delta^2 w(n) + q(n)G(y(n-\sigma)) + G(a)\Delta^2 w(n-m) \\ + G(a)q(n-m)G(y(n-m-\sigma)) = 0,\end{aligned}$$

that is,

$$\Delta^2 w(n) + G(a)\Delta^2 w(n-m) + \lambda Q(n)G(z(n-\sigma)) \leq 0,$$

due to (H_1) and (H_2). Since $\lim_{n\to\infty} \Delta w(n)$ exists and $z(n) > \alpha$, then summing the last inequality from $n = n_3 + \sigma$ to ∞, we get

$$\sum_{s=n_3+\sigma}^{\infty} Q(n) < \infty,$$

a contradiction. Let the latter hold. Then $\lim_{n\to\infty} w(n)$ exists and hence $\lim_{n\to\infty} z(n)$ exists. Assume that $\lim_{n\to\infty} z(n) = \mu, \mu \in [0,\infty)$. If $\mu = 0$, then $y(n) \le z(n)$ implies that $\lim_{n\to\infty} y(n) = 0$. Let $\mu \in (0,\infty)$. Then there exists $\varepsilon > 0$ and $n' > 0$ such that $z(n) > \mu - \varepsilon$ for $n \ge n'$. Let $n_3 = \max\{n_2, n'\}$. Using the same type of reasoning as above, we have a contradiction to (H_4).

Next, we assume that $y(n) < 0$ for $n \ge n_0$. Setting $x(n) = -y(n) > 0$ for $n \ge n_0$ and using (H_3), Eq. (1.1) can be written as

$$\Delta^2[x(n) + p(n)x(n-m)] + q(n)G(x(n-\sigma)) - r(n)H(x(n-\tau)) = 0. \tag{2.4}$$

Proceeding as above, we can find similar contradictions for Eq. (2.4). This completes the proof. □

Remark 2.2. In Theorem 2.1, for any function H, G is sublinear due to (H_5). Without (H_5), if we keep our attention on $\lim_{n\to\infty} z(n) =$ exists, then any solution $y(n)$ oscillates or tends to zero as it is seen in the last part of the theorem. Hence instead of (H_5), G could be linear, sublinear, or superlinear. Thus we have verified the following theorem.

Theorem 2.3. *Let* $0 \le p(n) \le a < \infty$. *If* ($H_1$)–($H_4$) *hold, then* (1.1) *is oscillatory or any solution* $y(n) \to 0$ *as* $n \to \infty$.

Remark 2.4. A close observation reveals that $y(n)$ is bounded in Theorem 2.1 when $\lim_{n\to\infty} z(n)$ exists. If $y(n)$ is unbounded, then $z(n) \ge y(n)$ implies that $z(n)$ is unbounded too. Hence the following corollary holds.

Corollary 2.5. *Let* $0 \le p(n) \le a < \infty$. *If* ($H_1$)–($H_4$) *hold, then every unbounded solution of* (1.1) *is oscillatory.*

The proof of Corollary 2.5 follows from the proof of Theorem 2.1, and hence the details are omitted.

Theorem 2.6. *Assume that* $-1 < b \le p(n) \le 0$ *and* $m < \sigma$. *If* (H_2), (H_3), (H_5), *and*

(H_6) $\sum_{n=0}^{\infty} q(n) = \infty$

hold, then every solution of (1.1) *either oscillates or tends to zero as* $n \to \infty$.

Proof. Proceeding as in the proof of Theorem 2.1, we get the inequality (2.2) for $n \geq n_1$. Hence $\Delta w(n)$ is nonincreasing on $[n_1, \infty)$. Assume that $\Delta w(n) < 0$ for $n \geq n_2 > n_1$. If $w(n) > 0$ for $n \geq n_3 > n_2$, then $z(n) > 0$ for $n \geq n_3$. Using the same type of reasoning as in Theorem 2.1, we get

$$\Delta z(n+1) + q(n)G(y(n-\sigma)) < 0.$$

Further, $z(n) \leq y(n)$ implies

$$\Delta z(n+1) + q(n)G(z(n-\sigma)) \leq 0$$

for $n \geq n_3$. Hence,

$$\frac{\Delta z(n+1)}{G(z(n-\sigma))} + q(n) \leq 0,$$

that is,

$$\int_{z(n+1)}^{z(n+2)} \frac{du}{G(z(n-\sigma))} + q(n) \leq 0.$$

Consequently, for $n \geq n_3$

$$\int_{z(n+1)}^{z(n+2)} \frac{du}{G(u)} + q(n) \leq 0,$$

where $z(n+2) < u < z(n+1)$ and $n - \sigma < n + 1$. Thus

$$\sum_{s=n_3}^{\infty} q(s) + \sum_{s=n_3}^{\infty} \int_{z(n+1)}^{z(n+2)} \frac{du}{G(u)} \leq 0,$$

that is,

$$\sum_{s=n_3}^{\infty} q(s) + \lim_{N\to\infty} \int_{z(n_3+1)}^{z(N+2)} \frac{du}{G(u)} \leq 0,$$

a contradiction to (H_6). Ultimately, $w(n) < 0$ for $n \geq n_3$. It follows that $\lim\limits_{n\to\infty} w(n) = -\infty$ and $\lim\limits_{n\to\infty} z(n) = -\infty$, that is, there exists a large n_4 such that $z(n) < 0$ for $n \geq n_4 > n_3$. On the other hand, $z(n) < 0$ implies that $y(n) < y(n-m)$ for $n \geq n_4$, that is, $y(n)$ is bounded and hence it is contradictory that $z(n)$ is bounded, $\lim\limits_{n\to\infty} z(n)$ exists.

Next, we assume that $\Delta w(n) > 0$ for $n \geq n_2$. If $w(n) > 0$ for $n \geq n_3 > n_2$, then there exists a constant $C > 0$ such that $w(n) \geq C$ for $n \geq n_4$, that is, $z(n) \geq C + k(n) > C$. Thus $y(n) \geq z(n) \geq C$ for $n \geq n_4$. Consequently, Eq. (2.2) yields

$$\Delta^2 w(n) + q(n)G(z(n-\sigma)) \leq 0, \tag{2.5}$$

for $n \geq n_4$. Summing the inequality (2.5) from n_4 to ∞, we have a contradiction to (H_6). Ultimately, $w(n) < 0$ for $n \geq n_3$. In what follows, $\lim_{n\to\infty} z(n)$ exists. Let $\lim_{n\to\infty} z(n) = d, d \in \mathbb{R}$. We assert that $d = 0$. If not, then we consider two cases viz., $z(n) > 0$ or $z(n) < 0$. If $z(n) > 0$, then $d \in (0, \infty)$. There exists $\delta > 0$ and $n^* > 0$ such that $z(n) > d - \delta$ for $n \geq n^*$. Let $n_4 = \max\{n_3, n^*\}$. Then for $n \geq n_4$ and $y(n) \geq z(n)$, it follows from (2.2) that

$$\sum_{n=n_4}^{\infty} q(n) < \infty,$$

a contradiction to (H_6). Assume that $z(n) < 0$. Then $d \in (-\infty, 0)$. Further, $z(n) \geq p(n)y(n-m) \geq by(n-m)$ implies that $y(n-\sigma) \geq \frac{1}{b}z(n+m-\sigma)$. Hence equation (2.2) yields

$$\Delta^2 w(n) + q(n)G\left(\frac{1}{b}z(n+m-\sigma)\right) \leq 0, \tag{2.6}$$

for $n \geq n_3$. Since $\lim_{n\to\infty} z(n) = d, d < 0$, then there exist $n^{**} > 0$ and $\beta < 0$ such that $z(n) \leq \beta$, for $n \geq n^{**}$. Consequently due to (H_2), the inequality (2.6) reduces to

$$\Delta^2 w(n) + G\left(\frac{\beta}{b}\right) q(n) \leq 0$$

for $n \geq n_4 > \max\{n_3, n^{**}\}$ and hence

$$\sum_{n=n_4}^{\infty} q(n) < \infty,$$

a contradiction to (H_6). Thus our assertion holds.

Finally, we need to show that $y(n)$ is bounded. Otherwise, there exists an increasing sequence $\{\xi_j\}_{j=1}^{\infty}$ such that $\{\xi_j\} \to \infty$ and $y(\xi_j) \to \infty$ as $j \to \infty$ and $y(\xi_j) = \max\{y(n) : n_3 \leq n \leq \xi_j\}$. Hence

$$w(\xi_j) \geq y(\xi_j) + by(\xi_j - m) - k(\xi_j) \geq (1+b)y(\xi_j) - k(\xi_j)$$

implies that $w(\xi_j) > 0$ for large j (since $1 + b > 0$), a contradiction to the fact that $w(n) < 0$, for $n \geq n_3$. Thus $y(n)$ is bounded. Therefore,

$$\begin{aligned}
0 \quad &= \lim_{n\to\infty} z(n) = \limsup_{n\to\infty}[y(n) + p(n)y(n-m)] \\
&\geq \limsup_{n\to\infty}[y(n) + by(n-m)] \\
&\geq \limsup_{n\to\infty} y(n) + \liminf_{n\to\infty}(by(n-m)) \\
&\geq \limsup_{n\to\infty} y(n) + b\liminf_{n\to\infty} y(n-m) \\
&= (1+b)\limsup_{n\to\infty} y(n),
\end{aligned}$$

that is, $\lim_{n\to\infty} y(n) = 0$. The case $y(n) < 0$ for $n \geq n_0$ can similarly be dealt with. Hence the proof is complete. $\square$

Theorem 2.7. *Let* $-1 < b \leq p(n) \leq 0$ *and* $m < \sigma$. *If* (H_2), (H_3), *and* (H_6) *hold, then every solution of* (1.1) *oscillates or tends to zero as* $n \to \infty$.

The proof of Theorem 2.7 follows from Theorem 2.3 and Theorem 2.6, and hence the details are omitted.

Theorem 2.8. *Let* $-1 < b \leq p(n) \leq 0$ *and* $m < \sigma$. *If* (H_2), (H_3), *and* (H_6) *hold, then every unbounded solution of* (1.1) *oscillates.*

Proof. Let $y(n)$ be an unbounded nonoscillatory solution of (1.1) such that $y(n) > 0$ for $n \geq n_0$. Then proceeding as in Theorem 2.6, we consider the case $\Delta w(n) < 0$ and $w(n) > 0$ for $n \geq n_3$. In what follows, $\lim_{n\to\infty} w(n)$ exists. While proceeding as in Theorem 2.6, it follows that $w(\xi_j) \to \infty$ as $j \to \infty$ is a contradiction. The rest of the proof follows from Theorem 2.6. Thus the proof is complete. $\square$

Theorem 2.9. *Let* $-\infty < b_1 \leq p(n) \leq b_2 < -1$. *If* (H_2), (H_3), *and* (H_6) *hold, then every bounded solution of* (1.1) *either oscillates or tends to zero as* $n \to \infty$.

Proof. Let $y(n)$ be an unbounded nonoscillatory solution of (1.1) such that $y(n) > 0$ for $n \geq n_0$. The case $y(n) < 0$ for $n \geq n_0$ is similar. Proceeding as in the proof of Theorem 2.6, we have equation (2.2) and $\Delta w(n)$ is nonincreasing on $[n_1, \infty)$. Assume that $\Delta w(n) < 0$ for $n \geq n_2 > n_1$. If $w(n) > 0$ for $n \geq n_3 > n_2$, then $\lim_{n\to\infty} w(n)$ exists. On the other hand $\Delta^2 w(n) \leq 0$, $\Delta w(n) < 0$ implies that there exist $n_4 > n_3$ and $L > 0$ such that $w(n) \leq -nL$ for $n \geq n_4$. Ultimately, $\lim_{n\to\infty} w(n) = -\infty$ is a contradiction. Hence $w(n) < 0$ for $n \geq n_3$. In what follows, $\lim_{n\to\infty} w(n) = -\infty$, that is, $\lim_{n\to\infty} z(n) = -\infty$ and $z(n) < 0$ for large n. When $z(n) < 0, z(n) = y(n) + p(n)y(n-m)$ implies that $z(n) \geq p(n)y(n-m) \geq b_1 y(n-m)$, for $n \geq n_4 > n_3$. Using the boundedness of y(n) and $y(n) \geq z(n+m)/b$, we conclude that $z(n)$ is bounded, which is a contradiction . Hence $\Delta w(n) < 0$ for $n \geq n_2$ is not possible.

Next we suppose that $\Delta w(n) > 0$ for $n \geq n_2$. If $w(n) > 0$ for $n \geq n_3 > n_2$, then using the same type of reasoning as in Theorem 2.6, we obtain a contradiction to (H_0). Hence $w(n) < 0$ for $n \geq n_3$. Again proceeding as in Theorem 2.6, $\lim_{n\to\infty} z(n) = 0$. Using the boundedness of $y(n)$, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} z(n) = \liminf_{n\to\infty}[y(n) + p(n)y(n-m)] \\ &\leq \liminf_{n\to\infty}[y(n) + b_2 y(n-m)] \\ &\leq \limsup_{n\to\infty} y(n) + \liminf_{n\to\infty}(b_2 y(n-m)) \\ &= \limsup_{n\to\infty} y(n) + b_2 \limsup_{n\to\infty} y(n-m) \\ &= (1+b_2)\limsup_{n\to\infty} y(n). \end{aligned}$$

Since $(1 + b_2) < 0$, then $\lim_{n\to\infty} y(n) = 0$. Hence the theorem is proved. □

Theorem 2.10. *Let* $-\infty < b \le p(n) \le -1$ *and* $m > \sigma + 2$. *If* (H$_2$), (H$_3$), *and*

(H$_7$) $\sum_{j=0}^{\infty} q(n_j^*) = \infty$ *for every* $\{n_j^*\}$ *of* $\{n\}$,

(H$_8$) $\int_{-\infty}^{-c} \frac{du}{G(u)} > -\infty$ *and* $\int_{c}^{\infty} \frac{du}{G(u)} < \infty$ *for all* $c > 0$

hold, then every unbounded solution of (1.1) *oscillates.*

Proof. Let $y(n)$ be an unbounded nonoscillatory solution of (1.1) such that $y(n) > 0$ for $n \ge n_0$. The case $y(n) < 0$ for $n \ge n_0$ is similar. Proceeding as in Theorem 2.1, we have Eq. (2.2) and $\Delta w(n)$ is nonincreasing on $[n_1, \infty)$. Hence we have four cases:

(i) $\Delta w(n) > 0, w(n) > 0, n \ge n_3 > n_2$,

(ii) $\Delta w(n) < 0, w(n) > 0, n \ge n_3 > n_2$,

(iii) $\Delta w(n) > 0, w(n) < 0, n \ge n_3 > n_2$,

(iv) $\Delta w(n) < 0, w(n) < 0, n \ge n_3 > n_2$.

Cases (i) and (ii) follow respectively from Theorem 2.9. For the case (i), we may note that (H$_7$) implies (H$_6$). Consider the case (iii), where $\lim_{n\to\infty} \Delta w(n)$ exists. Using the unboundedness of $y(n)$, we can find $\{n_j^*\} \subset \{n\}$ such that $n_j^* \to \infty$ and $y(n_j^*) \to \infty$ as $j \to \infty$. Hence for every $M > 0$, there exists $n_4 > n_3$ such that $n_j^* \ge n_4$ implies $y(n_j^*) > M$. Let $n_5 \ge n_4 + \sigma$. Hence

$$\sum_{n_j^*=n_5}^{\infty} q(n_j^*)G(y(n_j^* - \sigma)) > G(M) \sum_{n_j^*=n_5}^{\infty} q(n_j^*)$$

implies that

$$G(M) \sum_{n_j^*=n_5}^{\infty} q(n_j^*) < \sum_{n_j^*-n_5}^{\infty} \Delta^2 w(n_j^*) < \infty,$$

a contradiction to (H$_7$).

Finally, we consider the case (iv). Clearly, $\lim_{n\to\infty} w(n) = -\infty$ and hence $\lim_{n\to\infty} z(n) = -\infty$. Since $\Delta k(n) < 0$, then Eq. (1.1) reduces to

$$\Delta z(n+1) + q(n)G(y(n-\sigma)) < 0 \tag{2.7}$$

for $n \geq n_2 > n_1$. Further, $z(n) < 0$ implies $y(n-\sigma) \geq \dfrac{z(n+m-\sigma)}{b}$ and Eq. (2.7) can be written as

$$\Delta z(n+1) + q(n)G\left(\frac{1}{b}\right)G(z(n+m-\sigma)) < 0, \quad n \geq n_3 > n_2$$

due to (H_2). Consequently,

$$q(n) + \frac{\Delta z(n+1)}{G\left(\frac{1}{b}\right)G(z(n+m-\sigma))} < 0, \quad n \geq n_3$$

that is,

$$q(n) - \frac{1}{G\left(\frac{1}{b}\right)}\int_{z(n+2)}^{z(n+1)} \frac{du}{G(z(n+m-\sigma))} < 0, \quad n \geq n_3$$

where $z(n+2) < u < z(n+1)$. Since $n+m-\sigma > n+2, \Delta w(n) < 0$, that is, $\Delta z(n) < \Delta k(n) < 0$ the last inequality yields

$$q(n) - \frac{1}{G(\frac{1}{b})}\int_{z(n+2)}^{z(n+1)} \frac{du}{G(u)} < 0, \quad n \geq n_3.$$

Hence

$$\sum_{n=n_3}^{N} q(n) + \frac{1}{G\left(\frac{1}{b}\right)}\sum_{n=n_3}^{N}\int_{z(n+1)}^{z(n+2)} \frac{du}{G(u)} < 0, \quad n \geq n_3,$$

that is,

$$\sum_{n=n_3}^{\infty} q(n) + \frac{1}{G(\frac{1}{b})}\lim_{N\to\infty}\sum_{n=n_3}^{N}\int_{z(n+1)}^{z(n+2)} \frac{du}{G(u)} < 0, \quad n \geq n_3.$$

Consequently,

$$\sum_{n=n_3}^{\infty} q(n) + \frac{1}{G\left(\frac{1}{b}\right)}\lim_{N\to\infty}\int_{z(n_3+1)}^{z(N+2)} \frac{du}{G(u)} < 0$$

leads to a contradiction to (H_6) due to (H_8). This completes the proof. □

Theorem 2.11. *Let* $p(n) = -1$. *If* (H_7) *hold, then every solution of* (1.1) *either oscillates or tends to zero as* $n \to \infty$.

Proof. Let $y(n)$ be a nonoscillatory solution of (1.1) such that $y(n) > 0$ for $n \geq n_0$. Proceeding as in Theorem 2.10, case (i) and (ii) do not occur. For the case (iii), we assert that $\limsup_{n\to\infty} y(n) = 0$. If possible, let $\limsup_{n\to\infty} y(n) = \alpha, \alpha > 0$. Hence there exists $\{n_j^*\}$ of {n} such that $n_j^* \to \infty$ as $j \to \infty$ and $\lim_{j\to\infty} y(n_j^* - \sigma) = \alpha$. Thus there exist $\beta > 0$ and $n > n_4$ such that $y(n_j^* - \sigma) > \beta$ for $j > n_4$. Consequently, from the given hypothesis and Eq. (2.2), it follows that

$$G(\beta)\sum_{j=n_4}^{\infty} q(n_j^*)G(y(n_j^* - \sigma)) < -\sum_{j=n_4}^{\infty} \Delta^2 w(n_j^*) < \infty,$$

a contradiction to (H_7). Case (iv) follows from Theorem 2.6. The proof proceeds similarly when $y(n) < 0$ for $n \geq n_0$. Hence the proof is complete. □

Theorem 2.12. *Let* $-1 < -b \leq p(n) \leq a < 1, a > 0, b > 0$ *such that* $0 < a + b < 1$. *If* (H_1)–(H_4) *hold, then every solution of* (1.1) *either oscillates or tends to zero as* $n \to \infty$.

Proof. Let $y(n)$ be a nonoscillatory solution of (1.1) on $[n_0, \infty)$. Then there exists $n_1 > n_0$ such that $y(n) > 0$ for $n \geq n_1$. Setting as in (2.1), we get (2.2) for $n \geq n_1$. Then $\Delta w(n)$ is nonincreasing on $[n_1, \infty)$. Assume that $\Delta w(n) < 0$ for $n \geq n_2 > n_1$. If $w(n) > 0$ for $n \geq n_3 > n_2$, then the contradiction follows from Theorem 2.6. Let $w(n) < 0$ for $n \geq n_3$. Then $\lim_{n\to\infty} w(n) = -\infty$ and hence $\lim_{n\to\infty} z(n) = -\infty$. On the other hand, $z(n) < 0$ implies that $z(n) \geq y(n) - by(n-m)$ and thus $y(n) < y(n-m)$ for $n_4 \geq n_3$. Consequently, there exists $n > n_3 + m$ such that $y(n)$ is bounded for $n \geq n_4$, that is, $\lim_{n\to\infty} z(n)$ exists, a contradiction.

Suppose that $\Delta w(n) > 0$ for $n \geq n_2$. If $w(n) > 0$ for $n \geq n_3$, then there exists $n_4 > n_3$ such that $w(n) \geq \alpha$ for $n \geq n_4$, that is, $z(n) \geq k(n) + \alpha > \alpha$ for $n \geq n_4$. Hence

$$y(n) + ay(n-m) \geq z(n) > \alpha, \quad n \geq n_4. \tag{2.8}$$

Rewriting (2.2) for $(n-m)$ and then adding with (2.2), we get

$$\Delta^2 w(n) + q(n)G(y(n-\sigma)) + G(a)\Delta^2 w(n-m) + G(a)q(n-m)G(y(n-m-\sigma)) = 0$$

which reduces to

$$\Delta^2 w(n) + G(a)\Delta^2 w(n-m) + \lambda Q(n)G(z(n-\sigma)) \leq 0$$

due to (H_1), (H_2), and (2.8) for $n \geq n_4$. Summing the last inequality from n_5 to ∞, we obtain

$$\sum_{n=n_5}^{\infty} Q(n) < \infty, \quad n_5 > n_4 + \sigma,$$

a contradiction to (H_4). Ultimately, $w(n) < 0$ for $n \geq n_3$ and $\lim_{n\to\infty} w(n) =$ exists. We assert that $y(n)$ is bounded. If not, there exists $\{n_j^*\}$ of $\{n\}$ such that $n_j^* \to \infty, y(n_j^*) \to \infty$ as $j \to \infty$ and

$$y(n_j^*) = \max\{y(n) : n_3 \leq n \leq n_j^*\}.$$

In what follows,

$$w(n_j^*) \geq (1-b)y(n_j^*) - k(n_j^*) \to \infty \;\; as \;\; j \to \infty,$$

a contradiction to our supposition. Consequently, our assertion holds. Next, we claim that $\liminf_{n\to\infty} y(n) = 0$. Otherwise, $\liminf_{n\to\infty} y(n) = \alpha, 0 < \alpha < \infty$. Then there exists $\beta > 0$ and $n \geq n_5 > n_4 + \sigma$, so that (2.2) yields

$$\sum_{n=n_4}^{\infty} q(n)G(y(n-\sigma)) = -\sum_{n=n_4}^{\infty} \Delta^2 w(n),$$

that is,

$$G(\beta)\sum_{n=n_4}^{\infty} q(n) < -\sum_{n=n_4}^{\infty} \Delta^2 w(n) < \infty,$$

implies that

$$\sum_{n=n_4}^{\infty} Q(n) < -\sum_{n=n_4}^{\infty} \Delta^2 w(n) < \infty,$$

a contradiction to (H_4). So our claim holds. Using the boundedness of $y(n)$, it follows that

$$\begin{aligned}\lim_{n\to\infty} z(n) \quad &\geq \limsup_{n\to\infty}[y(n) - by(n-m)] \\ &\geq \limsup_{n\to\infty} y(n) + \liminf_{n\to\infty}(-by(n-m)) \\ &= \limsup_{n\to\infty} y(n) - b\limsup_{n\to\infty} y(n-m) \\ &= (1-b)\limsup_{n\to\infty} y(n).\end{aligned}$$

Moreover,

$$\begin{aligned}\lim_{n\to\infty} z(n) \quad &\leq \liminf_{n\to\infty}[y(n) + ay(n-m)] \\ &\leq \liminf_{n\to\infty} y(n) + \limsup_{n\to\infty}(ay(n-m)) \\ &= a\limsup_{n\to\infty} y(n)\end{aligned}$$

implies that

$$(1-b)\limsup_{n\to\infty} y(n) \le a \lim_{n\to\infty} \sup y(n).$$

Hence

$$0 \le (a+b-1)\limsup_{n\to\infty} y(n) \le 0,$$

that is, $\limsup_{n\to\infty} y(n) = 0$. Thus $\lim_{n\to\infty} y(n) = 0$. This completes the proof. □

Theorem 2.13. *Let* $-1 < -b \le p(n) \le a < 1, a > 0$ *and* $b > 0$. *If* (H_1)–(H_4) *hold, then every unbounded solution of* (1.1) *oscillates.*

The proof of Theorem 2.13 now follows from Theorem 2.10, and hence the details are omitted.

3 Examples and Discussion

Example 3.1. Consider

$$\Delta^2\left[y(n) + \frac{1}{3}(-1)^n y(n-1)\right] + \frac{4n+3}{n-3}y(n-3) - \frac{1}{(n-2)^3}y^3(n-2) = 0, \quad n \ge 4, \tag{3.1}$$

where $p(n) = \frac{1}{3}(-1)^n$ such that $-\frac{1}{3} \le p(n) \le \frac{1}{3}$ and $a+b = \frac{2}{3}$. Clearly, (H_0) and (H_4) are satisfied for $n \ge 4$. Hence by Theorem 2.13, every solution of (3.1) oscillates. In particular $y(n) = n(-1)^n$, for $n \ge 4$ is such an oscillatory solution of (3.1).

Example 3.2. Consider

$$\Delta^2[y(n) + y(n-1)] + \frac{1}{n-2}y(n-2) - \frac{5}{(n-4)^5}y^5(n-4) = 0, \quad n \ge 5. \tag{3.2}$$

It is easy to verify the conditions (H_0), (H_1), (H_2), and (H_4). Hence by Corollary 2.5, every unbounded solution of (3.2) oscillates. Indeed, $y(n) = n(-1)^n$ is an unbounded solution of (3.2).

Example 3.3. Consider

$$\Delta^2\left[y(n) - \frac{1}{3}y(n-3)\right] + \left(\frac{1}{n} + \frac{1}{n^2}\right)y(n-1) - \frac{1}{(n-4)^3}\left(8n + \frac{1}{n^2} + \frac{10}{3}\right)y^3(n-4) = 0, \quad n \ge 5. \tag{3.3}$$

It is easy to verify the conditions (H_2), (H_3), and (H_8). However, (H_7) does not hold. Therefore, Theorem 2.10 cannot be applied to (3.3). We note that $y(n) = n(-1)^n$ is an unbounded solution of (3.3). Apparently some extra condition or a different approach is required to improve Theorem 2.10.

As Eq. (1.1) is highly nonlinear, emphasis has been given to the nonlinear functions G and H. Its study is more interesting than the linear equations. We note that H could be linear, sublinear or superlinear. However, the prototype of G satisfying (H_1), (H_2), and (H_3) is of the type

$$G(u) = (a + b|u|^{\alpha})|u|^{\beta}\text{sgn}u,$$

where $\alpha \geq 0, \beta \geq 0, a \geq 0$ and $b > 0$ such that $a + b = 1$. In addition, (H_3) implies that $G(-1) = -G(1) < 0$.

The technique employed here illustrates the solution space of Eq. (1.1). When the solution is bounded, it seldom behaves asymptotically. But when the solution is unbounded, it oscillates although we are not sure of $p(n) \leq -1$, where G is superlinear only. It would be interesting to anticipate the answer to this fact soon. Explicitly our approach suggests that some more conditions or a different method is required to show that Eq. (1.1) is oscillatory for any nonlinear functions G and H.

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References

[1] R. P. Agarwal; *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.

[2] Q. Chuanxi, G. Ladas; *Oscillation in differential equations with positive and negative coefficients,* Canad. Math. Bull. **33** (1990), 442.

[3] K. Farrel, E. A. Grove, G. Ladas; *Neutral delay differential equations with positive and negative coefficients,* Appl. Anal., **27**, 181.

[4] I. Győri, G. Ladas; *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[5] B. Karpuz, O. Ocalan, M. K. Yildiz; *Oscillation of a class of difference equations of second order,* Math. Comput. Model **49** (2009), 912–917.

[6] G. Ladas; *Explicit condition for the oscillation of difference equations,* J. Math. Anal. Appl. **153** (1990), 276–286.

[7] J. Manojlovic, Y. Shoukaku, T. Tanigawa, N. Yoshida; *Oscillation criteria for second order differential equations with positive and negative coefficients,* Appl. Math. Comput, **18** (2006), 853–863.

[8] N. Parhi, S. Chand; *On second Order neutral delay differential equations with positive and negative coefficients*, Bull. Cal. Math. Soc. **94(1)** (2002), 7–16.

[9] R. B. Potts; *Exact solution of a difference approximation to Duffings equation*, J. Aust. Math. Soc. (series B) **23** (1981), 64–77.

[10] W. Shan, W. Ge; *Oscillation of neutral difference equations with positve and negative coefficients*, Comp. Math. Appl. **47** (2004), 1647–1657.

[11] B. Szmanda; *Oscillation theorems for nonlinear second order difference equations*, J. Math. Anal. Appl. **79** (1981), 90–95.

[12] X. H. Tang, J. S. Yu, D. H. Peng; *Oscillations and nonoscillation of neutral difference equations with deviating coefficients,* Comp. Math. Appl. **39** (2000), 169–181.

[13] A. K. Tripathy; *Some oscillation results for second order nonlinear dynamic equations of neutral type*, Nonlinear Analysis, **71** (2009), e1727–e1735.

[14] F. Weil; *Existence theorem for the difference equation* $y_{n+1} - 2y_n + y_{n+1} = h^2 f(y_n)$, Inter. J. Math. Sci. **3** (1980), 69–77.