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Asymptotic Behavior of the Solutions of a Class of Rational Difference Equations

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Abstract

In this paper we study the asymptotic behavior of the positive solutions of certain rational difference equations.

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1 Introduction

In [8] the author studied the global behavior of the second order rational difference equation having quadratic term

$$x_{n+1} = \frac{ax_{n-1}}{x_n x_{n-1} + b},\ a > 0,\ b > 0 \tag{1.1}$$

and the third order difference equation having quadratic term

$$x_{n+1} = \frac{ax_n x_{n-1}}{x_n + bx_{n-2}},\ a > 0,\ b > 0 \tag{1.2}$$

where x_{-2}, x_{-1}, x_0 are real numbers. For the study of equation (1.1) the author used the fact that (1.1) reduces to a linear nonhomogeneous equation. Moreover, for the study of (1.2) he showed that equation (1.2) reduces to (1.1).

Furthermore in [3] the authors investigated equation (1.1) with nonnegative initial values x_{-1}, x_0. Moreover if we get $b = 1$ in (1.1), then by dropping either the term x_n or x_{n-1} in the denominator of the equation (1.1), we obtain the equations

$$x_{n+1} = \frac{ax_{n-1}}{x_n + 1},\ \ x_{n+1} = \frac{ax_{n-1}}{x_{n-1} + 1}$$

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which have been studied in [2]. Finally, results concerning rational difference equations having quadratic terms are included in [1, 3–11] and the references cited therein.

Now in this paper we study the following equations

$$x_{n+1} = \frac{ax_{n-m(k+1)+1}}{\prod_{s=0}^{k} x_{n-m(s+1)+1} + 1}, \; n = 0, 1, \ldots \tag{1.3}$$

$$x_{n+1} = \frac{ax_{n-2k-1}\prod_{s=0}^{k} x_{n-2s}}{\prod_{s=0}^{2k+1} x_{n-s} + \prod_{s=0}^{k} x_{n-2s} + \prod_{s=0}^{k} x_{n-2s-1}} \tag{1.4}$$

and

$$x_{n+1} = \frac{ax_n x_{n-m(k+1)+1}}{x_n + x_{n-m(k+1)}}, \;\; n = 0, 1, \ldots, \tag{1.5}$$

where a is a positive number, $m, k \in \{1, 2, \ldots\}$ and the initial values of the above equations are positive numbers. More precisely, we study the existence of periodic solutions and the asymptotic behavior of the positive solutions for equations (1.3), (1.4), (1.5). We note that equations (1.3), (1.4), (1.5) have a common property: They reduces to a linear nonhomogeneous equation.

2 Study of Equation (1.3)

First we study the existence of positive periodic solutions of period $m(k+1)$ for equation (1.3).

Proposition 2.1. *Consider equation* (1.3)*. Suppose that*

$$a > 1. \tag{2.1}$$

Then equation (1.3) *has periodic solutions of period* $m(k + 1)$*.*

Proof. Suppose that x_n is a positive solution of (1.3) with initial values $x_{-m(k+1)+1}$, $x_{-m(k+1)+2}, \ldots, x_0 > 0$ such that

$$\prod_{s=0}^{k} x_{i-m(s+1)+1} = a - 1, \;\; i = 0, 1, \ldots, m - 1. \tag{2.2}$$

We prove that x_n is a periodic solution of (1.3) of period $m(k+1)$. From (1.3) and (2.2), we get

$$\begin{aligned} x_1 &= \frac{ax_{-m(k+1)+1}}{\prod_{s=0}^{k} x_{-m(s+1)+1}+1} = x_{-m(k+1)+1}, \\ x_2 &= \frac{ax_{-m(k+1)+2}}{\prod_{s=0}^{k} x_{-m(s+1)+2}+1} = x_{-m(k+1)+2}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m &= \frac{ax_{-mk}}{\prod_{s=0}^{k} x_{-ms}+1} = x_{-mk}. \end{aligned} \tag{2.3}$$

Then from (1.3) and (2.3), we obtain

$$\begin{aligned} x_{m+1} &= \frac{ax_{-mk+1}}{x_1 \prod_{s=1}^{k} x_{-ms+1}+1} = \frac{ax_{-mk+1}}{x_{-m(k+1)+1} \prod_{s=1}^{k} x_{-ms+1}+1} \\ &= \frac{ax_{-mk+1}}{\prod_{s=0}^{k} x_{-m(s+1)+1}+1} = x_{-mk+1}. \end{aligned}$$

Working inductively, we can prove that

$$x_{m+j} = x_{-mk+j}, \quad j = 2, 3, \ldots$$

and so the proof is completed. □

In the next proposition, we study the asymptotic behavior of the positive solutions of (1.3). We need the following lemma.

Lemma 2.2. *Let x_n be an arbitrary positive solution of* (1.3). *Then the following statements are true:*

(i) *If*

$$t_n = \prod_{s=0}^{k} x_{n-sm}^{-1}, \; n = 1, 2, \ldots \tag{2.4}$$

with

$$t_j = \prod_{s=0}^{k} x_{j-sm}^{-1}, \quad j = 1-m, 2-m, \ldots, 0, \tag{2.5}$$

then t_n satisfies the nonhomogeneous linear difference equation

$$y_{n+1} = \frac{1}{a} y_{n+1-m} + \frac{1}{a}, \quad n = 0, 1, \dots. \tag{2.6}$$

Moreover,

$$t_n = \begin{cases} B_n + \dfrac{n}{m}, \quad n = 1, 2, \dots & \text{if } a = 1 \\ \left(\dfrac{1}{a}\right)^{\frac{n}{m}} B_n + \dfrac{1}{a-1}, \quad n = 1, 2, \dots & \text{if } a \neq 1 \end{cases} \tag{2.7}$$

where

$$B_n = \sum_{i=0}^{r} c_i \cos\left(\frac{2\pi n i}{m}\right) + d_i \sin\left(\frac{2\pi n i}{m}\right), \quad r = \begin{cases} \dfrac{m-1}{2}, & \text{if } m \text{ is odd} \\ \dfrac{m}{2}, & \text{if } m \text{ is even} \end{cases} \tag{2.8}$$

and c_i, d_i, $i = 0, 1, \dots, r$ are constants which are derived from (2.5), (2.7) *and* (2.8).

(ii) *If*

$$y_n^{(j)} = x_{m(k+1)n+j}, \quad j = 0, 1, \dots, m(k+1) - 1, \tag{2.9}$$

then

$$y_n^{(j)} = y_0^{(j)} \prod_{s=0}^{n-1} \frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}, \quad j = 0, 1, \dots, m(k+1) - 1. \tag{2.10}$$

Proof. Let x_n be an arbitrary solution of (1.3). Then we get

$$x_{n+1} \prod_{s=1}^{k} x_{n+1-sm} = \frac{a x_{n-m(k+1)+1} \prod_{s=1}^{k} x_{n+1-sm}}{\prod_{s=0}^{k} x_{n+1-(s+1)m} + 1}$$

which implies that

$$\prod_{s=0}^{k} x_{n+1-sm} = \frac{a \prod_{s=0}^{k} x_{n+1-(s+1)m}}{\prod_{s=0}^{k} x_{n+1-(s+1)m} + 1}. \tag{2.11}$$

Then from (2.4) and (2.11), we have

$$\frac{1}{t_{n+1}} = \frac{\dfrac{a}{t_{n+1-m}}}{\dfrac{1}{t_{n+1-m}}+1}$$

which implies that t_n satisfies the difference equation (2.6). Then relations (2.7) and (2.8) follow immediately. This completes the proof of statement (i).

(ii) From (2.4), we have

$$\frac{t_n}{t_{n-m}} = \frac{x_n^{-1}x_{n-m}^{-1}\cdots x_{n-km}^{-1}}{x_{n-m}^{-1}x_{n-2m}^{-1}\cdots x_{n-(k+1)m}^{-1}} = \frac{x_{n-m(k+1)}}{x_n}$$

which implies that

$$x_n = \frac{t_{n-m}}{t_n}x_{n-m(k+1)}, \quad n = 1, 2, \ldots. \tag{2.12}$$

So, from (2.9) and (2.12) it holds

$$y_{n+1}^{(j)} = \frac{t_{m(k+1)(n+1)+j-m}}{t_{m(k+1)(n+1)+j}}y_n^{(j)}, \quad j = 0, 1, \ldots, m(k+1)-1. \tag{2.13}$$

Therefore relation (2.13) implies that (2.10) is true. This completes the proof. □

Proposition 2.3. *Consider equation* (1.3)*. Then the following statements are true.*

(i) *If*

$$0 < a \leq 1, \tag{2.14}$$

then every positive solution of (1.3) *tends to zero as* $n \to \infty$.

(ii) *If* (2.1) *holds, then every positive solution of* (1.3) *tends to a periodic solution of period* $m(k+1)$.

Proof. Let x_n be an arbitrary positive solution of (1.3).

(i) Suppose first that

$$0 < a < 1. \tag{2.15}$$

From (1.3), we get for $j = 0, 1, \ldots, m(k+1)-1$

$$x_{m(k+1)n+j} < ax_{m(k+1)(n-1)+j} < \cdots < a^n x_j. \tag{2.16}$$

Then from (2.15) and (2.16), we take

$$\lim_{n\to\infty} x_{m(k+1)n+j} = 0, \quad j = 0, 1, \ldots, m(k+1)-1$$

which implies that x_n tends to zero as $n \to \infty$.

Let now $a = 1$. We consider the functions

$$A_n^{(j)} = \ln\left(\prod_{s=0}^{n-1} \frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}\right) = \sum_{s=0}^{n-1} \ln\left(\frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}\right). \tag{2.17}$$

From (2.8) it is obvious that

$$B_{m(k+1)(s+1)+j-m} = B_{m(k+1)(s+1)+j}, \;\; s = 0, 1, \dots, \;\; j = 0, 1, \dots, m(k+1)-1. \tag{2.18}$$

Hence relations (2.7), (2.8) and (2.18) imply that

$$t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j} = -1, \;\; s = 0, 1, \dots, \;\; j = 0, 1, \dots, m(k+1) - 1. \tag{2.19}$$

In addition, if a is a real number such that $1 + a > 0$, then

$$\ln(1 + a) < a. \tag{2.20}$$

Then from (2.19) and (2.20), we get

$$\begin{aligned} &\sum_{s=0}^{n-1} \ln\left(1 + \frac{t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j}}{t_{m(k+1)(s+1)+j}}\right) \\ &\leq \sum_{s=0}^{n-1} \left(\frac{t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j}}{t_{m(k+1)(s+1)+j}}\right) = -\sum_{s=0}^{n-1} \frac{1}{t_{m(k+1)(s+1)+j}}. \end{aligned} \tag{2.21}$$

Since from (2.7)

$$\sum_{s=0}^{\infty} \frac{1}{t_{m(k+1)(s+1)+j}} = \infty,$$

we have from (2.21)

$$\sum_{s=0}^{\infty} \ln\left(\frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}\right) = -\infty. \tag{2.22}$$

Therefore, from (2.17) and (2.22), we have

$$\lim_{n\to\infty} A_n^{(j)} = -\infty, \;\; j = 0, 1, \dots, m(k+1) - 1$$

which implies that

$$\prod_{s=0}^{\infty} \frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}} = 0, \;\; j = 0, 1, \dots, m(k+1) - 1. \tag{2.23}$$

So from (2.10) and (2.23), we have that x_n tends to zero as $n \to \infty$. This completes the proof of statement (i).

(ii) If $a, b > 0$, then using (2.20), we can easily prove that

$$\left|\ln\left(\frac{a}{b}\right)\right| \leq |a-b| \max\left\{\frac{1}{a}, \frac{1}{b}\right\}. \tag{2.24}$$

Then from (2.24), we have for $j = 0, 1, \ldots, m(k+1)-1$

$$\begin{aligned}&\left|\sum_{s=0}^{n-1} \ln\left(\frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}\right)\right| \leq \sum_{s=0}^{n-1} \left|\ln\left(\frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}\right)\right| \\ &\leq \sum_{s=0}^{n-1} |t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j}| \max\left\{\frac{1}{t_{m(k+1)(s+1)+j}}, \frac{1}{t_{m(k+1)(s+1)+j-m}}\right\}.\end{aligned} \tag{2.25}$$

Furthermore, from (2.1), (2.7) and (2.18), we have

$$|t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j}| = \left(\frac{1}{a}\right)^{\frac{m(k+1)(s+1)+j}{m}} |B_{m(k+1)(s+1)+j}|(a-1). \tag{2.26}$$

Then using (2.7) and (2.26), we can prove that there exists a positive number $M > 0$ such that

$$\begin{aligned}&|t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j}| \max\left\{\frac{1}{t_{m(k+1)(s+1)+j}}, \frac{1}{t_{m(k+1)(s+1)+j-m}}\right\} \\ &\leq M\left(\frac{1}{a}\right)^{\frac{m(k+1)(s+1)+j}{m}}, \quad j = 0, 1, \ldots, m(k+1)-1.\end{aligned} \tag{2.27}$$

Therefore, from (2.1), (2.25) and (2.27), it follows that

$$\left|\sum_{s=0}^{\infty} \ln\left(\frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}\right)\right| < \infty. \tag{2.28}$$

Then using (2.17) and (2.28), it is obvious that there exist

$$\lim_{n\to\infty} A_n^{(j)} = l_j < \infty, \quad j = 0, 1, \ldots, m(k+1)-1. \tag{2.29}$$

Relations (2.9), (2.10), (2.17) and (2.29) imply that

$$\lim_{n\to\infty} x_{m(k+1)n+j} = p_j < \infty, \quad j = 0, 1, \ldots, m(k+1)-1.$$

This completes the proof. □

3 Study of Equation (1.4)

First we study the existence of positive solutions of period $2k+2$ for the equation (1.4).

Proposition 3.1. *Consider equation* (1.4) *where*

$$a > 2. \tag{3.1}$$

Then equation (1.4) *has positive periodic solutions of period* $2k+2$.

Proof. Let x_n be a positive solution of (1.4) with initial values such that

$$\prod_{s=0}^{k} x_{-2s} = \prod_{s=0}^{k} x_{-2s-1} = a - 2. \tag{3.2}$$

Then from (1.4) and (3.2), we get

$$x_1 = \frac{a x_{-2k-1} \prod_{s=0}^{k} x_{-2s}}{\prod_{s=0}^{2k+1} x_{-s} + \prod_{s=0}^{k} x_{-2s} + \prod_{s=0}^{k} x_{-2s-1}} = \frac{a(a-2)x_{-2k-1}}{(a-2)^2 + 2(a-2)} = x_{-2k-1},$$

$$\begin{aligned} x_2 &= \frac{a x_1 x_{-2k} \prod_{s=1}^{k} x_{1-2s}}{x_1 \prod_{s=1}^{2k+1} x_{1-s} + x_1 \prod_{s=1}^{k} x_{1-2s} + \prod_{s=0}^{k} x_{-2s}} \\ &= \frac{a x_{-2k-1} x_{-2k} \prod_{s=1}^{k} x_{1-2s}}{x_{-2k-1} \prod_{s=1}^{2k+1} x_{1-s} + x_{-2k-1} \prod_{s=1}^{k} x_{1-2s} + \prod_{s=0}^{k} x_{-2s}} \\ &= \frac{a x_{-2k} \prod_{s=0}^{k} x_{-2s-1}}{\prod_{s=0}^{2k+1} x_{-s} + \prod_{s=0}^{k} x_{-2s-1} + \prod_{s=0}^{k} x_{-2s}} = \frac{a(a-2)x_{-2k}}{(a-2)^2 + 2(a-2)} = x_{-2k}. \end{aligned}$$

Working inductively, we can prove that

$$x_n = x_{n-2k-2}, \; n = 3, 4, \ldots.$$

This completes the proof. $\square$

In the following proposition, we study the asymptotic behavior of the positive solutions of (1.4). We need the following lemma.

Lemma 3.2. *Let x_n be a positive solution of* (1.4). *Then the following statements are true:*

(i) *If*

$$t_n = \prod_{s=0}^{k} x_{n-2s}^{-1}, \quad n = 1, 2, \dots \tag{3.3}$$

with

$$t_j = \prod_{s=0}^{k} x_{j-2s}^{-1}, \quad j = -1, 0, \tag{3.4}$$

then t_n, $n = 1, 2, \dots$ satisfies the following difference equation

$$y_{n+1} = \frac{1}{a} y_n + \frac{1}{a} y_{n-1} + \frac{1}{a}, \quad n = 0, 1, \dots. \tag{3.5}$$

Moreover,

$$t_n = \begin{cases} c_1 \left(-\frac{1}{2}\right)^n + c_2 + \frac{1}{3} n, \quad n = 1, 2, \dots & \text{if } a = 2 \\ \\ c_1 p_1^n + c_2 p_2^n + \frac{1}{a-2}, \quad n = 1, 2, \dots & \text{if } a \neq 2 \end{cases} \tag{3.6}$$

where

$$p_1 = \frac{1}{2a}(1 - \sqrt{1+4a}), \quad p_2 = \frac{1}{2a}(1 + \sqrt{1+4a}), \tag{3.7}$$

c_1, c_2 *are defined from* (3.4) *and* (3.6).

(ii) *If*

$$y_n^{(j)} = x_{2(k+1)n+j}, \quad j = 0, 1, \dots, 2k+1, \tag{3.8}$$

then

$$y_n^{(j)} = y_0^{(j)} \prod_{s=0}^{n-1} \frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}}, \quad j = 0, 1, \dots, 2k+1. \tag{3.9}$$

Proof. (i) Let x_n be an arbitrary positive solution of (1.4). Then we get

$$x_{n+1} \prod_{s=1}^{k} x_{n-2s+1} = \frac{a x_{n-2k-1} \prod_{s=0}^{k} x_{n-2s} \prod_{s=1}^{k} x_{n-2s+1}}{\prod_{s=0}^{2k+1} x_{n-s} + \prod_{s=0}^{k} x_{n-2s} + \prod_{s=0}^{k} x_{n-2s-1}}$$

which implies that

$$\prod_{s=0}^{k} x_{n-2s+1} = \frac{a\displaystyle\prod_{s=0}^{2k+1} x_{n-s}}{\displaystyle\prod_{s=0}^{2k+1} x_{n-s} + \prod_{s=0}^{k} x_{n-2s} + \prod_{s=0}^{k} x_{n-2s-1}}. \tag{3.10}$$

Then relations (3.3) and (3.10) imply that

$$\frac{1}{t_{n+1}} = \frac{\dfrac{a}{t_n t_{n-1}}}{\dfrac{1}{t_n t_{n-1}} + \dfrac{1}{t_n} + \dfrac{1}{t_{n-1}}}$$

from which we conclude that t_n satisfies the difference equation (3.5). Then relation (3.6) follows immediately.

(ii) Using (3.3), we take

$$\frac{t_n}{t_{n-2}} = \frac{x_n^{-1} x_{n-2}^{-1} \cdots x_{n-2k}^{-1}}{x_{n-2}^{-1} x_{n-4}^{-1} \cdots x_{n-2k-2}^{-1}} = \frac{x_{n-2k-2}}{x_n}$$

which implies that

$$x_n = \frac{t_{n-2}}{t_n} x_{n-2k-2}, \quad n = 1, 2, \ldots. \tag{3.11}$$

So, from (3.8) and (3.11), it holds

$$y_n^{(j)} = \frac{t_{2(k+1)n+j-2}}{t_{2(k+1)n+j}} y_{n-1}^{(j)}, \quad j = 0, 1, \ldots, 2k+1. \tag{3.12}$$

From (3.12) relation (3.9) follows immediately. This completes the proof. □

Proposition 3.3. *Consider equation* (1.4)*. Then the following statements are true:*

(i) *If*

$$0 < a \leq 2, \tag{3.13}$$

then every positive solution of (1.4) *tends to zero as* $n \to \infty$.

(ii) *If*

$$a > 2, \tag{3.14}$$

then every positive solution of (1.4) *tends to a periodic solution of* (1.4) *of period* $2k+2$.

Proof. Let x_n be an arbitrary positive solution of (1.4).

(i) Suppose that (2.15) is satisfied. Relation (1.4) implies that for $j = 0, 1, \dots, 2k+1$

$$x_{2(k+1)n+j} < ax_{2(k+1)(n-1)+j} < \dots < a^n x_j. \tag{3.15}$$

Therefore, from (2.15) and (3.15), we get

$$\lim_{n\to\infty} x_{2(k+1)n+j} = 0, \quad j = 0, 1, \dots, 2k+1 \tag{3.16}$$

which imply that x_n tends to zero as $n \to \infty$.

Suppose that

$$1 \le a < 2. \tag{3.17}$$

From (3.7) and (3.17), we can easily prove that

$$|p_1| < 1, \quad 1 < p_2. \tag{3.18}$$

We set for $j = 0, 1, \dots, 2k+1$

$$B_n^{(j)} = \ln\left(\prod_{s=0}^{n-1} \frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}}\right). \tag{3.19}$$

Then from (2.20), we have for $j = 0, 1, \dots, 2k+1$

$$\begin{aligned} B_n^{(j)} &= \sum_{s=0}^{n-1} \ln\left(1 + \frac{t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}}\right) \\ &\le \sum_{s=0}^{n-1}\left(\frac{t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}}\right). \end{aligned} \tag{3.20}$$

Moreover, from (3.6) and (3.20), we can prove that

$$\frac{t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}} = \frac{c_1(p_1^{-2} - 1)\left(\dfrac{p_1}{p_2}\right)^{2(k+1)(s+1)+j} + c_2(p_2^{-2} - 1)}{c_1\left(\dfrac{p_1}{p_2}\right)^{2(k+1)(s+1)+j} + c_2 + \dfrac{1}{a-2} p_2^{-2(k+1)(s+1)+j}}. \tag{3.21}$$

Using (3.18) and (3.21), we have that

$$\lim_{s\to\infty}\left(\frac{t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}}\right) = p_2^{-2} - 1 < 0. \tag{3.22}$$

Therefore, from (3.20) and (3.22), we can prove that

$$\lim_{n\to\infty} B_n^{(j)} = -\infty, \quad j = 0, 1, \dots, 2k+1 \tag{3.23}$$

which from (3.19) imply that for $j = 0, 1, \dots, 2k+1$

$$\prod_{s=0}^{\infty} \frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}} = 0. \tag{3.24}$$

Hence, from (3.8), (3.9) and (3.24), we have that relations (3.16) are true and so x_n tends to zero as $n \to \infty$.

Suppose now that

$$a = 2. \tag{3.25}$$

So from (3.6) and (3.25), we get

$$\begin{aligned} &\frac{t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}} \\ &= \frac{3c_1\left(-\frac{1}{2}\right)^{2(k+1)(s+1)+j} - \frac{2}{3}}{c_2 + c_1\left(-\frac{1}{2}\right)^{2(k+1)(s+1)+j} + \frac{1}{3}(2(k+1)(s+1)+j)}. \end{aligned} \tag{3.26}$$

Then from (3.26), we can easily prove

$$\sum_{s=0}^{\infty} \left(\frac{t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}} \right) = -\infty. \tag{3.27}$$

Therefore, from (3.20), (3.27), we have that (3.23) is satisfied and so arguing as above, (3.16) holds, which implies that x_n tends to zero as $n \to \infty$.

(ii) Finally, suppose that (3.14) is satisfied. Then from (3.7) it is obvious that

$$\left|\frac{p_1}{p_2}\right| < 1, \ |p_1| < 1, \ \ p_2 < 1. \tag{3.28}$$

In addition, from (3.6), we have that for $j = 0, 1, \dots, 2k+1$

$$\begin{aligned} &t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j} \\ &= p_2^{2(k+1)(s+1)+j} \left(c_1(p_1^{-2} - 1)\left(\frac{p_1}{p_2}\right)^{2(k+1)(s+1)+j} + c_2(p_2^{-2} - 1) \right). \end{aligned} \tag{3.29}$$

In addition, from (2.24), we get for $j = 0, 1, \dots, 2k+1$

$$\begin{aligned} &\left| \ln\left(\frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}} \right) \right| \\ &\le |t_{2(k+1)(s+1)-2+j} - t_{2(k+1)(s+1)+j}| \max\left\{ \frac{1}{t_{2(k+1)(s+1)-2+j}}, \frac{1}{t_{2(k+1)(s+1)+j}} \right\}. \end{aligned} \tag{3.30}$$

Using (3.6), (3.28), (3.29) and (3.30), there exists a positive number N such that for $j = 0, 1, \dots, 2k+1$

$$\left| \ln \left(\frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}} \right) \right| \leq N p_2^{2(k+1)(s+1)+j}. \tag{3.31}$$

Therefore, from (3.19) and (3.31), we have that there exist

$$\lim_{n\to\infty} B_n^{(j)} = \mu_j < \infty, \;\; j = 0, 1, \dots, 2k+1. \tag{3.32}$$

Hence, relations (3.8), (3.9), (3.19) and (3.32) imply that

$$\lim_{n\to\infty} x_{2(k+1)n+j} = q_j < \infty, \;\; j = 0, 1, \dots, 2k+1,$$

and so the proof is completed. □

4 Study of Equation (1.5)

In the first proposition, we study the existence of positive periodic solutions of (1.5) of period $m(k+1)$.

Proposition 4.1. *Consider equation* (1.5) *where* (3.25) *holds. Let* x_n *be positive solution of* (1.5) *such that*

$$x_0 = x_{-m(k+1)}. \tag{4.1}$$

Then x_n *is a periodic solution of* (1.5) *with period* $m(k+1)$.

Proof. Let x_n be a positive solution of (1.5) such that (4.1) holds. Then from (1.5), (3.25), we get

$$x_1 = \frac{2x_0 x_{-m(k+1)+1}}{x_0 + x_{-m(k+1)}} = \frac{2x_0 x_{-m(k+1)+1}}{2x_0} = x_{-m(k+1)+1}$$

and working inductively, we can prove that

$$x_n = x_{n-m(k+1)}, \;\; n = 1, 2, \dots.$$

This completes the proof. □

In the last proposition of this paper, we study the asymptotic behavior of the positive solutions of (1.5). We need the following lemma.

Lemma 4.2. *Consider equation* (1.5)*. Let* x_n *be a positive solution of* (1.5)*. Then if* $a \neq 1$*, for* $j = 0, 1, \dots, m(k+1) - 1$ *and* $n = 0, 1, \dots$*, we have*

$$x_{nm(k+1)+j} = (a-1)^n x_j \prod_{s=1}^{n} \frac{1}{c(a-1)(\frac{1}{a})^{sm(k+1)+j} + 1} \tag{4.2}$$

where

$$c = \frac{x_{-m(k+1)}}{x_0} - \frac{1}{a-1},$$

and if $a = 1$*, for* $j = 0, 1, \dots, m(k+1) - 1$ *and* $n = 0, 1, \dots$*, we have*

$$x_{nm(k+1)+j} = x_j \prod_{s=1}^{n} \frac{1}{d + sm(k+1) + j}, \quad d = \frac{x_{-m(k+1)}}{x_0}. \tag{4.3}$$

Proof. We set

$$y_n = \frac{x_{n-m(k+1)}}{x_n}. \tag{4.4}$$

Then from (1.5) and (4.4), we get

$$y_{n+1} = \frac{1}{a} y_n + \frac{1}{a}, \quad n = 0, 1, \dots. \tag{4.5}$$

So from (4.4) and (4.5), relations (4.2) and (4.3) follow immediately. This completes the proof. □

Proposition 4.3. *Consider equation* (1.5)*. Then the following statements are true:*

(i) *If* $0 < a < 2$*, then every positive solution of* (1.5) *tends to zero as* $n \to \infty$*.*

(ii) *If* $a = 2$*, then every positive solution of* (1.5) *tends to a periodic solution of* (1.5) *of period* $m(k+1)$ *as* $n \to \infty$*.*

(iii) *If* $a > 2$*, then every positive solution of* (1.5) *tends to* ∞ *as* $n \to \infty$*.*

Proof. Let x_n be an arbitrary solution of (1.5).

(i) Suppose that (2.15) holds. Then using (1.5) and arguing as in Proposition 2.3, we can prove that x_n tends to zero as $n \to \infty$.

Suppose that

$$1 < a < 2. \tag{4.6}$$

Let for $j = 0, 1, \dots, m(k+1) - 1$

$$D_n^{(j)} = \prod_{s=1}^{n} \frac{1}{c(a-1)\left(\dfrac{1}{a}\right)^{sm(k+1)+j} + 1}. \tag{4.7}$$

We have for $j = 0, 1, \ldots, m(k+1) - 1$

$$\ln(D_n^{(j)}) = -\sum_{s=1}^{n} \ln\Big(c(a-1)\Big(\frac{1}{a}\Big)^{sm(k+1)+j} + 1\Big). \tag{4.8}$$

In addition, from (2.20), we get

$$|\ln(1+a)| \leq \max\Big\{a,\ \frac{-a}{1+a}\Big\}. \tag{4.9}$$

Using (4.8) and (4.9) and since

$$1 < a, \tag{4.10}$$

we can prove that

$$\lim_{n\to\infty} (\ln(D_n^{(j)}) = L_j < \infty,\ \ j = 0, 1, \ldots, m(k+1) - 1 \tag{4.11}$$

which implies that

$$\lim_{n\to\infty} D_n^{(j)} = M_j < \infty,\ \ j = 0, 1, \ldots, m(k+1) - 1. \tag{4.12}$$

Therefore, from (4.2), (4.6), (4.7) and (4.12), we have that

$$\lim_{n\to\infty} x_{nm(k+1)+j} = 0,\ \ j = 0, 1, \ldots, m(k+1) - 1 \tag{4.13}$$

and so x_n tends to zero as $n \to \infty$.

Let now $a = 1$. We set for $j = 0, 1, \ldots, m(k+1) - 1$

$$K_n^{(j)} = \prod_{s=1}^{n} \frac{1}{d + sm(k+1) + j}. \tag{4.14}$$

Then from (4.14) for $j = 0, 1, \ldots, m(k+1) - 1$, we take

$$\ln(K_n^{(j)}) = -\sum_{s=1}^{n} \ln\Big(d + sm(k+1) + j\Big). \tag{4.15}$$

So from (4.15), we can prove that

$$\lim_{n\to\infty} (\ln(K_n^{(j)}) = -\infty,\ \ j = 0, 1, \ldots, m(k+1) - 1$$

which implies that

$$\lim_{n\to\infty} K_n^{(j)} = 0,\ \ j = 0, 1, \ldots, m(k+1) - 1. \tag{4.16}$$

Then relations (4.3), (4.14), (4.16) imply that (4.13) are true, and so x_n tends to zero as $n \to \infty$.

(ii) Suppose now that $a = 2$. Then from (4.10), relations (4.12) are true. So from (4.2), we have

$$\lim_{n\to\infty} x_{nm(k+1)+j} = M_j x_j < \infty, \quad j = 0, 1, \dots, m(k+1) - 1,$$

and so x_n tends to a periodic solution of (1.5) of period $m(k+1)$ as $n \to \infty$.

(iii) Finally, suppose that $a > 2$. Then using (4.10), we have that relations (4.12) hold, and so from (4.2) it is clear that x_n tends to ∞ as $n \to \infty$. This completes the proof. □

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