Asymptotic Behavior of the Solutions of a Class of Rational Difference Equations

G. Papaschinopoulos and G. Stefanidou
Democritus University of Thrace
School of Engineering
67100 Xanthi, Greece
gpapas@env.duth.gr, gstefmath@yahoo.gr

Abstract

In this paper we study the asymptotic behavior of the positive solutions of certain rational difference equations.

AMS Subject Classifications: 39A10.

Keywords: Difference equation, periodic solution, convergence of the solutions.

1 Introduction

In [8] the author studied the global behavior of the second order rational difference equation having quadratic term

$$x_{n+1} = \frac{ax_{n-1}}{x_n x_{n-1} + b}, \quad a > 0, \quad b > 0$$

(1.1)

and the third order difference equation having quadratic term

$$x_{n+1} = \frac{ax_n x_{n-1}}{x_n + bx_{n-2}}, \quad a > 0, \quad b > 0$$

(1.2)

where $x_{-2}, x_{-1}, x_0$ are real numbers. For the study of equation (1.1) the author used the fact that (1.1) reduces to a linear nonhomogeneous equation. Moreover, for the study of (1.2) he showed that equation (1.2) reduces to (1.1).

Furthermore in [3] the authors investigated equation (1.1) with nonnegative initial values $x_{-1}, x_0$. Moreover if we get $b = 1$ in (1.1), then by dropping either the term $x_n$ or $x_{n-1}$ in the denominator of the equation (1.1), we obtain the equations

$$x_{n+1} = \frac{ax_{n-1}}{x_n + 1}, \quad x_{n+1} = \frac{ax_{n-1}}{x_{n-1} + 1}$$

Received April 14, 2010; Accepted May 26, 2010
Communicated by Gerasimos Ladas
which have been studied in \[2\]. Finally, results concerning rational difference equations
having quadratic terms are included in \[1, 3–11\] and the references cited therein.

Now in this paper we study the following equations

\[
x_{n+1} = \frac{a x_{n-m(k+1)+1}}{\prod_{s=0}^{k} x_{n-m(s+1)+1} + 1}, \quad n = 0, 1, \ldots
\]  

(1.3)

\[
x_{n+1} = \frac{a x_{n-2k-1} \prod_{s=0}^{k} x_{n-2s}}{\prod_{s=0}^{2k+1} x_{n-s} + \prod_{s=0}^{k} x_{n-2s} + \prod_{s=0}^{k} x_{n-2s-1}},
\]  

(1.4)

and

\[
x_{n+1} = \frac{a x_n x_{n-m(k+1)+1}}{x_n + x_{n-m(k+1)}}, \quad n = 0, 1, \ldots,
\]  

(1.5)

where \(a\) is a positive number, \(m, k \in \{1, 2, \ldots\}\) and the initial values of the above
equations are positive numbers. More precisely, we study the existence of periodic
solutions and the asymptotic behavior of the positive solutions for equations (1.3), (1.4),
(1.5). We note that equations (1.3), (1.4), (1.5) have a common property: They reduces
to a linear nonhomogeneous equation.

2 Study of Equation (1.3)

First we study the existence of positive periodic solutions of period \(m(k+1)\) for equation
(1.3).

Proposition 2.1. Consider equation (1.3). Suppose that

\[
a > 1.
\]  

(2.1)

Then equation (1.3) has periodic solutions of period \(m(k + 1)\).

Proof. Suppose that \(x_n\) is a positive solution of (1.3) with initial values \(x_{-m(k+1)+1},
\)
\(x_{-m(k+1)+2}, \ldots, x_0 > 0\) such that

\[
\prod_{s=0}^{k} x_{i-m(s+1)+1} = a - 1, \quad i = 0, 1, \ldots, m - 1.
\]  

(2.2)
We prove that \( x_n \) is a periodic solution of (1.3) of period \( m(k+1) \). From (1.3) and (2.2), we get
\[
\begin{align*}
x_1 &= \frac{ax^{-m(k+1)+1}}{k} \prod_{s=0}^{k} x^{-m(s+1)+1} + 1 = x^{-m(k+1)+1}, \\
x_2 &= \frac{ax^{-m(k+1)+2}}{k} \prod_{s=0}^{k} x^{-m(s+1)+2} + 1 = x^{-m(k+1)+2}, \\
\cdots \\
x_m &= \frac{ax^{-mk}}{k} \prod_{s=0}^{k} x^{-ms} + 1 = x^{-mk}.
\end{align*}
\] (2.3)

Then from (1.3) and (2.3), we obtain
\[
\begin{align*}
x_{m+1} &= \frac{ax^{-mk+1}}{k} \prod_{s=1}^{k} x^{-ms+1} + 1 = \frac{ax^{-mk+1}}{k} \prod_{s=1}^{k} x^{-ms+1} + 1 \\
&= \frac{ax^{-mk+1}}{k} \prod_{s=0}^{k} x^{-m(s+1)+1} + 1
\end{align*}
\]

Working inductively, we can prove that
\[
x_{m+j} = x^{-mk+j}, \quad j = 2, 3, \ldots
\]
and so the proof is completed.

In the next proposition, we study the asymptotic behavior of the positive solutions of (1.3). We need the following lemma.

**Lemma 2.2.** Let \( x_n \) be an arbitrary positive solution of (1.3). Then the following statements are true:

(i) If
\[
t_n = \prod_{s=0}^{k} x_{n-sm}^{-1}, \quad n = 1, 2, \ldots
\] (2.4)

with
\[
t_j = \prod_{s=0}^{k} x_{j-sm}^{-1}, \quad j = 1 - m, 2 - m, \ldots, 0,
\] (2.5)
then \( t_n \) satisfies the nonhomogeneous linear difference equation

\[
y_{n+1} = \frac{1}{a} y_{n+1-m} + \frac{1}{a}, \quad n = 0, 1, \ldots
\]  

(2.6)

Moreover,

\[
t_n = \begin{cases} 
B_n + \frac{n}{m}, & n = 1, 2, \ldots \quad \text{if } a = 1 \\
\left(\frac{1}{a}\right)^{\frac{m}{2}} B_n + \frac{1}{a - 1}, & n = 1, 2, \ldots \quad \text{if } a \neq 1 
\end{cases}
\]

(2.7)

where

\[
B_n = \sum_{i=0}^{r} c_i \cos \left(\frac{2\pi mi}{m}\right) + d_i \sin \left(\frac{2\pi mi}{m}\right), \quad r = \begin{cases} 
\frac{m - 1}{2}, & \text{if } m \text{ is odd} \\
\frac{m}{2}, & \text{if } m \text{ is even} 
\end{cases}
\]

(2.8)

and \( c_i, d_i, i = 0, 1, \ldots, r \) are constants which are derived from (2.5), (2.7) and (2.8).

(ii) If

\[
y_m^{(j)} = x_{m(k+1)n+j}, \quad j = 0, 1, \ldots, m(k+1) - 1, \quad \text{with } y_m^{(0)} = 1
\]

(2.9)

then

\[
y_m^{(j)} = y_0^{(j)} \prod_{s=0}^{n-1} \frac{x_{m(k+1)(s+1)+j-m}}{x_{m(k+1)(s+1)+j}}, \quad j = 0, 1, \ldots, m(k+1) - 1.
\]

(2.10)

Proof. Let \( x_n \) be an arbitrary solution of (1.3). Then we get

\[
x_{n+1} \prod_{s=1}^{k} x_{n+1-sm} = \frac{a x_{n-m(k+1)+1} \prod_{s=1}^{k} x_{n+1-sm}}{\prod_{s=0}^{k} x_{n+1-(s+1)m} + 1}
\]

which implies that

\[
\prod_{s=0}^{k} x_{n+1-sm} = a \prod_{s=0}^{k} x_{n+1-(s+1)m} \prod_{s=0}^{k} x_{n+1-(s+1)m} + 1
\]

(2.11)
Then from (2.4) and (2.11), we have
\[
\frac{1}{t_{n+1}} = \frac{1}{t_{n+1-m}} + 1
\]
which implies that \( t_n \) satisfies the difference equation (2.6). Then relations (2.7) and (2.8) follow immediately. This completes the proof of statement (i).

(ii) From (2.4), we have
\[
t_n = \frac{x_n^{-1}x_{n-m}^{-1} \cdots x_{n-km}^{-1}}{x_{n-m}^{-1}x_{n-2m}^{-1} \cdots x_{n-(k+1)m}^{-1}} = \frac{x_{n-m(k+1)}}{x_n}
\]
which implies that
\[
x_n = \frac{t_{n-m}}{t_n}x_{n-m(k+1)}, \quad n = 1, 2, \ldots .
\]
(2.12)
So, from (2.9) and (2.12) it holds
\[
y_{n+1}(j) = \frac{t_{m(k+1)(n+1)+j-m}}{t_{m(k+1)(n+1)+j}} y_n(j), \quad j = 0, 1, \ldots , m(k+1) - 1.
\]
(2.13)
Therefore relation (2.13) implies that (2.10) is true. This completes the proof.

**Proposition 2.3.** Consider equation (1.3). Then the following statements are true.

(i) If
\[
0 < a \leq 1,
\]
then every positive solution of (1.3) tends to zero as \( n \to \infty \).

(ii) If (2.1) holds, then every positive solution of (1.3) tends to a periodic solution of period \( m(k+1) \).

**Proof.** Let \( x_n \) be an arbitrary positive solution of (1.3).

(i) Suppose first that
\[
0 < a < 1.
\]
(2.15)
From (1.3), we get for \( j = 0, 1, \ldots , m(k+1) - 1 \)
\[
x_{m(k+1)n+j} < ax_{m(k+1)(n-1)+j} < \cdots < a^nx_j.
\]
(2.16)
Then from (2.15) and (2.16), we take
\[
\lim_{n \to \infty} x_{m(k+1)n+j} = 0, \quad j = 0, 1, \ldots , m(k+1) - 1
\]
which implies that \( x_n \) tends to zero as \( n \to \infty \).
Let now $a = 1$. We consider the functions

$$A_n^{(j)} = \ln \left( \prod_{s=0}^{n-1} \frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}} \right) = \sum_{s=0}^{n-1} \ln \left( \frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}} \right).$$ \hspace{1cm} (2.17)

From (2.8) it is obvious that

$$B_{m(k+1)(s+1)+j-m} = B_{m(k+1)(s+1)+j}, \quad s = 0, 1, \ldots, \quad j = 0, 1, \ldots, m(k+1)-1. \hspace{1cm} (2.18)$$

Hence relations (2.7), (2.8) and (2.18) imply that

$$t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j} = -1, \quad s = 0, 1, \ldots, \quad j = 0, 1, \ldots, m(k+1) - 1. \hspace{1cm} (2.19)$$

In addition, if $a$ is a real number such that $1 + a > 0$, then

$$\ln(1 + a) < a. \hspace{1cm} (2.20)$$

Then from (2.19) and (2.20), we get

$$\sum_{s=0}^{n-1} \ln \left( 1 + \frac{t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j}}{t_{m(k+1)(s+1)+j}} \right) \leq \sum_{s=0}^{n-1} \frac{1}{t_{m(k+1)(s+1)+j}}. \hspace{1cm} (2.21)$$

Since from (2.7)

$$\sum_{s=0}^{\infty} \frac{1}{t_{m(k+1)(s+1)+j}} = \infty,$$

we have from (2.21)

$$\sum_{s=0}^{\infty} \ln \left( \frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}} \right) = -\infty. \hspace{1cm} (2.22)$$

Therefore, from (2.17) and (2.22), we have

$$\lim_{n \to \infty} A_n^{(j)} = -\infty, \quad j = 0, 1, \ldots, m(k+1) - 1$$

which implies that

$$\prod_{s=0}^{\infty} \frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}} = 0, \quad j = 0, 1, \ldots, m(k+1) - 1. \hspace{1cm} (2.23)$$

So from (2.10) and (2.23), we have that $x_n$ tends to zero as $n \to \infty$. This completes the proof of statement (i).
(ii) If $a, b > 0$, then using (2.20), we can easily prove that

$$\left| \ln \left( \frac{a}{b} \right) \right| \leq |a - b| \max \left\{ \frac{1}{a}, \frac{1}{b} \right\}. \quad (2.24)$$

Then from (2.24), we have for $j = 0, 1, \ldots, m(k + 1) - 1$

$$\left| \sum_{s=0}^{n-1} \ln \left( \frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}} \right) \right| \leq \sum_{s=0}^{n-1} \left| \ln \left( \frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}} \right) \right|
\leq \sum_{s=0}^{n-1} \left| t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j} \right| \max \left\{ \frac{1}{t_{m(k+1)(s+1)+j}}, \frac{1}{t_{m(k+1)(s+1)+j-m}} \right\}. \quad (2.25)$$

Furthermore, from (2.1), (2.7) and (2.18), we have

$$|t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j}| = \left( \frac{1}{a} \right)^{m(k+1)(s+1)+j} |B_{m(k+1)(s+1)+j}|(a-1). \quad (2.26)$$

Then using (2.17) and (2.28), it is obvious that there exist

$$\lim_{n \to \infty} A_n^{(j)} = l_j < \infty, \quad j = 0, 1, \ldots, m(k + 1) - 1. \quad (2.29)$$

Relations (2.9), (2.10), (2.17) and (2.29) imply that

$$\lim_{n \to \infty} x_{m(k+1)n+j} = p_j < \infty, \quad j = 0, 1, \ldots, m(k + 1) - 1.$$  

This completes the proof. □
3 Study of Equation (1.4)

First we study the existence of positive solutions of period $2k + 2$ for the equation (1.4).

**Proposition 3.1.** Consider equation (1.4) where

$$a > 2.$$  \hspace{1cm} (3.1)

Then equation (1.4) has positive periodic solutions of period $2k + 2$.

**Proof.** Let $x_n$ be a positive solution of (1.4) with initial values such that

$$\prod_{s=0}^{k} x_{-2s} = \prod_{s=0}^{k} x_{-2s-1} = a - 2.$$  \hspace{1cm} (3.2)

Then from (1.4) and (3.2), we get

$$x_1 = \frac{ax_{-2k-1} \prod_{s=0}^{k} x_{-2s}}{\prod_{s=0}^{2k+1} x_{-s} + \prod_{s=0}^{k} x_{-2s} + \prod_{s=0}^{k} x_{-2s-1}} = \frac{a(a - 2)x_{-2k-1}}{(a - 2)^2 + 2(a - 2)} = x_{-2k-1},$$

$$x_2 = \frac{ax_1 x_{-2k} \prod_{s=1}^{k} x_{1-2s}}{x_1 \prod_{s=1}^{2k+1} x_{1-s} + x_1 \prod_{s=1}^{k} x_{1-2s} + \prod_{s=0}^{k} x_{-2s}} = \frac{a(a - 2)x_{-2k}}{(a - 2)^2 + 2(a - 2)} = x_{-2k}.$$  

Working inductively, we can prove that

$$x_n = x_{n-2k-2}, \hspace{0.2cm} n = 3, 4, \ldots.$$  

This completes the proof.  \qed
In the following proposition, we study the asymptotic behavior of the positive solutions of (1.4). We need the following lemma.

**Lemma 3.2.** Let \( x_n \) be a positive solution of (1.4). Then the following statements are true:

(i) If

\[
t_n = \prod_{s=0}^{k-1} x_{n-2s}, \quad n = 1, 2, \ldots
\]

(3.3)

with

\[
t_j = \prod_{s=0}^{k-1} x_{j-2s}, \quad j = -1, 0,
\]

(3.4)

then \( t_n, n = 1, 2, \ldots \) satisfies the following difference equation

\[
y_{n+1} = \frac{1}{a} y_n + \frac{1}{a} y_{n-1} + \frac{1}{a}, \quad n = 0, 1, \ldots
\]

(3.5)

Moreover,

\[
t_n = \begin{cases} 
  c_1 \left( -\frac{1}{2} \right)^n + c_2 + \frac{1}{3} n, & n = 1, 2, \ldots \quad \text{if } a = 2 \\
  c_1 p_1^n + c_2 p_2^n + \frac{1}{a-2}, & n = 1, 2, \ldots \quad \text{if } a \neq 2
\end{cases}
\]

(3.6)

where

\[
p_1 = \frac{1}{2a} (1 - \sqrt{1 + 4a}), \quad p_2 = \frac{1}{2a} (1 + \sqrt{1 + 4a}),
\]

(3.7)

\( c_1, c_2 \) are defined from (3.4) and (3.6).

(ii) If

\[
y^{(j)}_n = x_{2(k+1)n+j}, \quad j = 0, 1, \ldots, 2k + 1,
\]

(3.8)

then

\[
y^{(j)}_n = y_0^{(j)} \prod_{s=0}^{n-1} \frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}}, \quad j = 0, 1, \ldots, 2k + 1.
\]

(3.9)

**Proof.** (i) Let \( x_n \) be an arbitrary positive solution of (1.4). Then we get
which implies that
\[
\prod_{s=0}^{k} x_{n-2s+1} = \frac{2k+1}{a} \prod_{s=0}^{k} x_{n-s} \prod_{s=0}^{k} x_{n-2s} + \prod_{s=0}^{k} x_{n-2s-1}.
\] (3.10)

Then relations (3.3) and (3.10) imply that
\[
\frac{1}{t_{n+1}} = \frac{a}{t_n t_{n-1}} + \frac{1}{t_n} + \frac{1}{t_{n-1}}
\]
from which we conclude that \( t_n \) satisfies the difference equation (3.5). Then relation (3.6) follows immediately.

(ii) Using (3.3), we take
\[
\prod_{s=0}^{k} x_{n-2s+1} = \frac{x_{n-2} x_{n-2} \cdots x_{n-2k}}{x_{n-2k} \cdots x_n}
\]
which implies that
\[
x_n = \frac{t_{n-2}}{t_n} x_{n-2k-2}, \ n = 1, 2, \ldots \quad (3.11)
\]
from which we conclude that \( t_n \) satisfies the difference equation (3.5). Then relation (3.6) follows immediately.

Proposition 3.3. Consider equation (1.4). Then the following statements are true:

(i) If
\[
0 < a \leq 2,
\] (3.13)
then every positive solution of (1.4) tends to zero as \( n \to \infty \).

(ii) If
\[
a > 2,
\] (3.14)
then every positive solution of (1.4) tends to a periodic solution of (1.4) of period \( 2k + 2 \).
Proof. Let $x_n$ be an arbitrary positive solution of (1.4).

(i) Suppose that (2.15) is satisfied. Relation (1.4) implies that for $j = 0, 1, \ldots, 2k+1$

$$x_{2(k+1)n+j} < ax_{2(k+1)(n-1)+j} < \ldots < a^n x_j.$$  (3.15)

Therefore, from (2.15) and (3.15), we get

$$\lim_{n \to \infty} x_{2(k+1)n+j} = 0, \quad j = 0, 1, \ldots, 2k + 1$$  (3.16)

which imply that $x_n$ tends to zero as $n \to \infty$.

Suppose that $1 \leq a < 2$.  (3.17)

From (3.7) and (3.17), we can easily prove that

$$|p_1| < 1, \quad 1 < p_2.$$  (3.18)

We set for $j = 0, 1, \ldots, 2k + 1$

$$B_n^{(j)} = \ln \left( \prod_{s=0}^{n-1} \frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}} \right).$$  (3.19)

Then from (2.20), we have for $j = 0, 1, \ldots, 2k + 1$

$$B_n^{(j)} = \sum_{s=0}^{n-1} \ln \left( 1 + \frac{t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}} \right) \leq \sum_{s=0}^{n-1} \frac{t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}}.$$  (3.20)

Moreover, from (3.6) and (3.20), we can prove that

$$\frac{t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}} = \frac{c_1(p_1^{-2} - 1)(P_1^{2(k+1)(s+1)+j}) + c_2(p_2^{-2} - 1)}{c_1(P_1^{2(k+1)(s+1)+j}) + c_2 + \frac{1}{a - 2} p_2^{-2} - 2(k+1)(s+1)+j}. \quad (3.21)$$

Using (3.18) and (3.21), we have that

$$\lim_{s \to \infty} \left( \frac{t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}} \right) = p_2^{-2} - 1 < 0.$$  (3.22)

Therefore, from (3.20) and (3.22), we can prove that

$$\lim_{n \to \infty} B_n^{(j)} = -\infty, \quad j = 0, 1, \ldots, 2k + 1.$$  (3.23)
which from (3.19) imply that for \(j = 0, 1, \ldots, 2k + 1\)

\[
\prod_{s=0}^{\infty} \frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}} = 0. 
\] (3.24)

Hence, from (3.8), (3.9) and (3.24), we have that relations (3.16) are true and so \(x_n\) tends to zero as \(n \to \infty\).

Suppose now that \(a = 2\).

(ii) Finally, suppose that (3.14) is satisfied. Then from (3.7) it is obvious that

\[
\left| \frac{p_1}{p_2} \right| < 1, \quad |p_1| < 1, \quad p_2 < 1. 
\] (3.28)

In addition, from (3.6), we have that for \(j = 0, 1, \ldots, 2k + 1\)

\[
t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j} \\
= p_2^{2(k+1)(s+1)+j} \left( c_1 (p_1^{-2} - 1) \left( p_1 \right)^{2(k+1)(s+1)+j} + c_2 (p_2^{-2} - 1) \right). 
\] (3.29)

In addition, from (2.24), we get for \(j = 0, 1, \ldots, 2k + 1\)

\[
\left| \ln \left( \frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}} \right) \right| \\
\leq |t_{2(k+1)(s+1)-2+j} - t_{2(k+1)(s+1)+j}| \max \left\{ \frac{1}{t_{2(k+1)(s+1)-2+j}}, \frac{1}{t_{2(k+1)(s+1)+j}} \right\}. 
\] (3.30)
Using (3.6), (3.28), (3.29) and (3.30), there exists a positive number \( N \) such that for \( j = 0, 1, \ldots, 2k + 1 \)

\[
\left| \ln \left( \frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}} \right) \right| \leq Np_2^{2(k+1)(s+1)+j}. \tag{3.31}
\]

Therefore, from (3.19) and (3.31), we have that there exist

\[
\lim_{n \to \infty} B_{n}^{(j)} = \mu_j < \infty, \quad j = 0, 1, \ldots, 2k + 1. \tag{3.32}
\]

Hence, relations (3.8), (3.9), (3.19) and (3.32) imply that

\[
\lim_{n \to \infty} x_{2(k+1)n+j} = q_j < \infty, \quad j = 0, 1, \ldots, 2k + 1,
\]

and so the proof is completed.

\[\Box\]

4 Study of Equation (1.5)

In the first proposition, we study the existence of positive periodic solutions of (1.5) of period \( m(k+1) \).

**Proposition 4.1.** Consider equation (1.5) where (3.25) holds. Let \( x_n \) be positive solution of (1.5) such that

\[
x_0 = x_{-m(k+1)}. \tag{4.1}
\]

Then \( x_n \) is a periodic solution of (1.5) with period \( m(k+1) \).

**Proof.** Let \( x_n \) be a positive solution of (1.5) such that (4.1) holds. Then from (1.5), (3.25), we get

\[
x_1 = \frac{2x_0x_{-m(k+1)+1}}{x_0 + x_{-m(k+1)}} = \frac{2x_0x_{-m(k+1)+1}}{2x_0} = x_{-m(k+1)+1}
\]

and working inductively, we can prove that

\[
x_n = x_{n-m(k+1)}, \quad n = 1, 2, \ldots.
\]

This completes the proof. \[\Box\]

In the last proposition of this paper, we study the asymptotic behavior of the positive solutions of (1.5). We need the following lemma.
Lemma 4.2. Consider equation (1.5). Let $x_n$ be a positive solution of (1.5). Then if $a \neq 1$, for $j = 0, 1, \ldots, m(k+1) - 1$ and $n = 0, 1, \ldots$, we have

$$x_{nm(k+1)+j} = (a - 1)^n x_j \prod_{s=1}^{n} \frac{1}{c(a-1)(\frac{1}{a})^{sm(k+1)+j} + 1}$$

(4.2)

where

$$c = \frac{x_{-m(k+1)}}{x_0} - \frac{1}{a - 1}$$

and if $a = 1$, for $j = 0, 1, \ldots, m(k+1) - 1$ and $n = 0, 1, \ldots$, we have

$$x_{nm(k+1)+j} = x_j \prod_{s=1}^{n} \frac{1}{d + sm(k+1) + j}, \quad d = \frac{x_{-m(k+1)}}{x_0}.$$  

(4.3)

Proof. We set

$$y_n = \frac{x_{n-m(k+1)}}{x_n}.$$  

(4.4)

Then from (1.5) and (4.4), we get

$$y_{n+1} = \frac{1}{a} y_n + \frac{1}{a}, \quad n = 0, 1, \ldots.$$  

(4.5)

So from (4.4) and (4.5), relations (4.2) and (4.3) follow immediately. This completes the proof.

Proposition 4.3. Consider equation (1.5). Then the following statements are true:

(i) If $0 < a < 2$, then every positive solution of (1.5) tends to zero as $n \to \infty$.

(ii) If $a = 2$, then every positive solution of (1.5) tends to a periodic solution of (1.5) of period $m(k+1)$ as $n \to \infty$.

(iii) If $a > 2$, then every positive solution of (1.5) tends to $\infty$ as $n \to \infty$.

Proof. Let $x_n$ be an arbitrary solution of (1.5).

(i) Suppose that (2.15) holds. Then using (1.5) and arguing as in Proposition 2.3, we can prove that $x_n$ tends to zero as $n \to \infty$.

Suppose that

$$1 < a < 2.$$  

(4.6)

Let for $j = 0, 1, \ldots, m(k+1) - 1$

$$D^{(j)}_n = \prod_{s=1}^{n} \frac{1}{c(a-1)(\frac{1}{a})^{sm(k+1)+j} + 1}.$$  

(4.7)
Asymptotic Behavior of the Solutions of Rational Difference Equations

We have for $j = 0, 1, \ldots, m(k + 1) - 1$

$$\ln(D_n^{(j)}) = -\sum_{s=1}^{n} \ln\left(c(a-1)\left(\frac{1}{a}\right)^{sm(k+1)+j} + 1\right). \quad (4.8)$$

In addition, from (2.20), we get

$$|\ln(1 + a)| \leq \max\left\{a, \frac{-a}{1+a}\right\}. \quad (4.9)$$

Using (4.8) and (4.9) and since $1 < a$, (4.10)

we can prove that

$$\lim_{n \to \infty} (\ln(D_n^{(j)})) = L_j < \infty, \quad j = 0, 1, \ldots, m(k + 1) - 1 \quad (4.11)$$

which implies that

$$\lim_{n \to \infty} D_n^{(j)} = M_j < \infty, \quad j = 0, 1, \ldots, m(k + 1) - 1. \quad (4.12)$$

Therefore, from (4.2), (4.6), (4.7) and (4.12), we have that

$$\lim_{n \to \infty} x_{nm(k+1)+j} = 0, \quad j = 0, 1, \ldots, m(k + 1) - 1 \quad (4.13)$$

and so $x_n$ tends to zero as $n \to \infty$.

Let now $a = 1$. We set for $j = 0, 1, \ldots, m(k + 1) - 1$

$$K_n^{(j)} = \prod_{s=1}^{n} \frac{1}{d + sm(k + 1) + j}. \quad (4.14)$$

Then from (4.14) for $j = 0, 1, \ldots, m(k + 1) - 1$, we take

$$\ln(K_n^{(j)}) = -\sum_{s=1}^{n} \ln\left(d + sm(k + 1) + j\right). \quad (4.15)$$

So from (4.15), we can prove that

$$\lim_{n \to \infty} (\ln(K_n^{(j)})) = -\infty, \quad j = 0, 1, \ldots, m(k + 1) - 1$$

which implies that

$$\lim_{n \to \infty} K_n^{(j)} = 0, \quad j = 0, 1, \ldots, m(k + 1) - 1. \quad (4.16)$$

Then relations (4.3), (4.14), (4.16) imply that (4.13) are true, and so $x_n$ tends to zero as $n \to \infty$. 

(ii) Suppose now that \( a = 2 \). Then from (4.10), relations (4.12) are true. So from (4.2), we have
\[
\lim_{n \to \infty} x_{nm(k+1)+j} = M_j x_j < \infty, \quad j = 0, 1, \ldots, m(k+1) - 1,
\]
and so \( x_n \) tends to a periodic solution of (1.5) of period \( m(k+1) \) as \( n \to \infty \).

(iii) Finally, suppose that \( a > 2 \). Then using (4.10), we have that relations (4.12) hold, and so from (4.2) it is clear that \( x_n \) tends to \( \infty \) as \( n \to \infty \). This completes the proof.

References


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