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# Oscillation of Second Order Functional Dynamic Equations

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**Abstract**

In this paper we investigate the oscillation of a second order nonlinear functional dynamic equation. Our results extend and improve many known results for oscillation of second order dynamic equations. Some examples are given to illustrate the main results.

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**Keywords:** Oscillation, nonlinear functional dynamic equations, time scales.

## 1 Introduction

Not only does the theory of so-called "dynamic equations" unify the theories of differential equations and difference equations, but also extends these classical cases to cases "in between", e.g., to so-called $q$-difference equations when $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$ (which has important applications in quantum theory (see [26])) and can be applied on different types of time scales like $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \mathbb{N}_0^2$ and $\mathbb{T} = \{H_n\}$ the set of harmonic numbers. In this work a knowledge and understanding of time scales and time scale notation is

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assumed; for an excellent introduction to the calculus on time scales, see Bohner and Peterson [6, 7].

We are concerned with the oscillatory behavior of the solutions of the second order nonlinear functional dynamic equation

$$\left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\sum_{i=0}^{n} p_i(t)\Phi_{\alpha_i}\left(x\left(g_i(t)\right)\right)=0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where $\gamma$ is a quotient of odd positive integers and $\Phi_c(u)=|u|^c$ sgn $u,\ c>0$ with $\alpha_0=\gamma$ and where $a$ is a positive rd-continuous function on $\mathbb{T}$ and $p_i,\ i=0,1,2,\ldots,n$ are nonnegative rd-continuous functions on $\mathbb{T}$ such that not all of the $p_i(t)$ vanish in a neighborhood of infinity. The functions $g_i:\mathbb{T}\to\mathbb{T},\ i=0,1,2,\ldots,n$ satisfy $\lim_{t\to\infty} g_i(t)=\infty$, $i=0,1,2,\ldots,n$. Both of the two cases

$$\int_{t_0}^{\infty}\frac{\Delta t}{a^{1/\gamma}(t)}=\infty, \tag{1.2}$$

and

$$\int_{t_0}^{\infty}\frac{\Delta t}{a^{1/\gamma}(t)}<\infty, \tag{1.3}$$

are considered. Equation (1.1) can be viewed as a perturbation of the important equation

$$\left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+p_0x^{\gamma}(g_0(t))=0,$$

by nonlinear terms which may be superlinear or sublinear.

Since we are interested in the oscillatory behavior of solutions near infinity, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. By a solution of (1.1) we mean a nontrivial real-valued function $x\in C^1_{rd}[T_x,\infty)_{\mathbb{T}}$, $T_x\geq t_0$ which has the property that $a\left(x^{\Delta}\right)^{\gamma}\in C^1_{rd}[T_x,\infty)$ and $x$ satisfies equation (1.1) on $[T_x,\infty)_{\mathbb{T}}$, where $C_{rd}$ is the space of rd-continuous functions. The solutions vanishing identically in some neighborhood of infinity will be excluded from our consideration. A solution $x$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is said to be nonoscillatory.

We note that (1.1) includes a large class of differential equations/difference equations with delay term. For example, if $n=0$, and $\alpha_0=\gamma$ then the equation becomes the second order superlinear, linear, or sublinear equation with delay for which there are many oscillation criteria. In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations, we refer the reader to the papers [1–3, 5, 8–18, 20–24] and the references cited therein.

Before stating our main results, we begin with the following lemma which will play an important role in the proof of our main results.

**Lemma 1.1.** *Let* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *be an* $n$*-tuple satisfying* $\alpha_1 > \alpha_2 > \ldots > \alpha_m > \gamma > \alpha_{m+1} > \ldots > \alpha_n > 0$. *Then there exists an* $n$*-tuple* $(\eta_1, \eta_2, \ldots, \eta_n)$ *with* $0 < \eta_i < 1$ *satisfying*

$$\sum_{i=1}^{n} \alpha_i \eta_i = \gamma, \quad \sum_{i=1}^{n} \eta_i = 1. \tag{1.4}$$

The proof of Lemma 1.1 is as in [22, Lemma 2.1].

Throughout this paper, we assume $g(t) := \min\{t,\ g_1(t),\ g_2(t), \ldots, g_n(t)\}$ and, for all sufficiently large $T_1 \in [t_0, \infty)_{\mathbb{T}}$,

$$A(t, T_1) := \frac{B(t, T_1)}{B(g(t), T_1)}, \qquad \text{for } g(t) \neq T_1,$$

where

$$B(u, T_1) := \int_{T_1}^{u} \frac{\Delta s}{a^{1/\gamma}(s)}.$$

We also assume $\alpha_1 > \alpha_2 > \ldots > \alpha_m > \gamma > \alpha_{m+1} > \ldots > \alpha_n > 0$.

# 2 Oscillation Criteria for (1.1) when (1.2) holds

In this section, we establish oscillation criteria for equation (1.1) when (1.2) holds. In the next theorem, we assume $g_i(t) \equiv \tau(t)$, $i = 1, 2, \ldots, n$, $\tau(t) \leq t$, and we assume that $\tau$ is a nondecreasing and delta differentiable function with $\tau \circ \sigma = \sigma \circ \tau$ on $[t_0, \infty)_{\mathbb{T}}$. This condition is satisfied for a number of important time scales (e.g., $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ or any isolated time scales). In fact, many of the results dealing with delay dynamic equations for the case when $\mathbb{T}$ is an isolated time scale are stated for the case when $\tau(t) = \rho^k(t)$ and in this case, the above assumptions clearly hold. In the first result we use the notation $d_+ = \max\{d, 0\}$.

**Theorem 2.1.** *Let* $g_i(t) \equiv \tau(t)$, $i = 1, 2, \ldots, n$, $\tau(t) \leq t$, *and let* $\tau$ *be nondecreasing and delta differentiable with* $\tau(\sigma(t)) = \sigma(\tau(t))$ *on* $[t_0, \infty)_{\mathbb{T}}$. *Assume that* (1.2) *holds, and assume for any given positive differentiable function* $\phi(t)$ *and all sufficiently large* $T$*, we have*

$$\limsup_{t\to\infty} \int_{T}^{t} \left[ P_1(s) - \frac{\left(\left(\phi^{\Delta}(s)\right)_+\right)^{\gamma+1} a(\tau(s))}{(\gamma+1)^{\gamma+1} \left(\phi(s)\tau^{\Delta}(s)\right)^{\gamma}} \right] \Delta s = \infty, \tag{2.1}$$

*where*

$$P_1(t) := \phi(t) \left[ p_0(t) + \prod_{i=1}^{n} \left(\eta_i^{-1} p_i(t)\right)^{\eta_i} \right].$$

*Then every solution of equation* (1.1) *is oscillatory.*

*Proof.* Assume (1.1) has a nonoscillatory solution $x$ on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1\in[t_0,\infty)_{\mathbb{T}}$, sufficiently large, such that $x(t)>0$ and $x(\tau(t))>0$ on $[t_1,\infty)_{\mathbb{T}}$ and not all of the $p_i(t)$ are identically zero on $[t_1,\infty)_{\mathbb{T}}$. From (1.1), we have

$$\left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}=-\sum_{i=0}^{n}p_i(t)x^{\alpha_i}(\tau(t))<0. \tag{2.2}$$

Then $a(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. As in [10, Lemma 2.1], we get $x^{\Delta}(t)>0$ on $[t_1,\infty)_{\mathbb{T}}$. Consider the generalized Riccati substitution

$$w(t):=\phi(t)\left[\frac{a(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(\tau(t))}\right].$$

By the product rule and then the quotient rule

$$\begin{aligned}w^{\Delta}(t) &= \frac{\phi(t)}{x^{\gamma}(\tau(t))}\left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}\\
&\quad+\left[\frac{\phi(t)}{x^{\gamma}(\tau(t))}\right]^{\Delta}(a(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\sigma}\\
&= \phi(t)\frac{\left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}}{x^{\gamma}(\tau(t))}\\
&\quad+\left[\frac{\phi^{\Delta}(t)}{(x\circ\tau)^{\gamma\sigma}(t)}-\frac{\phi(t)(x^{\gamma}(\tau(t)))^{\Delta}}{x^{\gamma}(\tau(t))\,(x\circ\tau)^{\gamma\sigma}(t)}\right](a(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\sigma}.\end{aligned}$$

From (1.1), we have

$$\begin{aligned}w^{\Delta}(t) &= -\phi(t)\sum_{i=0}^{n}p_i(t)x^{\alpha_i-\gamma}(\tau(t))\\
&\quad+\phi^{\Delta}(t)\left[\frac{a(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(\tau(t))}\right]^{\sigma}-\frac{\phi(t)(x^{\gamma}(\tau(t)))^{\Delta}}{x^{\gamma}(\tau(t))\,(x\circ\tau)^{\gamma\sigma}(t)}(a(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\sigma}.\end{aligned}$$

Using the arithmetic-geometric mean inequality [4, Page 17]

$$\sum_{i=1}^{n}\eta_i u_i\geq\prod_{i=1}^{n}u_i^{\eta_i},\quad \text{where } u_i\geq 0,$$

we get, for $t\geq T$

$$\begin{aligned}\sum_{i=1}^{n}p_i(t)x^{\alpha_i-\gamma}(\tau(t)) &= \sum_{i=1}^{n}\eta_i\left(\eta_i^{-1}p_i(t)x^{\alpha_i-\gamma}(\tau(t))\right)\\
&\geq \prod_{i=1}^{n}\left(\eta_i^{-1}p_i(t)\right)^{\eta_i}x^{\eta_i(\alpha_i-\gamma)}(\tau(t))\overset{(1.4)}{=}\prod_{i=1}^{n}\left(\eta_i^{-1}p_i(t)\right)^{\eta_i},\end{aligned}$$

which implies

$$\begin{aligned} w^{\Delta}(t) &\leq -P_1(t) + \phi^{\Delta}(t)\left[\frac{a(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}\left(\tau(t)\right)}\right]^{\sigma} \\ &\quad - \frac{\phi(t)(x^{\gamma}\left(\tau(t)\right))^{\Delta}}{x^{\gamma}\left(\tau(t)\right)\left(x\circ\tau\right)^{\gamma\sigma}(t)}(a(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\sigma}. \end{aligned}$$

Since $x$ and $\tau$ are differentiable functions, and $\tau$ is nondecreasing and $\tau \circ \sigma = \sigma \circ \tau$ on $[t_0, \infty)_{\mathbb{T}}$, we have $x \circ \tau$ is a differentiable function and $(x\left(\tau(t)\right))^{\Delta} = x^{\Delta}\left(\tau(t)\right)\tau^{\Delta}(t)$. Then, by the Pötzsche chain rule (see [6, Theorem 1.90]), we obtain

$$\begin{aligned} (x^{\gamma}\left(\tau(t)\right))^{\Delta} &= \gamma\int_0^1 \left[x\left(\tau(t)\right) + h\mu(t)(x\left(\tau(t)\right))^{\Delta}\right]^{\gamma-1} dh\ (x\left(\tau(t)\right))^{\Delta} \\ &= \gamma\int_0^1 \left[(1-h)\,x\left(\tau(t)\right) + h\left(x\circ\tau\right)^{\sigma}(t)\right]^{\gamma-1} dh\ x^{\Delta}\left(\tau(t)\right)\tau^{\Delta}(t) \\ &\geq \begin{cases} \gamma\left(x\circ\tau\right)^{\sigma(\gamma-1)}(t)x^{\Delta}\left(\tau(t)\right)\tau^{\Delta}(t), & 0<\gamma\leq 1 \\ \gamma x^{\gamma-1}\left(\tau(t)\right)x^{\Delta}\left(\tau(t)\right)\tau^{\Delta}(t), & \gamma\geq 1. \end{cases} \end{aligned}$$

If $0 < \gamma \leq 1$, we have that

$$\begin{aligned} w^{\Delta}(t) &\leq -P_1(t) + \phi^{\Delta}(t)\left[\frac{a(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}\left(\tau(t)\right)}\right]^{\sigma} \\ &\quad -\gamma\phi(t)x^{\Delta}\left(\tau(t)\right)\tau^{\Delta}(t)\frac{(a(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\sigma}}{\left(x\circ\tau\right)^{\sigma(\gamma+1)}(t)}\left(\frac{\left(x\circ\tau\right)^{\sigma}(t)}{x\left(\tau(t)\right)}\right)^{\gamma}, \end{aligned}$$

whereas if $\gamma \geq 1$, we have that

$$\begin{aligned} w^{\Delta}(t) &\leq -P_1(t) + \phi^{\Delta}(t)\left[\frac{a(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}\left(\tau(t)\right)}\right]^{\sigma} \\ &\quad -\gamma\phi(t)x^{\Delta}\left(\tau(t)\right)\tau^{\Delta}(t)\frac{(a(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\sigma}}{\left(x\circ\tau\right)^{\sigma(\gamma+1)}(t)}\frac{\left(x\circ\tau\right)^{\sigma}(t)}{x\left(\tau(t)\right)}. \end{aligned}$$

Using the fact that $x^{\Delta}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, we obtain, for $\gamma > 0$,

$$\begin{aligned} w^{\Delta}(t) &\leq -P_1(t) + \phi^{\Delta}(t)\left[\frac{a(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}\left(\tau(t)\right)}\right]^{\sigma} \\ &\quad -\gamma\phi(t)x^{\Delta}\left(\tau(t)\right)\tau^{\Delta}(t)\frac{(a(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\sigma}}{\left(x\circ\tau\right)^{\sigma(\gamma+1)}(t)}. \end{aligned} \tag{2.3}$$

Then using the fact that $a(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$, we get that

$$x^{\Delta}(\tau(t)) \geq \left[\frac{\left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}}{a(\tau(t))}\right]^{1/\gamma}, \quad \text{for} \quad t \geq t_1. \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\begin{aligned} w^{\Delta}(t) \ &\leq \ -P_1(t) + \phi^{\Delta}(t)\left[\frac{a(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(\tau(t))}\right]^{\sigma} \\ &\quad -\frac{\gamma\phi(t)\tau^{\Delta}(t)}{a^{1/\gamma}(\tau(t))}\left[\frac{\left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}}{(x\circ\tau)^{\sigma\gamma}(t)}\right]^{\lambda}, \end{aligned}$$

where $\lambda := \dfrac{\gamma+1}{\gamma}$. From the definition of $w(t)$, we get

$$w^{\Delta}(t) \leq -P_1(t) + \frac{\left(\phi^{\Delta}(t)\right)_+}{\phi^{\sigma}(t)}w^{\sigma}(t) - \frac{\gamma\phi(t)\tau^{\Delta}(t)}{a^{1/\gamma}(\tau(t))\,\phi^{\lambda\sigma}(t)}w^{\lambda\sigma}(t), \quad \text{for} \quad t \geq t_1. \tag{2.5}$$

Define $X \geq 0$ and $Y \geq 0$ by

$$X^{\lambda} := \frac{\gamma\phi(t)\tau^{\Delta}(t)}{a^{1/\gamma}(\tau(t))\,\phi^{\lambda\sigma}(t)}w^{\lambda\sigma}(t), \quad Y^{\lambda-1} := \frac{\left(\phi^{\Delta}(t)\right)_+ a^{\frac{1}{\gamma+1}}(\tau(t))}{\lambda\left(\gamma\phi(t)\tau^{\Delta}(t)\right)^{\frac{1}{\lambda}}},$$

and using the inequality (see [19])

$$\lambda XY^{\lambda-1} - X^{\lambda} \leq (\lambda-1)Y^{\lambda}, \tag{2.6}$$

we have

$$\frac{\left(\phi^{\Delta}(t)\right)_+}{\phi^{\sigma}(t)}w^{\sigma}(t) - \frac{\gamma\phi(t)\tau^{\Delta}(t)}{a^{1/\gamma}(\tau(t))\,\phi^{\lambda\sigma}(t)}w^{\lambda\sigma}(t) \leq \frac{\left(\left(\phi^{\Delta}(t)\right)_+\right)^{\gamma+1} a(\tau(t))}{(\gamma+1)^{\gamma+1}\left(\phi(t)\tau^{\Delta}(t)\right)^{\gamma}}.$$

From this last inequality and (2.5), we have

$$\int_{t_1}^{t}\left[P_1(s) - \frac{\left(\left(\phi^{\Delta}(s)\right)_+\right)^{\gamma+1} a(\tau(s))}{(\gamma+1)^{\gamma+1}\left(\phi(s)\tau^{\Delta}(s)\right)^{\gamma}}\right]\Delta s \leq w(t_1),$$

which contradicts assumption (2.1). This completes the proof. □

**Example 2.2.** Consider the nonlinear delay dynamic equation

$$\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\frac{\beta}{t^{\gamma+1}}x^{\gamma}\left(\tau(t)\right)+\frac{\eta_1}{\eta_2 t^{(\gamma+1)/\gamma}}\Phi_{\alpha_1}\left(x\left(\tau(t)\right)\right)$$
$$+\frac{1}{t^{(\gamma+1)/\gamma}}\Phi_{\alpha_2}\left(x\left(\tau(t)\right)\right)=0, \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}}, \tag{2.7}$$

where $\gamma \geq 1$ is a quotient of odd positive integers, and $\beta,\ \eta_1$, $\eta_2$, $\alpha_1$ and $\alpha_2$ are positive constants with $\alpha_1>\gamma>\alpha_2$. We have

$$a(t)=1, \quad p_0(t)=\frac{\beta}{t^{\gamma+1}}, \quad p_1(t)=\frac{\eta_1}{\eta_2 t^{(\gamma+1)/\gamma}}, \quad p_2(t)=\frac{1}{t^{(\gamma+1)/\gamma}}.$$

Let

$$\eta_1=\frac{\alpha_2-\gamma}{\alpha_2-\alpha_1} \quad \text{and} \quad \eta_2=\frac{\gamma-\alpha_1}{\alpha_2-\alpha_1},$$

so that condition (1.4) is satisfied. It is clear that condition (1.2) holds, since

$$\int_{t_0}^{\infty}\frac{\Delta t}{a^{1/\gamma}(t)}=\int_{t_0}^{\infty}\Delta t=\infty. \tag{2.8}$$

For a time scale $\mathbb{T}$, we can choose $\tau(t)$ such that $\tau(t)\leq t$, and $\tau^{\Delta}(t)\geq 1$ and $\tau(\sigma(t))=\sigma(\tau(t))$, for example when $\mathbb{T}=\mathbb{R}$ or $\mathbb{T}=\mathbb{Z}$, we choose $\tau(t)=t-\tau,\ \tau\geq 0$, ($\tau$ is a nonnegative integer, if $\mathbb{T}=\mathbb{Z}$). For any isolated time scale we can chose $\tau(t)=\rho^k(t)$, etc. Let us take $\phi(t)=t^{\gamma}$, then, by the Pötzsche chain rule

$$\phi^{\Delta}(t)=(t^{\gamma})^{\Delta}=\gamma\int_0^1(t+h\mu(t))^{\gamma-1}dh\leq\gamma\sigma^{\gamma-1}(t).$$

Thus, we assume $\mathbb{T}$ is a time scale satisfying $\sigma(t)\leq kt$, for some $k>0$, $t\geq T_k>T_*$. Therefore

$$\begin{aligned}
&\limsup_{t\to\infty}\int_T^t\left[P_1(s)-\frac{\left(\left(\phi^{\Delta}(s)\right)_+\right)^{\gamma+1}a\left(\tau(s)\right)}{(\gamma+1)^{\gamma+1}\left(\phi(s)\tau^{\Delta}(s)\right)^{\gamma}}\right]\Delta s\\
\geq\ &\limsup_{t\to\infty}\int_T^t\left[\frac{\eta}{s}-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{k^{\gamma^2-1}}{s}\right]\Delta s\\
=\ &\left(\eta-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}k^{\gamma^2-1}\right)\limsup_{t\to\infty}\int_{T_k}^t\frac{\Delta s}{s},
\end{aligned}$$

where $\eta=\beta+\eta_2^{\frac{-1}{\gamma}}$. Therefore, condition (2.1) is satisfied if $\eta>\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}k^{\gamma^2-1}$. We conclude that if $[t_0,\infty)_{\mathbb{T}}$ is a time scale where $\sigma(t)\leq kt$, for some $k>0$, $t\geq T_k$, then, by Theorem 2.1, every solution of (2.7) is oscillatory if $\eta>\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}k^{\gamma^2-1}$.

**Theorem 2.3.** *Assume that* (1.2) *holds. If, for any given positive differentiable function* $\phi(t)$ *and all sufficiently large* $T_1 \in [t_0, \infty)_{\mathbb{T}}$*, there is a* $T > T_1$ *such that* $g(t) > T_1$*, for* $t \geq T$ *and*

$$\limsup_{t\to\infty} \int_T^t \left[ Q_1(s, T_1) - \frac{\left( \left(\phi^\Delta(s)\right)_+ \right)^{\gamma+1} a(s)}{(\gamma+1)^{\gamma+1} \phi^\gamma(s)} \right] \Delta s = \infty, \tag{2.9}$$

*where*

$$Q_1(t, T_1) := \phi(t) \left[ p_0(t) A^{\alpha_0}(t, T_1) + \prod_{i=1}^{n} \left( \eta_i^{-1} p_i(t) A^{\alpha_i}(t, T_1) \right)^{\eta_i} \right],$$

*then every solution of equation* (1.1) *is oscillatory.*

*Proof.* Assume (1.1) has a nonoscillatory solution $x$ on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ and $x(g_i(t)) > 0,\ i = 1, 2, \ldots, n$ on $[t_1, \infty)_{\mathbb{T}}$ and not all of the $p_i(t)$'$s$ are identically zero on $[t_1, \infty)_{\mathbb{T}}$. As in the proof of Theorem 2.1, we get

$$x^\Delta(t) > 0, \quad \text{and} \quad \left(a(t)\left(x^\Delta(t)\right)^\gamma\right)^\Delta < 0, \quad \text{on } [t_1, \infty)_{\mathbb{T}}.$$

Consider the generalized Riccati substitution

$$w(t) := \phi(t) \left[ \frac{a(t)\left(x^\Delta(t)\right)^\gamma}{x^\gamma(t)} \right].$$

As in the proof of Theorem 2.1, we get

$$\begin{aligned} w^\Delta(t) &\leq -\phi(t) \sum_{i=0}^{n} p_i(t) \frac{x^{\alpha_i}(g_i(t))}{x^\gamma(t)} \\ &\quad + \phi^\Delta(t) w^\sigma(t) - \frac{\gamma \phi(t)}{a^{1/\gamma}(t)} w^{\lambda\sigma}(t) \\ &\leq -\phi(t) \sum_{i=0}^{n} p_i(t) \frac{x^{\alpha_i}(g(t))}{x^\gamma(t)} \\ &\quad + \frac{\phi^\Delta(t)}{\phi^\sigma(t)} w^\sigma(t) - \frac{\gamma \phi(t)}{a^{1/\gamma}(t) \phi^{\lambda\sigma}(t)} w^{\lambda\sigma}(t). \end{aligned} \tag{2.10}$$

Using the fact that $a(t)\left(x^\Delta(t)\right)^\gamma$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$, we get

$$\begin{aligned} x(t) - x(g(t)) &= \int_{g(t)}^{t} \frac{\left[a(s)\left(x^\Delta(s)\right)^\gamma\right]^{1/\gamma}}{a^{1/\gamma}(s)} \Delta s \\ &\leq \left[a(g(t))\left(x^\Delta(g(t))\right)^\gamma\right]^{1/\gamma} \int_{g(t)}^{t} \frac{\Delta s}{a^{1/\gamma}(s)}, \end{aligned}$$

and so

$$\frac{x(t)}{x(g(t))} \leq 1 + \frac{\left[a(g(t))\left(x^{\Delta}(g(t))\right)^{\gamma}\right]^{1/\gamma}}{x(g(t))} \int_{g(t)}^{t} \frac{\Delta s}{a^{1/\gamma}(s)}. \tag{2.11}$$

We can choose $t_2 > t_1$ so that $g(t) > t_1$, for $t \geq t_2$. Then, we obtain

$$\begin{aligned} x(g(t)) &> x(g(t)) - x(t_1) = \int_{t_1}^{g(t)} \frac{\left[a(s)\left(x^{\Delta}(s)\right]^{\gamma}\right)^{1/\gamma}}{a^{1/\gamma}(s)} \Delta s \\ &\geq \left[a(g(t))\left(x^{\Delta}(g(t))\right)^{\gamma}\right]^{1/\gamma} \int_{t_1}^{g(t)} \frac{\Delta s}{a^{1/\gamma}(s)}, \end{aligned}$$

and hence

$$\frac{\left[a(g(t))\left(x^{\Delta}(g(t))\right)^{\gamma}\right]^{1/\gamma}}{x(g(t))} \leq \left(\int_{t_1}^{g(t)} \frac{\Delta s}{a^{1/\gamma}(s)}\right)^{-1}, \quad \text{for } t \geq t_2. \tag{2.12}$$

Therefore, (2.11) and (2.12) imply

$$\frac{x(t)}{x(g(t))} \leq \int_{t_1}^{t} \frac{\Delta s}{a^{1/\gamma}(s)} \left(\int_{t_1}^{g(t)} \frac{\Delta s}{a^{1/\gamma}(s)}\right)^{-1} = A(t, t_1),$$

and hence we get the desired inequality

$$x(g(t)) \geq A(t, t_1)\, x(t), \quad \text{on } [t_2, \infty)_{\mathbb{T}}. \tag{2.13}$$

Using (2.13) in (2.10), we get

$$w^{\Delta}(t) \leq -\phi(t) \sum_{i=0}^{n} p_i(t) A^{\alpha_i}(t, t_1) x^{\alpha_i - \gamma}(t) + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)} w^{\sigma}(t) - \frac{\gamma \phi(t)}{a^{1/\gamma}(t) \phi^{\lambda\sigma}(t)} w^{\lambda\sigma}(t),$$

and the rest of the proof is the same as the proof of Theorem 2.1 with $p_i(t)$ replaced by $p_i(t)A^{\alpha_i}(t, t_1)$. □

**Example 2.4.** Consider the nonlinear dynamic equation

$$\left(t^{\gamma-1}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \frac{1}{t^{1/(\gamma+1)} A^{\gamma}(t, t_0)} x^{\gamma}(g_0(t)) + \sum_{i=1}^{n} p_i(t) \Phi_{\alpha_i}(x(g_i(t))) = 0, \tag{2.14}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, and $g(t) \neq t_0$, where $\gamma$ is a quotient of odd positive integers, $\alpha_i$, $i = 1, 2, \ldots, n$ are positive constants and $p_i$, $i = 1, 2, \ldots, n$ are nonnegative rd-continuous functions on $\mathbb{T}$. Here

$$a(t) = t^{\gamma-1}, \quad p_0(t) = \frac{1}{t^{1/(\gamma+1)} A^{\gamma}(t, t_0)}.$$

Suppose there exists an $n$-tuple $(\eta_1, \eta_2, \ldots, \eta_n)$ with $0 < \eta_i < 1$ satisfying (1.4). The condition (1.2) holds since

$$\int_{t_0}^{\infty} \frac{\Delta t}{a^{1/\gamma}(t)} = \int_{t_0}^{\infty} \frac{\Delta t}{t^{1-1/\gamma}} = \infty,$$

by [7, Example 5.60]. Also, by choosing $\phi(t) \equiv 1$, we have

$$\begin{aligned} & \limsup_{t\to\infty} \int_T^t \left[ P_2\left(s, T_1\right) - \frac{\left(\left(\phi^{\Delta}(s)\right)_+\right)^{\gamma+1} a(s)}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)} \right] \Delta s \\ \geq \; & \limsup_{t\to\infty} \int_T^t p_0(s) A^{\alpha_0}(s, T_1) \Delta s \\ = \; & \limsup_{t\to\infty} \int_T^t \frac{1}{s^{1/(\gamma+1)}} \Delta s = \infty, \end{aligned}$$

since $\displaystyle\int_{t_0}^{\infty} \frac{\Delta t}{a^{1/\gamma}(t)} = \infty$ implies $\displaystyle\lim_{t\to\infty} \frac{A\left(t, T_1\right)}{A\left(t, t_0\right)} = 1$. Then by Theorem 2.3, every solution of (2.14) is oscillatory.

Under condition (1.2) and when $\phi(t) \equiv 1$, we get the following oscillation criterion.

**Theorem 2.5.** *Assume* (1.2) *holds and $a(t)$ is a (delta) differentiable function with $a^{\Delta}(t) \geq 0$. Suppose $l > 0$ and, for all sufficiently large $T_1 \in [t_0, \infty)_{\mathbb{T}}$, there is a $T > T_1$ such that $g(T) > T_1$ and*

$$\liminf_{t\to\infty} \frac{t^{\gamma}}{a(t)} \int_{\sigma(t)}^{\infty} Q_2(s, T_1) \Delta s > \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}, \tag{2.15}$$

*where*

$$Q_2\left(t, T_1\right) := p_0(t) A^{\alpha_0}(t, T_1) + \prod_{i=1}^{n} \left(\eta_i^{-1} p_i(t) A^{\alpha_i}(t, T_1)\right)^{\eta_i}$$

*and $l := \liminf_{t\to\infty} \dfrac{t}{\sigma(t)}$. Then every solution of equation* (1.1) *is oscillatory.*

*Proof.* Assume (1.1) has a nonoscillatory solution $x$ on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ and $x\left(g_i(t)\right) > 0,\ i = 1, 2, \ldots, n$ on $[t_1, \infty)_{\mathbb{T}}$ and not all of the $p_i(t)$ are identically zero on $[t_1, \infty)_{\mathbb{T}}$. As in the proof of Theorem 2.1, we get

$$x^{\Delta}(t) > 0, \quad \text{and} \quad \left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0, \qquad \text{on } [t_1, \infty)_{\mathbb{T}},$$

and, as in the proof of Theorems 2.1 and 2.3, we get

$$w^{\Delta}(t) \leq -Q_2(s, T_1) - \frac{\gamma}{a^{1/\gamma}(t)} \left(w^{\sigma}(t)\right)^{(\gamma+1)/\gamma}, \tag{2.16}$$

where

$$w(t) = \frac{a(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(t)}.$$

Using the fact that $a(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$, it can be shown that

$$w(t) < \left(\int_{t_1}^{t} \frac{\Delta s}{a^{1/\gamma}(s)}\right)^{-\gamma}, \quad \text{for} \quad t \in (t_1, \infty)_{\mathbb{T}}.$$

Then from (1.2), we have that $\lim_{t\to\infty} w(t) = 0$. Integrating (2.16) from $\sigma(t)$ to $\infty$ and using $\lim_{t\to\infty} w(t) = 0$, we have

$$w^{\sigma}(t) \geq \int_{\sigma(t)}^{\infty} Q_2(s, T_1)\Delta s + \gamma \int_{\sigma(t)}^{\infty} \frac{(w^{\sigma}(s))^{1/\gamma} w^{\sigma}(s)}{a^{1/\gamma}(s)} \Delta s. \tag{2.17}$$

It follows from (2.17) that

$$\frac{t^{\gamma} w^{\sigma}(t)}{a(t)} \geq \frac{t^{\gamma}}{a(t)} \int_{\sigma(t)}^{\infty} Q_2(s, T_1)\Delta s + \gamma \frac{t^{\gamma}}{a(t)} \int_{\sigma(t)}^{\infty} \frac{(w^{\sigma}(s))^{1/\gamma} w^{\sigma}(s)}{a^{1/\gamma}(s)} \Delta s. \tag{2.18}$$

Let $\epsilon > 0$. We can pick $t_2 \in [t_1, \infty)_{\mathbb{T}}$, sufficiently large, so that, for $t \in [t_2, \infty)_{\mathbb{T}}$

$$\frac{t^{\gamma}}{a(t)} \int_{\sigma(t)}^{\infty} Q_2(s, T_1)\Delta s \geq Q_* - \epsilon, \quad \text{and} \quad \frac{t^{\gamma} w^{\sigma}(t)}{a(t)} \geq a_* - \epsilon, \tag{2.19}$$

where $Q_* := \liminf_{t\to\infty} \frac{t^{\gamma}}{a(t)} \int_{\sigma(t)}^{\infty} Q_2(s, T_1)\Delta s$ and $a_* := \liminf_{t\to\infty} \frac{t^{\gamma} w^{\sigma}(t)}{a(t)}$. From (2.18) and (2.19) and using the fact $a^{\Delta}(t) \geq 0$, we get that

$$\begin{aligned}
\frac{t^{\gamma} w^{\sigma}(t)}{a(t)} &\geq (Q_* - \epsilon) + \gamma \frac{t^{\gamma}}{a(t)} \int_{\sigma(t)}^{\infty} \frac{s\left(w^{\sigma}(s)\right)^{1/\gamma} s^{\gamma} w^{\sigma}(s)}{s^{\gamma+1} a^{1/\gamma}(s)} \Delta s \\
&\geq (Q_* - \epsilon) + (a_* - \epsilon)^{1+1/\gamma} \frac{t^{\gamma}}{a(t)} \int_{\sigma(t)}^{\infty} \frac{\gamma a(s)}{s^{\gamma+1}} \Delta s \\
&\geq (Q_* - \epsilon) + (a_* - \epsilon)^{1+1/\gamma} t^{\gamma} \int_{\sigma(t)}^{\infty} \frac{\gamma}{s^{\gamma+1}} \Delta s. \qquad (2.20)
\end{aligned}$$

Using the Pötzsche chain rule (see [6, Theorem 1.90]), we get

$$\begin{aligned}
\left(\frac{-1}{s^{\gamma}}\right)^{\Delta} &= \gamma \int_0^1 \frac{1}{[s + h\mu(s)]^{\gamma+1}} dh \\
&\leq \int_0^1 \left(\frac{\gamma}{s^{\gamma+1}}\right) dh = \frac{\gamma}{s^{\gamma+1}}. \qquad (2.21)
\end{aligned}$$

Then from (2.20) and (2.21), we have

$$\frac{t^\gamma w^\sigma(t)}{a(t)} \geq (Q_* - \epsilon) + (a_* - \epsilon)^{1+1/\gamma} \left(\frac{t}{\sigma(t)}\right)^\gamma.$$

Taking the $\liminf$ of both sides as $t \to \infty$ we get that

$$a_* \geq Q_* - \epsilon + (a_* - \epsilon)^{1+1/\gamma}\, l^\gamma.$$

Since $\epsilon > 0$ is arbitrary, we get

$$Q_* \leq a_* - a_*^\lambda l^\gamma,$$

where $\lambda = \dfrac{\gamma+1}{\gamma}$. By using the inequality (2.6) with

$$X^\lambda := a_*^\lambda l^\gamma \quad \text{and} \quad Y^{\lambda-1} := \frac{1}{\lambda l^{\frac{\gamma}{\lambda}}},$$

we get

$$Q_* \leq \frac{\gamma^\gamma}{l^{\gamma^2}(\gamma+1)^{\gamma+1}},$$

which contradicts (2.15). □

**Theorem 2.6.** *Let $g(t)$ be nondecreasing on $[t_0, \infty)_{\mathbb{T}}$. Assume that* (1.2) *holds. If, for all sufficiently large $T_1$, there is a $T > T_1$ such that $g(T) > T_1$ and*

$$\limsup_{t\to\infty} B^\gamma(g(t), T_1) R(t) > 1, \tag{2.22}$$

*where*

$$R(t) := \int_t^\infty p_0(s)\Delta s + \prod_{i=1}^n \left(\eta_i^{-1} \int_t^\infty p_i(s)\Delta s\right)^{\eta_i},$$

*then every solution of equation* (1.1) *is oscillatory.*

*Proof.* Assume (1.1) has a nonoscillatory solution $x$ on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ and $x\left(g_i(t)\right) > 0,\ i = 1, 2, \ldots, n$ on $[t_1, \infty)_{\mathbb{T}}$ and not all of the $p_i(t)$ are identically zero on $[t_1, \infty)_{\mathbb{T}}$. As in the proof of Theorem 2.1, we get

$$x^\Delta(t) > 0, \quad \text{and} \quad (a(t)\left(x^\Delta(t)\right)^\gamma)^\Delta < 0, \qquad \text{on } [t_1, \infty)_{\mathbb{T}}.$$

Integrating both sides of the dynamic equation (1.1) from $t$ to $\infty$ and then using $x(t)$ is strictly increasing and $g(t)$ is nondecreasing, we obtain

$$\sum_{i=0}^n x^{\alpha_i}(g(t)) \int_t^\infty p_i(s)\Delta s \leq a(t)\left(x^\Delta(t)\right)^\gamma. \tag{2.23}$$

Since $a(t)\left(x^\Delta(t)\right)^\gamma$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$, we get

$$\begin{aligned} x(t) &> x(t)-x(t_1)=\int_{t_1}^t \frac{\left[a(s)\left(x^\Delta(s)\right)^\gamma\right]^{1/\gamma}}{a^{1/\gamma}(s)}\Delta s \\ &\geq \left[a(t)\left(x^\Delta(t)\right)^\gamma\right]^{1/\gamma}\int_{t_1}^t \frac{\Delta s}{a^{1/\gamma}(s)}=\left[a(t)\left(x^\Delta(t)\right)^\gamma\right]^{1/\gamma}B(t,t_1), \end{aligned}$$

and so

$$x^\gamma(t)\geq B^\gamma(t,t_1)\left[a(t)\left(x^\Delta(t)\right)^\gamma\right], \quad \text{for } t\geq t_1. \tag{2.24}$$

Pick $t_2>t_1$, sufficiently large, so that $g(t)>t_1$, for $t\geq t_2$. Then from (2.24), we get, for $t\geq t_2$

$$x^\gamma(g(t))\geq B^\gamma(g(t),t_1)\left[a\left(x^\Delta\right)^\gamma(g(t))\right]\geq B^\gamma(g(t),t_1)\left[a(t)\left(x^\Delta(t)\right)^\gamma\right]. \tag{2.25}$$

Using (2.25) in (2.23), we find, for $t\geq t_2$,

$$\sum_{i=0}^n x^{\alpha_i}(g(t))\int_t^\infty p_i(s)\Delta s\leq\left(\frac{x(g(t))}{B(g(t),t_1)}\right)^\gamma,$$

which yields from the fact $x^\Delta(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$ that

$$\int_t^\infty p_0(s)\Delta s+\sum_{i=1}^n x^{\alpha_i-\gamma}(g(t))\int_t^\infty p_i(s)\Delta s<\frac{1}{B^\gamma(g(t),t_1)}.$$

As in Theorem 2.1, we obtain

$$B^\gamma(g(t),t_1)Q(t)<1, \quad \text{for } t\geq t_2.$$

Then

$$\limsup_{t\to\infty} B^\gamma(g(t),t_1)Q(t)\leq 1,$$

which leads to a contradiction to (2.22). □

Under the condition that $a(t)$ is a (delta) differentiable function with

$$a^\Delta(t)\geq 0 \quad \text{and} \quad \sum_{i=0}^n\int_{t_0}^\infty p_i(t)g^{\alpha_i}(t)\Delta t=\infty. \tag{2.26}$$

we get the following oscillation criteria for equation (1.1).

**Theorem 2.7.** *Assume that* (1.2) *holds. If, for any given positive differentiable function* $\phi(t)$ *and for all sufficiently large* $T$, *we have*

$$\limsup_{t\to\infty}\int_T^t\left[P_2(s)-\frac{\left(\left(\phi^\Delta(s)\right)_+\right)^{\gamma+1}a(s)}{(\gamma+1)^{\gamma+1}\phi^\gamma(s)}\right]\Delta s=\infty, \tag{2.27}$$

*where*

$$P_2(t) := \phi(t)\left[p_0(t)\left(\frac{g(t)}{t}\right)^{\alpha_0} + \prod_{i=1}^{n}\left(\eta_i^{-1}p_i(t)\left(\frac{g(t)}{t}\right)^{\alpha_i}\right)^{\eta_i}\right],$$

*then every solution of equation* (1.1) *is oscillatory.*

*Proof.* Assume (1.1) has a nonoscillatory solution $x$ on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1 \in [t_0,\infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ and $x\left(g_i(t)\right) > 0,\ i = 1, 2, \ldots, n$ on $[t_1,\infty)_{\mathbb{T}}$ and not all of the $p_i(t)$'$s$ are identically zero on $[t_1,\infty)_{\mathbb{T}}$. As in the proof of Theorem 2.1, we get

$$x^{\Delta}(t) > 0, \quad \text{and} \quad (a(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} < 0, \quad \text{on } [t_1,\infty)_{\mathbb{T}},$$

and, as in the proof of Theorems 2.1 and 2.3, we have

$$\begin{aligned} w^{\Delta}(t) \;&\leq\; -\phi(t)\sum_{i=0}^{n} p_i(t)\frac{x^{\alpha_i}(g(t))}{x^{\gamma}(t)} \\ &\quad + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)}w^{\sigma}(t) - \frac{\gamma\phi(t)}{a^{1/\gamma}(t)\phi^{\lambda\sigma}(t)}w^{\lambda\sigma}(t). \end{aligned} \tag{2.28}$$

From (2.26), we get (see [15, Lemma 2.1])

$$\left(\frac{x(t)}{t}\right)^{\Delta} < 0, \quad \text{on } [t_1,\infty)_{\mathbb{T}},$$

so

$$x\left(g(t)\right) \geq \frac{g(t)}{t}x(t), \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}}. \tag{2.29}$$

It follows from (2.28) and (2.29) that

$$\begin{aligned} w^{\Delta}(t) \;&\leq\; -\phi(t)\sum_{i=0}^{n} p_i(t)\left(\frac{g(t)}{t}\right)^{\alpha_i} x^{\alpha_i-\gamma}(t) \\ &\quad + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)}w^{\sigma}(t) - \frac{\gamma\phi(t)}{a^{1/\gamma}(t)\phi^{\lambda\sigma}(t)}w^{\lambda\sigma}(t), \end{aligned}$$

and the rest of the proof is as in the proof of Theorem 2.1 and hence is omitted. □

**Theorem 2.8.** *Assume* (1.2) *holds and* $a(t)$ *is a (delta) differentiable function with* $a^{\Delta}(t) \geq 0$. *If* $l > 0$ *and*

$$\liminf_{t\to\infty} \frac{t^{\gamma}}{a(t)}\int_{\sigma(t)}^{\infty} P_3(s)\Delta s > \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}},$$

*where*

$$P_3(t) := p_0(t)\left(\frac{g(t)}{t}\right)^{\alpha_0} + \prod_{i=1}^{n}\left(\eta_i^{-1}p_i(t)\left(\frac{g(t)}{t}\right)^{\alpha_i}\right)^{\eta_i},$$

*and* $l := \liminf_{t\to\infty} \dfrac{t}{\sigma(t)}$, *then every solution of equation* (1.1) *is oscillatory.*

**Theorem 2.9.** *Let $g$ be nondecreasing on $[t_0,\infty)_{\mathbb{T}}$. Assume that* (1.2) *and* (2.26) *hold. If*

$$\limsup_{t\to\infty}\left[\frac{g(t)}{t}\right]^{\gamma} R(t) > 1, \tag{2.30}$$

*then every solution of equation* (1.1) *is oscillatory.*

# 3 Oscillation Criteria for (1.1) when (1.3) holds

Before stating our next results, we begin with the following lemma.

**Lemma 3.1.** *Assume that*

$$\int_{t_0}^{\infty}\left[\frac{1}{a(t)}\sum_{i=0}^{n}\int_{t_0}^{t} p_i(s)\beta_i^{\gamma}(s)\Delta s\right]^{\frac{1}{\gamma}}\Delta t = \infty \tag{3.1}$$

*holds, where* $\beta_i(t) := \displaystyle\int_{g_i(t)}^{\infty}\frac{\Delta s}{a^{1/\gamma}(s)},\ i = 0,1,2,\dots,n$ *and that* (1.1) *has a positive solution $x$ on $[t_0,\infty)_{\mathbb{T}}$. Then there exists a $T\in[t_0,\infty)_{\mathbb{T}}$, sufficiently large, so that $x^{\Delta}(t) > 0$ on $[T,\infty)_{\mathbb{T}}$.*

*Proof.* Pick $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(g_i(t)) > 0,\ i = 1,2,\dots,n$ on $[t_1,\infty)_{\mathbb{T}}$. As in the proof of Theorem 2.1, we get $a(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. We claim that $x^{\Delta}(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$. If not, then there exists $t_2\geq t_1$ such that $x^{\Delta}(t_2) < 0$. We choose $t_3\geq t_2$ so that $g_i(t)\geq t_2$, for $i = 0,1,2,\dots,n$ and $t\geq t_3$. Using the fact that $a(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is decreasing, we obtain

$$\begin{aligned}
-x(g_i(t)) &< x(\infty) - x(g_i(t)) = \int_{g_i(t)}^{\infty}\frac{\left(a(s)\left(x^{\Delta}(s)\right)^{\gamma}\right)^{\frac{1}{\gamma}}}{a^{1/\gamma}(s)}\Delta s\\
&\leq \left(a(g_i(t))\left(x^{\Delta}(g_i(t))\right)^{\gamma}\right)^{\frac{1}{\gamma}}\int_{g_i(t)}^{\infty}\frac{\Delta s}{a^{1/\gamma}(s)}\\
&\leq \left(a(t_2)\left(x^{\Delta}(t_2)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\int_{g_i(t)}^{\infty}\frac{\Delta s}{a^{1/\gamma}(s)} = -L\beta_i(t),
\end{aligned}$$

so

$$x^{\alpha_i}(g_i(t)) > L^{\alpha_i}\beta_i^{\alpha_i}(t),\ \ \text{for } i = 1,2,\dots,n \text{ and } t\geq t_3,$$

where $L := -\left(a(t_2)\left(x^{\Delta}(t_2)\right)^{\gamma}\right)^{\frac{1}{\gamma}} > 0$. From (1.1), we get, for $t\geq t_3$

$$\left(a(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} = -\sum_{i=0}^{n} p_i(t)x^{\alpha_i}\left(g_i(t)\right) \leq -\sum_{i=0}^{n} L^{\alpha_i}p_i(t)\beta_i^{\alpha_i}(t).$$

Hence, for $t \geq t_3$, we have

$$a(t)(x^{\Delta}(t))^{\gamma} \leq a(t_3)(x^{\Delta}(t_3))^{\gamma} - \sum_{i=0}^{n} L^{\alpha_i} \int_{t_3}^{t} p_i(u)\beta_i^{\gamma}(u)\Delta u.$$

It follows from this last inequality that

$$x(t) - x(t_3) \leq -\int_{t_3}^{t} \left[ \sum_{i=0}^{n} \frac{L^{\alpha_i}}{a(s)} \int_{t_3}^{s} p_i(u)\beta_i^{\gamma}(u)\Delta u \right]^{\frac{1}{\gamma}} \Delta s.$$

Hence by (3.1), we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x$ is a positive solution of (1.1). Thus $x^{\Delta}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. □

From Lemma 3.1 and Theorems 2.1, 2.3, 2.6, 2.7 and 2.9, we get the next five oscillation criteria for equation (1.1).

**Corollary 3.2.** *Let $g_i(t) \equiv \tau(t)$, $i = 1, 2, \ldots, n,\ \tau(t) \leq t$, and $\tau$ be nondecreasing and delta differentiable with $\tau \circ \sigma = \sigma \circ \tau$ on $[t_0, \infty)_{\mathbb{T}}$. Assume that* (1.3), (2.1) *and* (3.1) *hold. Then every solution of equation* (1.1) *is oscillatory.*

**Corollary 3.3.** *Assume that* (1.3), (2.9) *and* (3.1) *hold. Then every solution of equation* (1.1) *is oscillatory.*

**Corollary 3.4.** *Let $a(t)$ be a (delta) differentiable function. Assume that* (1.3), (2.26), (2.27) *and* (3.1) *hold. Then every solution of equation* (1.1) *is oscillatory.*

**Corollary 3.5.** *Let $g(t)$ be nondecreasing on $[t_0, \infty)_{\mathbb{T}}$. Assume that* (1.3), (2.22) *and* (3.1) *hold. Then every solution of equation* (1.1) *is oscillatory.*

**Corollary 3.6.** *Let $g(t)$ be nondecreasing on $[t_0, \infty)_{\mathbb{T}}$. Assume that* (1.3), (2.30) *and* (3.1) *hold. Then every solution of equation* (1.1) *is oscillatory.*

**Example 3.7.** Consider the nonlinear dynamic equation

$$\left((t\sigma(t))^{\gamma} \left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \sum_{i=0}^{n} p_i(t)\Phi_{\alpha_i}\left(x\left(g_i(t)\right)\right) = 0, \tag{3.2}$$

where $\gamma$ is the quotient of odd positive integers and $p_0(t) := t^{\gamma}$, where $0 < \gamma = \alpha_0 \leq 1$ is a quotient of odd positive integers, $\alpha_i$, $i = 1, 2, \ldots, n$ are positive constants and $p_i$, $i = 1, 2, \ldots, n$ are nonnegative rd-continuous functions on $\mathbb{T}$ and where $g_0(t) \leq t$ on $[t_0, \infty)_{\mathbb{T}}$. Let $(\eta_1, \eta_2, \ldots, \eta_n)$ with $0 < \eta_i < 1$ be an $n$-tuple satisfying (1.4).

It is clear that $a(t) = (t\sigma(t))^{\gamma}$ satisfies

$$\int_{t_0}^{\infty} \frac{\Delta t}{a^{1/\gamma}(t)} \leq \int_{t_0}^{\infty} \frac{1}{t\sigma(t)}\Delta t = \int_{t_0}^{\infty} \left(\frac{-1}{t}\right)^{\Delta} \Delta t < \infty,$$

for those time scales $[t_0, \infty)_{\mathbb{T}}$, $t_0 > 0$. To see that (3.1) holds note that

$$\begin{aligned}\int_{t_0}^{\infty} \left[ \frac{1}{a(t)} \sum_{i=0}^{n} \int_{t_0}^{t} p_i(s) \beta_i^{\gamma}(s) \Delta s \right]^{\frac{1}{\gamma}} \Delta t \;&\geq\; \int_{t_0}^{\infty} \left[ \frac{1}{(t\sigma(t))^{\gamma}} \int_{t_0}^{t} (s\beta_0(s))^{\gamma} \Delta s \right]^{\frac{1}{\gamma}} \Delta t \\ &\geq\; \int_{t_0}^{\infty} \left[ \frac{t - t_0}{(t\sigma(t))^{\gamma}} \right]^{\frac{1}{\gamma}} \Delta t,\end{aligned}$$

since

$$\beta_0(t) = \int_{g_0(t)}^{\infty} \frac{\Delta s}{a^{\frac{1}{\gamma}}(s)} = \int_{g_0(t)}^{\infty} \frac{1}{s\sigma(s)} = \int_{g_0(t)}^{\infty} \left( \frac{-1}{s} \right)^{\Delta} \Delta s = \frac{1}{g_0(t)} \geq \frac{1}{t}.$$

We can find $0 < k < 1$ such that $t - t_0 > kt$, for $t \geq t_k > t_0$. Therefore, we get

$$\int_{t_0}^{\infty} \left[ \frac{1}{a(t)} \sum_{i=0}^{n} \int_{t_0}^{t} p_i(s) \beta_i^{\gamma}(s) \Delta s \right]^{\frac{1}{\gamma}} \Delta t > k^{\frac{1}{\gamma}} \int_{t_K}^{\infty} \frac{\Delta t}{t^{1-\frac{1}{\gamma}} \sigma(t)} = \infty$$

for those time scales (see [7, Theorem 5.68]) where this last integral diverges. To apply Corollary 3.3, it remains to prove that condition (2.9) holds. Then by putting $\phi(t) \equiv 1$,

$$\limsup_{t \to \infty} \int_{T}^{t} \left[ Q_1(s, T_1) - \frac{\left( \left( \phi^{\Delta}(s) \right)_+ \right)^{\gamma+1} a(s)}{(\gamma+1)^{\gamma+1} \phi^{\gamma}(s)} \right] \Delta s \geq \limsup_{t \to \infty} \int_{T}^{t} s^{\gamma} \Delta s = \infty.$$

We conclude that if $[t_0, \infty)_{\mathbb{T}}$, $t_0 > 0$ is a time scale interval with

$$\int_{t_0}^{\infty} \frac{\Delta t}{t^{1-\frac{1}{\gamma}} \sigma(t)} = \infty, \quad \text{for } 0 < \gamma \leq 1. \tag{3.3}$$

Then, by Corollary 3.3, every solution of (3.2) is oscillatory.

*Remark* 3.8. The results in this paper are extendable to the neutral case and to the damped case, see [10, 11].

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