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Oscillatory Behaviour of a Class of Nonlinear Systems of First-Order Difference Equations

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Abstract

In this paper, sufficient conditions are obtained so that every vector solution of a class of nonlinear systems of difference equations

$$\begin{aligned} x(n+1) &= a(n)f(x(n)) + b(n)g(y(n)) \\ y(n+1) &= c(n)f(x(n)) + d(n)g(y(n)) \end{aligned}$$

oscillates. Its associated nonhomogeneous system

$$\begin{aligned} x(n+1) &= a(n)f(x(n)) + b(n)g(y(n)) + h_1(n) \\ y(n+1) &= c(n)f(x(n)) + d(n)g(y(n)) + h_2(n) \end{aligned}$$

is also studied.

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1 Introduction

Consider the nonlinear system of first-order difference equations of the form

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} \begin{bmatrix} f(x(n)) \\ g(y(n)) \end{bmatrix}, \tag{1.1}$$

where $a(n)$, $b(n)$, $c(n)$ and $d(n)$ are real-valued functions defined for $n \in N(n_0) = \{n_0, n_0+1,\ n_0+2, \cdots\}$, $n_0 \geq 0$ such that $a(n)d(n) - b(n)c(n) \neq 0$ and $f, g \in C(\mathbb{R}, \mathbb{R})$ with $uf(u) > 0$, $ug(u) > 0$ for $u \neq 0$.

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By a solution of (1.1) we mean a real-valued vector function $X(n) = [x(n), y(n)]^T$ which satisfies (1.1), for $n \geq n_0$. We say that the solution $X(n)$ oscillates componen-twise or simply oscillates if each component oscillates. Otherwise, $X(n)$ is nonoscil-latory. Therefore, a solution of (1.1) is nonoscillatory if it has a component which is eventually positive or eventually negative. A solution $x(n)$ of (1.1) is said to be discon-jugate on $[n_0, \infty)$ if each of the components is disconjugate on $[n_0, \infty)$.

During the last several years, oscillatory, nonoscillatory and asymptotic behaviour of solutions of nonlinear/linear systems of first-order difference equations have been studied extensively and many interesting results have appeared in the literature, see e.g., [1–6]. A close observation reveals that almost all works in this direction are the discrete analogues of differential systems. It seems that not much work has been done on the matrix equation (1.1).

In a recent paper [7], the author has studied the linear system

$$X(n+1) = A(n)X$$

and its associated nonhomogeneous system of equations

$$X(n+1) = A(n)X + H(n),$$

where $A(n)$ is the coefficient matrix of (1.1) and $H(n) = [h_1(n), h_2(n)]^T$, and obtained some oscillation results. It is interesting to note that such results are not the discrete analogues of differential systems. Motivated by the object of the work in [7], an attempt is made here to study the oscillatory behaviour of solutions of the system of equations (1.1) and its corresponding nonhomogeneous system of equations

$$X(n+1) = A(n)F(X(n)) + H(n), \tag{1.2}$$

where $F(X(n)) = [f(x(n)), g(y(n))]^T$.

In [3, 5, 6], authors have studied the oscillatory and asymptotic behaviour of the vector solutions of the two dimensional matrix equation

$$\begin{bmatrix} \Delta x_n \\ \Delta y_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & b_n \\ -a_n & 0 \end{bmatrix} \begin{bmatrix} f(x_n) \\ g(y_n) \end{bmatrix}, \tag{1.3}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of reals and $f, g \in C(\mathbb{R}, \mathbb{R})$ with $uf(u) > 0$, $ug(u) > 0$ for $u \neq 0$. Keeping in view the discrete analogue of the two dimensional matrix differential equation

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & b(t) \\ a(t) & 0 \end{bmatrix} \begin{bmatrix} f(x(t)) \\ g(y(t)) \end{bmatrix},$$

they have obtained similar results for (1.1). However, the purpose of this work is alto-gether in a different approach.

2 Oscillation of System (1.1)

This section deals with the study of the oscillatory behaviour of solutions of the matrix difference equation (1.1). We use the notation $X(n)$ as any solution of (1.1) such that $X(n) = [x(n), y(n)]^T$ defined for $n \geq n_0 \geq 0$.

Lemma 2.1. *Assume that $a(n) \equiv 0 \equiv d(n)$ for all n. If $b(n)c(n) > 0$, then $x(n)$ is nonoscillatory if and only if $y(n)$ is nonoscillatory, where $x(n)$ and $y(n)$ are the components of the vector $X(n)$.*

Proof. The system (1.1) can be written as

$$x(n+1) = b(n)g(y(n)), \tag{2.1}$$

$$y(n+1) = c(n)f(x(n)). \tag{2.2}$$

Let $x(n)$ be the nonoscillatory component of $X(n)$. Without loss of generality, we may assume that $x(n) > 0$ for $n \geq n_0$. Hence (2.2) yields that $y(n+1) > 0$ for $n \geq n_0$. On the other hand, Equ. (2.1) implies that $y(n) > 0$ for $n \geq n_0$. Therefore $y(n) > 0$ for $n \geq n_0$ when we consider the case $b(n) > 0$ and $c(n) > 0$. It is easy to verify the above fact when we consider the case $b(n) < 0$ and $c(n) < 0$ for $n \geq n_0$. The converse part can be dealt with similarly. The proof is complete. □

Now it is immediate to prove the following lemma:

Lemma 2.2. *Assume that all the conditions of Lemma 2.1 hold. Then $x(n)$ is oscillatory if and only if $y(n)$ is oscillatory.*

Lemma 2.3. *Let $a(n) \leq 0$ and $d(n) \leq 0$ for $n \geq n_0$. Then for any $b(n) \neq 0$ and $c(n) \neq 0$, $x(n)$ is oscillatory if and only if $y(n)$ is oscillatory.*

Proof. Let $x(n)$ be the oscillatory component of $X(n)$ defined for $n \geq n_0$. We claim that $y(n)$ is oscillatory. If not, there exists $n_1 > n_0$ such that $y(n) > 0$ or < 0 for $n \geq n_1$. Assume that $y(n) > 0$ for $n \geq n_1$. Then

$$0 < y(n+1) - d(n)g(y(n)) = c(n)f(x(n))$$

implies that $x(n)$ is nonoscillatory for any $c(n) \neq 0$, a contradiction. So our claim holds. The case $y(n) < 0$ for $n \geq n_1$ is similar. Next, we assume that $y(n)$ is oscillatory. If $x(n)$ is nonoscillatory for $n \geq n_1 > n_0$, then

$$0 < x(n+1) - a(n)f(x(n)) = b(n)g(y(n))$$

implies that $y(n)$ is nonoscillatory for any $b(n) \neq 0$, a contradiction. Hence the lemma is proved. □

Lemma 2.4. *Assume that any one of the conditions*

(a) $a(n) \leq 0,\ b(n) \leq 0,\ c(n) \leq 0$ *and* $d(n) \geq 0$,

(b) $a(n) \leq 0,\ b(n) \geq 0,\ c(n) \geq 0$ *and* $d(n) \geq 0$

holds, where $a(n) \not\equiv 0 \not\equiv b(n)$ *and* $c(n) \not\equiv 0 \not\equiv d(n)$. *If* $x(n)$ *is nonoscillatory, then* $y(n)$ *is nonoscillatory.*

Proof. Suppose that $x(n)$ is a nonoscillatory component of $X(n)$. Without loss of generality, we may assume that $x(n) > 0$ for $n \geq n_0$. Then

$$0 < x(n+1) - a(n)f(x(n)) = b(n)g(y(n))$$

implies that $b(n)$ and $g(y(n))$ are of same sign for $n \geq n_0$. If $b(n) \geq 0$ for all n, then

$$y(n+1) = c(n)f(x(n)) + d(n)g(y(n)) \geq 0,$$

if $c(n) \geq 0$ and $d(n) \geq 0$ and hence $y(n)$ is a nonoscillatory component of $X(n)$. If $b(n) \leq 0$ for all n, then

$$y(n+1) = c(n)f(x(n)) + d(n)g(y(n)) \leq 0$$

when $c(n) \leq 0$ and $d(n) \geq 0$, that is, $y(n)$ is a nonoscillatory component of $X(n)$. This completes the proof. □

Lemma 2.5. *Assume that any one of the conditions*

(a) $d(n) \leq 0,\ a(n) \geq 0,\ b(n) \leq 0$ *and* $c(n) \leq 0$,

(b) $d(n) \leq 0,\ a(n) \geq 0,\ b(n) \geq 0$ *and* $c(n) \geq 0$

holds, where $a(n) \not\equiv 0 \not\equiv b(n)$ *and* $c(n) \not\equiv 0 \not\equiv d(n)$. *If* $y(n)$ *is nonoscillatory, then* $x(n)$ *is nonoscillatory.*

Proof. The proof follows from Lemma 2.4. Hence the details are omitted. □

Remark 2.6. When Lemma 2.4(a) holds, then the nonoscillatory solutions $X(n) = [x(n), y(n)]^T$ lie in the second or fourth open quadrant and when Lemma 2.4(b) or Lemma 2.5(b) holds, then the nonoscillatory solutions $X(n)$ lie in the first or third open quadrant.

Theorem 2.7. *Let* $b(n) \equiv 0 \equiv c(n)$ *for all* n. *Then the following statements hold.*

(i) *If* $a(n) > 0$ *and* $d(n) > 0$ *for* $n \geq 0$, *then the system* (1.1) *is disconjugate on* $[0, \infty)$.

(ii) *If* $a(n) < 0$ *and* $d(n) < 0$ *for* $n \geq n_0$, *then the system* (1.1) *is oscillatory.*

(iii) *If* $a(n)$ *and* $d(n)$ *changes sign, then the vector solution of* (1.1) *is oscillatory.*

Proof. We first prove (i). The system (1.1) can be written as

$$x(n+1) = a(n)f(x(n)), \tag{2.3}$$
$$y(n+1) = d(n)g(y(n)). \tag{2.4}$$

Let $k \in [0,\infty) = \{0,1,2,3,\cdots\}$ be a generalized zero of a solution $x(n)$ of (2.3). If $k = 0$, then $x(0) = 0$ and hence $x(n)$ is a trivial solution of (2.3). Assume that $k \in (0,\infty)$. If $x(k) = 0$, then $x(n) = 0$ for $n \geq k$, that is, $x(n)$ is a trivial solution of (2.3). If $x(k) \neq 0$, then $x(k-1)x(k) < 0$. If we consider $x(k) > 0$, then $x(k-1) < 0$. Ultimately,

$$0 < x(k) = a(k-1)f(x(k-1)) < 0,$$

which is absurd. Altering the sign of $x(k)$ and $x(k-1)$, we have the same observation. Hence $x(n)$ has no generalized zero in $[0,\infty)$, that is, (2.3) is disconjugate on $[0,\infty)$. Proceeding as above, we can show that a solution $y(n)$ of (2.4) has no generalized zero on $[0,\infty)$. Thus the system (1.1) is disconjugate on $[0,\infty)$.

Now we show (ii). It is enough to show that (2.3) and (2.4) are oscillatory. Let $x(n)$ be a nonoscillatory solution of (2.3). Without loss of generality, we may assume that $x(n) > 0$ for $n \geq n_0$. Then

$$0 < x(n+1) = a(n)f(x(n)) < 0,$$

a contradiction. Hence (2.3) is oscillatory. Similarly, we can show that (2.4) is oscillatory. Therefore, the vector solution $X(n) = [x(n),\ y(n)]^T$ is oscillatory.

Finally, we prove (iii). Without loss of generality, let us assume that $x(n)$ is a nonoscillatory solution of (2.3) such that $x(n) > 0$ for $n \geq n_0$. Since $a(n)$ changes sign, there exists $k \geq n_1 \geq n_0 + 1$ such that $a(k-1)a(k) < 0$. If $a(k) < 0$, then

$$0 < x(k+1) = a(k)f(x(k)) < 0,$$

a contradiction. If $a(k) > 0$, then $a(k-1) < 0$ and hence

$$0 < x(k) = a(k-1)f(x(k-1)) < 0,$$

a contradiction. Thus (2.3) is oscillatory. Proceeding as above, we can show that (2.4) is oscillatory. Consequently, the vector solution $X(n)$ of (1.1) is oscillatory. □

Corollary 2.8. *If $a(n)d(n) < 0$, then the system* (1.1) *is nonoscillatory.*

Proof. The proof follows from the Theorem 2.7. Hence the details are omitted. □

Theorem 2.9. *Let $b(n) \equiv 0 \equiv c(n)$ for all n. Assume that*

(H_1) $f(uv) = f(u)f(v)$, $g(uv) = g(u)g(v)$ *for* $u, v \in \mathbb{R}$.

Furthermore, assume that $x(0) = x_0 \neq 0 \neq y_0 = y(0)$. *Then any vector solution* $X(n)$ *of* (1.1) *is oscillatory if and only if*

$$a(n-1)\prod_{j=1}^{n-1} f^j(a(n-j-1)) \quad \textit{and} \quad d(n-1)\prod_{j=1}^{n-1} g^j(d(n-j-1))$$

are oscillatory, where $f^n(x) = f\left(f^{n-1}(x)\right)$ *and* $g^n(x) = g\left(g^{n-1}(x)\right)$ *for* $n \geq 1$.

Proof. Using (H_1), it is easy to verify that

$$x(n) = f^n(x_0)a(n-1)\prod_{j=1}^{n-1} f^j(a(n-j-1))$$

and

$$y(n) = g^n(y_0)d(n-1)\prod_{j=1}^{n-1} g^j(d(n-j-1))$$

are the solutions of (2.3) and (2.4) respectively. Hence the proof follows. □

Theorem 2.10. *Let* $a(n) \equiv 0 \equiv d(n)$ *for all* n. *If* $b(n)c(n) > 0$, *then* (1.1) *is either oscillatory or nonoscillatory.*

Proof. From Lemma 2.1, it follows that the components of $X(n)$ are either oscillatory or nonoscillatory and hence any solution $X(n)$ of (1.1) is either oscillatory or nonoscillatory. This completes the proof. □

When $a(n) \equiv 0 \equiv d(n)$ for all n, then the system (1.1) can be written as

$$x(n+1) = b(n)g(y(n)), \tag{2.5}$$

$$y(n+1) = c(n)f(x(n)). \tag{2.6}$$

Using (2.6), (2.5) yields that

$$x(n+2) = b(n+1)g[c(n)f(x(n))], \tag{2.7}$$

and using (2.5), (2.6) becomes

$$y(n+2) = c(n+1)f[b(n)g(y(n))]. \tag{2.8}$$

Theorem 2.11. *Let* $a(n) \equiv 0 \equiv d(n)$ *for all* n. *Assume that* $b(n)c(n) < 0$ *for every* $n \geq n_0 > 0$ *such that* $\liminf_{n\to\infty} b(n) \neq 0 \neq \liminf_{n\to\infty} c(n)$. *Furthermore, suppose that there exist constants* $M > 0$ *and* $N > 0$ *such that*

(H_2) $\dfrac{f(x)}{x} \geq M$, $\dfrac{g(x)}{x} \geq N$, *for* $x \in \mathbb{R}$.

Then the system (1.1) *is oscillatory.*

Proof. In order to prove that the system (1.1) is oscillatory, it is sufficient to show that the nonlinear second-order difference equations (2.7) and (2.8) are oscillatory. Suppose on the contrary that, $x(n)$ is a nonoscillatory solution of (2.7) such that $x(n) > 0$ for $n \geq n_0$. Without loss of generality, we may assume that $b(n) < 0$ and $c(n) > 0$ for all $n \geq n_0$. Let $\liminf_{n\to\infty} b(n) = \alpha$ and $\liminf_{n\to\infty} c(n) = \beta$. Using ($H_2$), (2.7) becomes

$$x(n+2) \leq MNb(n+1)c(n)x(n)$$

for $n \geq n_1 > n_0$. Hence for $n \geq n_1$,

$$\frac{x(n+2)}{x(n)} \leq MNb(n+1)c(n),$$

that is,

$$\liminf_{n\to\infty} \left[\frac{x(n+2)}{x(n+1)} \frac{x(n+1)}{x(n)}\right] \leq MN \liminf_{n\to\infty} [b(n+1)c(n)]. \tag{2.9}$$

If $\lambda = \liminf_{n\to\infty} \dfrac{x(n+1)}{x(n)}$, then (2.9) yields that

$$\lambda^2 - MN\alpha\beta \leq 0.$$

Let $F(\lambda) = \lambda^2 - MN\alpha\beta$. Clearly, $F(\lambda)$ attains minimum at $\lambda = 0$. Consequently, $\min F(\lambda) \leq F(\lambda)$ implies that $MN\alpha\beta \geq 0$, a contradiction. The proof is similar if we assume that $x(n) < 0$ for $n \geq n_0$. Hence (2.7) is oscillatory. Proceeding as above, we can show that (2.8) is oscillatory. Hence the system (1.1) is oscillatory. This completes the proof. □

Remark 2.12. Theorem 2.11 generalizes [7, Theorem 2.4].

Remark 2.13. The prototype of f and g satisfying (H_2) could be of the type

$$f(u) = (M + a|u|^\lambda)|u|\text{sgn}u$$

and

$$g(v) = (N + b|v|^\mu)|v|\text{sgn}v,$$

where $M > 0$, $N > 0$, $a \geq 0$, $b \geq 0$, $\lambda \geq 0$ and $\mu \geq 0$.

Example 2.14. Consider the system of equations

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} f(x(n)) \\ g(y(n)) \end{bmatrix}, \tag{2.10}$$

where

$$f(x(n)) = \left(\frac{3}{16} + \frac{1}{16}|x(n)|^{\lambda}\right)|x(n)|\text{sgn}x$$

and

$$f(y(n)) = \left(\frac{2}{9} + \frac{1}{9}|y(n)|^{\mu}\right)|y(n)|\text{sgn}y.$$

Clearly, $M = \dfrac{3}{16}$ and $N = \dfrac{2}{9}$ and (2.10) satisfies all the conditions of Theorem 2.11. Hence (2.10) is oscillatory. In particular, $X(n) = [(-1)^n, (-1)^n]^T$ is such an oscillatory solution of (2.10).

Theorem 2.15. *If $a(n) \leq 0$, $d(n) \leq 0$ and $b(n) \neq 0 \neq c(n)$ for every $n \geq n_0$, then every vector solution of* (1.1) *is oscillatory.*

Proof. The proof is an immediate consequence of Lemma 2.3, and hence the details are omitted. □

Example 2.16. Consider the system of equations

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} -3+(-1)^n & 2-(-1)^n \\ 3-(-1)^n & -4+(-1)^n \end{bmatrix} \begin{bmatrix} f(x(n)) \\ g(y(n)) \end{bmatrix}, \tag{2.11}$$

where $-4 \leq a(n) \leq -2$, $-5 \leq d(n) \leq -3$, $1 \leq b(n) \leq 3$, $2 \leq c(n) \leq 4$, $f(x(n)) = x^3(n)$ and $g(y(n)) = y^5(n)$. Applying Theorem 2.15, every vector solution of (2.11) is oscillatory. In particular, $X(n) = [(-1)^n, (-1)^n]^T$ is such an oscillatory solution of (2.11).

Theorem 2.17. *Assume that $a(n) \leq 0$, $b(n) \leq 0$, $c(n) \leq 0$ and $d(n) \leq 0$ for all $n \geq n_0$. If one of the conditions*

(H_3) $\liminf_{n\to\infty} \left(\dfrac{\det A(n)}{a(n)}\right) < 0$,

(H_4) $\liminf_{n\to\infty} \left(\dfrac{\det A(n)}{d(n)}\right) < 0$

holds, then the system (1.1) *is oscillatory.*

Proof. Let $X(n) = [x(n), y(n)]^T$ be the nonoscillatory solution of (1.1), for $n \geq n_0$, where at least one vector component is nonoscillatory. Assume that $x(n) > 0$ for $n \geq n_0$. Then $y(n)$ could be nonoscillatory or oscillatory for $n \geq n_0$. Let $y(n) < 0$ for $n \geq n_0$. Clearly,

$$x(n+1) - a(n)f(x(n)) = b(n)g(y(n)) \tag{2.12}$$

implies that

$$-a(n)f(x(n)) < b(n)g(y(n)), \quad n \geq n_0.$$

Consequently,

$$y(n+1) = c(n)f(x(n)) + d(n)g(y(n))$$

yields that

$$y(n+1) > \left(d(n) - \frac{b(n)c(n)}{a(n)}\right) g(y(n)),$$

that is,

$$0 < \frac{y(n+1)}{g(y(n))} < \frac{\det A(n)}{a(n)},$$

a contradiction to (H_3). If $y(n) > 0$ or $y(n)$ is oscillatory for $n \geq n_0$, then (2.12) gives a contradiction.

Next, we assume that $x(n) < 0$ for $n \geq n_0$. We verify the above three cases for $y(n)$. If $y(n) > 0$ for $n \geq n_0$. then

$$0 < y(n+1) - d(n)g(y(n)) = c(n)f(x(n))$$

implies that,

$$-d(n)g(y(n)) < c(n)f(x(n)), \quad n \geq n_0.$$

Hence

$$\begin{aligned} x(n+1) &= a(n)f(x(n)) + b(n)g(y(n)) \\ &> a(n)f(x(n)) - \frac{c(n)}{d(n)}b(n)f(x(n)) \\ &= \left(\frac{\det A(n)}{d(n)}\right) f(x(n)) \end{aligned}$$

yields that

$$0 < \frac{x(n+1)}{f(x(n))} < \frac{\det A(n)}{d(n)},$$

a contradiction to (H_4). If $y(n) < 0$ or $y(n)$ is oscillatory for $n \geq n_0$, then (2.12) gives a contradiction. Hence $x(n)$ is oscillatory.

If our supposition is such that $y(n)$ is nonoscillatory, then we can proceed as above to find similar contradictions keeping in view the three cases of $x(n)$. Hence the proof is complete. □

Theorem 2.18. *Suppose that* $b(n)c(n) \geq 0$, $a(n)d(n) \leq 0$ *and* $\det A(n) \neq 0$ *for all* $n \geq n_0$. *Then there exists a nonoscillatory solution to the system* (1.1).

Proof. The proof follows from Lemma 2.4 and Lemma 2.5. Hence the details are omitted. □

Example 2.19. Theorem 2.17 is applicable to the system

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{5}\left\{-3-\frac{(-1)^n}{5}\right\} & \frac{1}{5}\left\{-2+\frac{(-1)^n}{5}\right\} \\ \frac{1}{6}\left\{-3+\frac{(-1)^n}{5}\right\} & \frac{1}{6}\left\{-3-\frac{(-1)^n}{5}\right\} \end{bmatrix} \begin{bmatrix} f(x(n)) \\ g(y(n)) \end{bmatrix},$$

where $f(x(n)) = x^3(n)$ and $g(y(n)) = y^3(n)$. Clearly, $\det A(n) = \frac{1}{150}[15 + 11(-1)^n]$ and

$$\liminf_{n\to\infty}\left[\frac{\det A(n)}{a(n)}\right] = \frac{\liminf\limits_{n\to\infty} \det A(n)}{\limsup\limits_{n\to\infty} a(n)} = -\frac{1}{21}$$

implies that the above system is oscillatory and in particular, $X(n) = [(-1)^n, (-1)^n]^T$ is such an oscillatory solution.

3 Oscillation of System (1.2)

In this section, sufficient conditions are obtained for the oscillation of the vector solutions of the system of difference equations (1.2).

Theorem 3.1. *Assume that* $b(n) \equiv 0 \equiv c(n)$ *for all* n.

(i) *If* $a(n) < 0$, $d(n) < 0$ *and* $h_i(n)$ *changes sign for* $i = 1, 2$, *then* (1.2) *is oscillatory.*

(ii) *If* $a(n) > 0$, $d(n) > 0$ *and* $h_i(n) \geq 0$ *for* $i = 1, 2$, *then there exists a nonoscillatory solution to* (1.2).

Proof. We first prove (i). Suppose on the contrary that (1.2) has a nonoscillatory solution $X(n) = [x(n), y(n)]^T$ for $n \geq n_0$, where the components are the solutions of the equations

$$x(n+1) = a(n)f(x(n)) + h_1(n), \tag{3.1}$$

$$y(n+1) = d(n)g(y(n)) + h_2(n), \tag{3.2}$$

respectively. If $x(n)$ is a nonoscillatory solution of (3.1), then for $n \geq n_0$,

$$h_1(n) = x(n+1) - a(n)f(x(n)) > 0 \ \text{ or } \ < 0,$$

depending on $x(n)$ whether $x(n) > 0$ or < 0, a contradiction. Similarly, if $y(n)$ is a nonoscillatory solution of (3.2), then for $n \geq n_0$,

$$h_2(n) = y(n+1) - d(n)g(y(n)) > 0 \ \text{ or } \ < 0,$$

depending on $y(n)$ whether $y(n) > 0$ or < 0, a contradiction. Hence $X(n)$ is oscillatory.

Now we show (ii). For any $x(0) = x_0 \neq 0$ and $y(0) = y_0 \neq 0$, it is easy to verify that $x(n)$ and $y(n)$ are the nonoscillatory solutions of (3.1) and (3.2) respectively. □

Theorem 3.2. *Let $a(n) \equiv 0 \equiv d(n)$ for all n. Assume that $b(n)c(n) < 0$ for every $n \geq n_0 > 0$ such that $\liminf_{n\to\infty} b(n) \neq 0 \neq \liminf_{n\to\infty} c(n)$ and $h_i(n)$, $i = 1, 2$, $n \geq n_0$ is eventually of one sign such that*

$$c(n+1)f(h_1(n)) + h_2(n+1) \leq 0,\ b(n+1)g(h_2(n)) + h_1(n+1) \leq 0.$$

Furthermore, suppose that (H_2) *and*

(H_5) *there exist $\lambda, \mu > 0$ such that $f(x) + f(y) \geq \lambda f(x+y)$, $g(x) + g(y) \geq \mu g(x+y)$ hold for all $x, y \in \mathbb{R}$.*

Then every solution of (1.2) *oscillates.*

Proof. When $a(n) \equiv 0 \equiv d(n)$ for all n, then (1.2) becomes

$$\begin{aligned} x(n+1) &= b(n)g(y(n)) + h_1(n), \\ y(n+1) &= c(n)f(x(n)) + h_2(n). \end{aligned}$$

It is easy to verify that $x(n)$ and $y(n)$ are the solutions of the nonlinear second-order difference equations

$$x(n+2) = b(n+1)g[c(n)f(x(n)) + h_2(n)] + h_1(n+1), \tag{3.3}$$

$$y(n+2) = c(n+1)f[b(n)g(y(n)) + h_1(n)] + h_2(n+1), \tag{3.4}$$

respectively. Suppose on the contrary that $X(n) = [x(n), y(n)]^T$ is a nonoscillatory solution of (1.2). Without loss of generality, we may assume that $x(n) > 0$ for $n \geq n_0$. Then $x(n)$ is a solution of (3.3). Using (H_5) and then (H_2), (3.3) yields that

$$x(n+2) \leq \frac{MN}{\mu} b(n+1)g(c(n))x(n) + b(n+1)g(h_2(n)) + h_1(n+1),$$

that is,

$$x(n+2) - \frac{MN}{\mu} b(n+1)g(c(n))x(n) \leq b(n+1)g(h_2(n)) + h_1(n+1) \leq 0,$$

for $n \geq n_0$. Consequently,

$$\frac{x(n+2)}{x(n)} - \frac{MN}{\mu} b(n+1)g(c(n)) \leq 0.$$

Hence

$$\liminf_{n\to\infty} \left[\frac{x(n+2)}{x(n+1)} \cdot \frac{x(n+1)}{x(n)} \right] - \frac{MN\alpha\beta}{\mu} \leq 0,$$

where $\liminf_{n\to\infty} b(n) = \alpha$ and $\liminf_{n\to\infty} c(n) = \beta$. Let $\sigma = \liminf_{n\to\infty}\left[\frac{x(n+1)}{x(n)}\right]$ and define $f(\sigma) = \sigma^2 - \frac{MN\alpha\beta}{\mu}$. Then $f(\sigma)$ attains its minimum at $\sigma = 0$, and hence $\min f(\sigma) \leq f(\sigma)$ implies that $-\frac{MN\alpha\beta}{\mu} \leq 0$, a contradiction. Ultimately, $x(n)$ is oscillatory.

If $y(n)$ is a nonoscillatory solution of (3.4), then using (H_4) and (H_2), we get

$$y(n+2) \leq \frac{MN}{\lambda} c(n+1) f(b(n)) y(n) + c(n+1) f(h_1(n)) + h_2(n+1),$$

for $n \geq n_0$. Proceeding as above, we have a contradiction. Hence the proof is complete. □

Theorem 3.3. *Let* $a(n) \equiv 0 \equiv d(n)$ *for all* n. *Assume that* f *and* g *are Lipschitzian on a compact interval of the form* $[a, b]$, $0 < a < b < \infty$. *If*

$$(H_6)\quad \sum_{n=0}^{\infty} n|h_1(n)| < \infty, \sum_{n=0}^{\infty} n|b(n+1)h_2(n)| < \infty, \sum_{n=0}^{\infty} n[1 + |c(n)b(n+1)|] < \infty$$

holds, then there exists a nonoscillatory vector solution to the system (1.2).

Proof. It is enough to show that either (3.3) or (3.4) has a nonoscillatory solution. As a result, it is easy to verify that if $u(n)$ is a solution of

$$u(n) = 1 + \sum_{i=n}^{\infty} (i - n + 1)[u(i) - 2u(i+1) + h_1(i+1) + b(i+1) g\{c(i) f(u(i)) + h_2(i)\}],$$

then $u(n)$ is a solution of (3.3). This can be understood by writing (3.3) as

$$\begin{aligned} x(n+2) - 2x(n+1) + x(n) &= x(n) - 2x(n+1) + h_1(n+1) \\ &+ b(n+1) g[c(n) f(x(n)) + h_2(n)] \end{aligned}$$

for all n. It is possible to find $N > 0$ large enough such that

$$\begin{aligned} &\sum_{i=N}^{\infty} (i - N + 1)|h_1(i)| \leq \frac{1}{6}, \quad n \geq N \\ &\sum_{i=N}^{\infty} (i - N + 1)|b(i+1)h_2(i)| \leq \frac{1}{6K}, \quad n \geq N \\ &\sum_{i=N}^{\infty} (i - N + 1)[1 + |c(i)b(i+1)|] \leq \frac{1}{6K}, \quad n \geq N, \end{aligned}$$

where K is the Lipschitz constant of g on $\left[\frac{1}{2}, \frac{3}{2}\right]$ and $K^* = \max\left\{\frac{1}{2}, Kf\left(\frac{3}{2}\right)\right\}$ due to (H_6). Let X be the set of all bounded real-valued functions $u(n)$ with sup norm defined by

$$\|u\| = \sup\{|u(n)| : n \geq N\}.$$

Then X is a Banach space. Define a subset Ω of X as

$$\Omega = \left\{u \in X : \frac{1}{2} \leq u(n) \leq \frac{3}{2}, n \geq N\right\}.$$

Clearly, Ω is a bounded, convex and closed subset of X. Furthermore, define an operator $T : \Omega \to X$ by putting $(Tu)_n$ equal to

$$1 + \sum_{i=n}^{\infty}(i-n+1)\left[u(i) - 2u(i+1) + h_1(i+1) + b(i+1)g\left\{c(i)f(u(i)) + h_2(i)\right\}\right]$$

for $n \geq N$. The mapping T has the following properties. First of all, T maps Ω into Ω. Indeed, for $u \in \Omega$,

$$\begin{aligned}
&\left|\sum_{i=n}^{\infty}(i-n+1)\left[u(i) - 2u(i+1) + h_1(i+1) + b(i+1)g\{c(i)f(u(i)) + h_2(i)\}\right]\right| \\
&\leq \sum_{i=N}^{\infty}(i-N+1)\left[\left|\frac{3}{2} - 2\left(\frac{1}{2}\right) + h_1(i+1) + b(i+1)g\{c(i)f(u(i)) + h_2(i)\}\right|\right] \\
&\leq \sum_{i=N}^{\infty}(i-N+1)\left[\frac{1}{2} + |h_1(i+1)| + |b(i+1)|\,|g\{c(i)f(u(i)) + h_2(i)\}|\right] \\
&\leq \sum_{i=N}^{\infty}(i-N+1)\left[\frac{1}{2} + |h_1(i+1)| + |b(i+1)|\,|K|c(i)f(u(i)) + h_2(i)|\right] \\
&\leq \sum_{i=N}^{\infty}(i-N+1)\left[\frac{1}{2} + |h_1(i+1)| + K|b(i+1)|\left\{|c(i)|f\left(\frac{3}{2}\right) + |h_2(i)|\right\}\right] \\
&= \sum_{i=N}^{\infty}(i-N+1)|h_1(i+1)| + K\sum_{i=N}^{\infty}(i-N+1)|b(i+1)h_2(i)| \\
&\qquad + \sum_{i=N}^{\infty}\left[\frac{1}{2} + K\ \ f\left(\frac{3}{2}\right)|c(i)b(i+1)|\right](i-N+1) \\
&\leq \sum_{i=N}^{\infty}(i-N+1)|h_1(i+1)| + K\sum_{i=N}^{\infty}(i-N+1)|b(i+1)h_2(i)| \\
&\qquad + K^*\sum_{i=N}^{\infty}(i-N+1)\left[1 + |c(i)b(i+1)|\right] \\
&\leq \frac{1}{2}
\end{aligned}$$

implies that $\frac{1}{2} \le (Tu)_n \le \frac{3}{2}$.

Next, we show that T is continuous. Let $u^{(k)} \in \Omega$ such that $\lim_{k\to\infty} \|u^{(k)} - u\| = 0$. Since Ω is closed, then $u \in \Omega$ and hence

$$\begin{aligned}
\left|\left(Tu^{(k)}\right)_n - (Tu)_n\right| &= \left|\sum_{i=n}^{\infty}(i-n+1)\left[u^{(k)}(i) - u(i) + 2\left(u(i+1) - u^{(k)}(i+1)\right)\right.\right. \\
&\quad \left.\left. + b(i+1)\left\{g\left\{c(i)f(u^k(i)) + h_2(i)\right\} - g\left\{c(i)f(u(i)) + h_2(i)\right\}\right\}\right]\right| \\
&\le \sum_{i=n}^{\infty}(i-n+1)\left[\left|u^{(k)}(i) - u(i)\right| + 2\left|u^{(k)}(i+1) - u(i)\right|\right] \\
&\quad + \sum_{i=n}^{\infty}(i-n+1)|b(i+1)|\left|g\left\{c(i)f(u^k(i)) + h_2(i)\right\}\right. \\
&\qquad \left. - g\left\{c(i)f(u(i)) + h_2(i)\right\}\right|.
\end{aligned}$$

By the continuity of f and g and using Lebesgue's dominated convergence theorem, it follows that

$$\lim_{k\to\infty} \left\|Tu^{(k)} - Tu\right\| = 0.$$

Thus T is continuous.

In order to apply Schauder's fixed point theorem we need to show that $T\Omega$ is precompact. Let $u \in \Omega$ and $m, n \ge N$. Then for $m > n$

$$\begin{aligned}
|(Tu)_m - (Tu)_n| &= \left|\sum_{i=n}^{m-1}(i-n+1)\left[u(i) - 2u(i+1) + h_1(i+1)\right.\right. \\
&\qquad \left.\left. + b(i+1)g\left\{c(i)f(u(i)) + h_2(i)\right\}\right]\right| \\
&\le \left|\sum_{i=n}^{\infty}(i-n+1)\left[u(i) - 2u(i+1) + h_1(i+1)\right.\right. \\
&\qquad \left.\left. + b(i+1)g\left\{c(i)f(u(i)) + h_2(i)\right\}\right]\right| \\
&< \infty
\end{aligned}$$

due to (H_6). Hence $T\Omega$ is precompact.

By Schauder's fixed point theorem, there exists $u \in \Omega$ such that $Tu = u$. This completes the proof. □

4 Summary

Systems of equations like (1.1) and (1.2) naturally apply to various fields of scientific endeavor, like biology (the study of competitive species in population dynamics), physics (the study of the motions of interacting bodies), the study of control systems, neurology and electricity. The results established here may be helpful to study specially the nonlinear biological systems for the discrete time intervals. Keeping in view the above fact, the present work may initiate further study in this direction.

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