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# Asymptotic Behavior of Solutions of a Class of Nonlinear Difference Systems

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**Abstract**

In this paper we study the asymptotic behavior of the two-dimensional nonlinear difference system

$$\begin{cases} \Delta x(n) = f(n, x(n), y(n)) \\ \Delta y(n) = g(n, y(n), x(n)), \end{cases}$$

and necessary as well as sufficient conditions are established.

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## 1 Introduction

The problem of asymptotic behavior is one of the most important topic in the qualitative study of nonlinear scalars and systems of difference equations and has been the subject

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of many investigations. Recently, Li [5] and Agarwal, Li and Pang [1] studied a class of two-dimensional nonlinear difference systems of the form

$$\begin{cases} \Delta x(n) = a(n)f(y(n)), \\ \Delta y(n) = b(n)g(x(n)). \end{cases}$$

They provided a classification scheme for positive solutions of the above system and gave conditions for the existence of solutions with designated asymptotic properties. Moreover, Li and Raffoul [6] studied the classification and existence of positive solutions of the Volterra nonlinear difference system

$$\begin{cases} \Delta x(n) = h(n)x(n) + \Sigma_{i=0}^{n} a(n,i)f(y(i)), \\ \Delta y(n) = p(n)y(n) + \Sigma_{i=0}^{n} b(n,i)g(x(i)), n > 0. \end{cases}$$

In this paper, we shall consider the general nonlinear difference system

$$\begin{cases} \Delta x(n) = f(n, x(n), y(n)), \\ \Delta y(n) = g(n, y(n), x(n)), n \geq n_0, \end{cases} \tag{1.1}$$

where $f, g : \mathbb{N}(n_0) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \mathbb{N}(n_0) = \{n_0, n_0 + 1, n_0 + 2, \cdots\}$ ($n_0$ a nonnegative integer, $x(n_0) = x_0, y(n_0) = y_0$), and $\Delta$ denotes the forward difference operator, that is, $\Delta x(n) = x(n+1) - x(n)$ for a sequence $x(n)$. Next, we state the following definition.

**Definition 1.1.** We say that (1.1) has an *asymptotic equilibrium* if

(i) there exist $\xi \in \mathbb{R}$ and $r > 0$ such that any solution $x(n, n_0, x_0)$ of (1.1) with $\mid x_0 \mid < r$ satisfies

$$x(n) = \xi + o(1) \quad \text{as} \quad n \to \infty; \tag{1.2}$$

(ii) corresponding to each $\xi \in \mathbb{R}$, there is a solution of (1.1) satisfying (1.2).

# 2 Main Results

First, we need the following comparison principle for difference inequalities.

**Lemma 2.1 (See [3]).** *Let $\psi(n, r)$ be a nonnegative nondecreasing function in $r$ for any fixed $n \in \mathbb{N}(n_0)$. Suppose that for any $n \geq n_0$, nonnegative functions $u(n)$ and $v(n)$ defined on $\mathbb{N}(n_0)$ satisfy the inequality*

$$v(n) - \sum_{s=n_0}^{n-1} \psi(s, v(s)) < u(n) - \sum_{s=n_0}^{n-1} \psi(s, u(s)).$$

*If $v(n_0) < u(n_0)$, then $v(n) < u(n)$ for all $n \geq n_0$.*

**Theorem 2.2.** *Assume that for $n \in \mathbb{N}(n_0), x, y \in \mathbb{R}$, and the functions $f$ and $g$ satisfy*

$$|f(n,x,y)| \le \psi(n,|x|), \quad |g(n,y,x)| \le \psi(n,|y|), \tag{2.1}$$

*where $\psi : \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing with respect to the second variable for each $n \in \mathbb{N}(n_0)$. Also, suppose that each solution $u(n)$ of the difference equation*

$$\Delta u(n) = \psi(n, u(n)), \quad u(n_0) = u_0, \quad n \ge n_0 \tag{2.2}$$

*is bounded on $\mathbb{N}(n_0)$. Then any solution $\{(x(n), y(n))\}$ of* (1.1) *with $\sqrt{|x|^2 + |y|^2} < r$ for each $n \ge n_0$ satisfies the asymptotic property* (1.2).

*Proof.* Let $\{(x(n), y(n))\}$ be any solution of (1.1). Then we have

$$|x(n)| \le |x(n_0)| + \sum_{s=n_0}^{n-1} |f(s, x(s), y(s))| \le |x_0| + \sum_{s=n_0}^{n-1} \psi(s, |x(s)|)$$

and

$$|y(n)| \le |y(n_0)| + \sum_{s=n_0}^{n-1} |g(s, y(s), x(s))| \le |y_0| + \sum_{s=n_0}^{n-1} \psi(s, |y(s)|).$$

If $u_0 \ge |x_0|, \quad u_0 \ge |y_0|$, then

$$|x(n)| - \sum_{s=n_0}^{n-1} \psi(s, |x(s)|) \le |x_0| \le u_0 = u(n) - \sum_{s=n_0}^{n-1} \psi(s, u(s))$$

and

$$|y(n)| - \sum_{s=n_0}^{n-1} \psi(s, |y(s)|) \le |y_0| \le u_0 = u(n) - \sum_{s=n_0}^{n-1} \psi(s, u(s)).$$

By Lemma 2.1, we obtain

$$|x(n)| < u(n), \quad |y(n)| < u(n), \quad n \ge n_0. \tag{2.3}$$

Since every solution $u(n)$ of (2.2) is bounded on $\mathbb{N}(n_0)$, it follows from (2.3) that the solution $\{(x(n), y(n))\}$ of (1.1) is also bounded on $\mathbb{N}(n_0)$. Furthermore, for any $n > m \ge n_0$, we have

$$\begin{aligned} |x(n) - x(m)| &\le \sum_{s=m}^{n-1} |f(s, x(s), y(s))| \le \sum_{s=m}^{n-1} \psi(s, |x(s)|) \\ &\le \sum_{s=m}^{n-1} \psi(s, u(s)) = u(n) - u(m), \\ |y(n) - y(m)| &\le \sum_{s=m}^{n-1} |g(s, y(s), x(s))| \le \sum_{s=m}^{n-1} \psi(s, |y(s)|) \\ &\le \sum_{s=m}^{n-1} \psi(s, u(s)) = u(n) - u(m). \end{aligned} \tag{2.4}$$

Since every solution $u(n)$ of (2.2) is nondecreasing and bounded on $\mathbb{N}(n_0)$, this implies that, given any $\epsilon > 0$, we can choose a $T > 0$ sufficiently large so that $0 \leq u(n) - u(m) < \epsilon$ for all $n \geq T$. It then follows from (2.4) that $|x(n) - x(m)| < \epsilon$, and $|y(n) - y(m)| < \epsilon$ for all $n \geq m \geq T$, which shows that the solution $\{(x(n), y(n))\}$ of (1.1) converges to $\{(\xi_1, \xi_2)\}$ as $n \to \infty$. This completes the proof. □

**Theorem 2.3.** *Let the assumptions of Theorem 2.2 hold. In addition, assume that $f(n, x, y)$ and $g(n, y, x)$ are continuous in $x$ and $y$ for any fixed $n \in \mathbb{N}(n_0)$ and*

$$\sup_{i\geq 0} |x_i(n_0)| = \widehat{u}_0 \leq \rho, \quad \sup_{j\geq 0} |y_j(n_0)| = \widehat{v}_0 \leq \rho.$$

*Then for each $\rho > 0$ there is an $n_0$ large enough such that for every $\xi_i \in \mathbb{R}$ with $|\xi_i| \leq \rho$, there exists a solution $\{(x(n), y(n))\}$ of (1.1) tending to $\{(\xi_1, \xi_2)\}$ as $n \to \infty$.*

*Proof.* Let $\xi_i \in \mathbb{R}$ with $2|\xi_i| < \rho$. Define

$$\mathbb{B}_\rho = \{(x, y) = (x(n), y(n)) | \|x\| \leq \rho, \|y\| \leq \rho\},$$

where

$$\|x\| = \sup_{n\geq n_0} |x(n)| \text{ and } \|y\| = \sup_{n\geq n_0} |y(n)|.$$

Also, we definite the operator $\mathbb{S}$ by

$$\mathbb{S}\begin{pmatrix} x(n) \\ y(n) \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} - \begin{pmatrix} \displaystyle\sum_{s=n_0}^{\infty} f(s, x(s), y(s)) \\ \displaystyle\sum_{s=n_0}^{\infty} g(s, y(s), x(s)) \end{pmatrix}, \quad n \geq n_0. \tag{2.5}$$

Then we can show that the following properties (i)–(iii) hold, which are needed in the application of S. S. Cheng and W. T. Patula [2].

(i) There exists an $n_0$ such that $\mathbb{S}$ maps $\mathbb{B}_\rho$ into itself. In fact, from the proof of Theorem 2.2, we see

$$\sum_{s=n_0}^{n-1} \psi(s, u(s)) \leq u(n) - |x_0|.$$

This means $\displaystyle\sum_{s=n_0}^{\infty} \psi(s, u(s)) \leq \infty$, which is possible since $u(n)$ is convergent. Therefore, we take $n_0$ large enough so that

$$\sum_{s=n_0}^{\infty} \psi(s, u(s)) \leq \frac{\rho}{2}.$$

Then $(x, y) \in \mathbb{B}_\rho$ implies that for $n \geq n_0$,

$$\begin{aligned}
\begin{pmatrix} |\mathbb{S}x(n)| \\ |\mathbb{S}y(n)| \end{pmatrix} &\leq \begin{pmatrix} |\xi_1| \\ |\xi_2| \end{pmatrix} + \begin{pmatrix} \displaystyle\sum_{s=n_0}^{\infty} \psi(s, |x(s)|) \\ \displaystyle\sum_{s=n_0}^{\infty} \psi(s, |y(s)|) \end{pmatrix} \\
&\leq \begin{pmatrix} \rho/2 \\ \rho/2 \end{pmatrix} + \begin{pmatrix} \displaystyle\sum_{s=n_0}^{\infty} \psi(s, u(s)) \\ \displaystyle\sum_{s=n_0}^{\infty} \psi(s, u(s)) \end{pmatrix} \leq \begin{pmatrix} \rho \\ \rho \end{pmatrix}.
\end{aligned} \tag{2.6}$$

(ii) The operator $\mathbb{S}$ is continuous. Let $(x, y) \in \mathbb{B}_\rho$, and $\{x_i\}_{i=0}^{\infty}, \{y_i\}_{i=0}^{\infty}$ be arbitrary sequences of elements of $\mathbb{B}_\rho$ such that $\lim\limits_{i\to\infty} \|x_i - x\| = 0$ and $\lim\limits_{i\to\infty} \|y_i - y\| = 0$. Since the solutions $\overline{u}(n)$ and $u(n)$ of (2.2) are convergent for any $\epsilon > 0$, we can choose $n_1 \in \mathbb{N}(n_0)$ large enough that

$$\sum_{s=n_1}^{\infty} \psi(s, \overline{u}(s)) + \sum_{s=n_1}^{\infty} \psi(s, u(s)) < \frac{\epsilon}{2}.$$

We obtain

$$\begin{aligned}
&|\mathbb{S}x_i(n) - \mathbb{S}x(n)| \leq \sum_{s=n_0}^{\infty} |f(s, x_i(s), y_i(s)) - f(s, x(s), y(s))| \\
&\leq \sum_{s=n_0}^{n_1-1} |f(s, x_i(s), y_i(s)) - f(s, x(s), y(s))| + \sum_{s=n_1}^{\infty} \psi(s, |x_i(s)|) + \sum_{s=n_1}^{\infty} \psi(s, |x(s)|) \\
&\leq \sum_{s=n_0}^{n_1-1} |f(s, x_i(s), y_i(s)) - f(s, x(s), y(s))| + \sum_{s=n_1}^{\infty} \psi(s, \overline{u}(s)) + \sum_{s=n_1}^{\infty} \psi(s, u(s)) \\
&\leq \sum_{s=n_0}^{n_1-1} |f(s, x_i(s), y_i(s)) - f(s, x(s), y(s))| + \frac{\epsilon}{2},
\end{aligned}$$

from which, as a result of the continuity of $f$, it follows that

$$\lim_{i\to\infty} \left[ \sum_{s=n_0}^{n_1-1} |f(s, x_i(s), y_i(s)) - f(s, x(s), y(s))| \right] = 0.$$

Consequently,

$$\lim_{i\to\infty} \sup{}_{n \geq n_1} |\mathbb{S}x_i(n) - \mathbb{S}x(n)| = 0.$$

Showing that

$$\lim_{i\to\infty} \sup{}_{n\geq n_1} |\mathbb{S}y_i(n) - \mathbb{S}y(n)| = 0$$

is similar and hence we omit it. Hence, $\mathbb{S}$ is continuous.

(iii) $\mathbb{S}\mathbb{B}_\rho$ is relatively compact. It suffices to show that $\mathbb{S}\mathbb{B}_\rho$ is bounded and equiconvergent to $(\xi_1, \xi_2)$. Since $|x(n_0)| \leq u_0 \leq \rho, \quad |y(n_0)| \leq u_0 \leq \rho$ for any $(x, y) \in \mathbb{B}_\rho$. We have

$$|\mathbb{S}x(n)| \leq |\xi_1| + \sum_{s=n_0}^{\infty} \psi(s, u(s)) < \infty$$

and

$$|\mathbb{S}y(n)| \leq |\xi_2| + \sum_{s=n_0}^{\infty} \psi(s, u(s)) < \infty.$$

Therefore, the set $\mathbb{S}\mathbb{B}_\rho$ is a uniformly bounded subset of the Banach Space. Moreover, it is equiconvergent to $(\xi_1, \xi_2)$, since for every $\epsilon > 0$, there exists a $n_2 = n_2(\xi_1, \xi_2)$ such that

$$|\mathbb{S}x(n) - \xi_1| \leq \sum_{s=n_2}^{\infty} \psi(s, u(s)) < \varepsilon,$$

and

$$|\mathbb{S}y(n) - \xi_2| \leq \sum_{s=n_2}^{\infty} \psi(s, u(s)) < \varepsilon,$$

for every $n \geq n_2$ and all $(x, y) \in \mathbb{B}_\rho$. Thus, $\mathbb{S}\mathbb{B}_\rho$ is relatively compact.

Therefore, by S. S. Cheng and W. T. Patula's fixed-point theorem, there exists $(x, y) \in \mathbb{B}_\rho$ such that $\mathbb{S}(x, y) = (x, y)$. That is, there exists a solution $\{(x(n), y(n))\}$ of (2.5). Obviously, $\{(x(n), y(n))\}$ is a solution of the problem (1.1). The proof is complete. □

In the following paragraphs, we consider the asymptotic behavior of (1.1) under the following conditions (H):

$$|f(n, x(n), y(n))| \geq \sum_{i=n_0}^{n} \lambda_i(n)\omega_i(|y(n)|),$$

and

$$|g(n, y(n), x(n))| \geq \sum_{i=n_0}^{n} r_i(n)u_i(|x(n)|),$$

where $f, g : \mathbb{N}(n_0) \times \mathbb{R} \to \mathbb{R}, f(n, u, v) > 0, g(n, u, v) > 0$ for $u, v > 0$. $\lambda_i(n)$ and $r_i(n)$ are two sequences of nonnegative real numbers, $\omega_i(s)$ and $u_i(s)$ are nondecreasing and strictly positive for $s \geq n_0$.

Let $\mathbb{F}$ be the set of all sequences and define

$$\Omega = \{\{(x(n), y(n))\} \in \mathbb{F} : x(n), y(n) > 0, \quad \text{for} \quad n > n_0\}.$$

and

$$K(\alpha,\beta)=\{\{(x(n),y(n))\}\in\Omega:\lim_{n\to\infty}x(n)=\alpha \text{ and } \lim_{n\to\infty}y(n)=\beta\}.$$

($\alpha,\beta$ are two positive numbers). Now we have the following result.

**Theorem 2.4.** *Suppose the conditions* (H). *Then*

$$A=\lim_{n\to\infty}\sum_{s=n_0}^{n-1}\sum_{i=n_0}^{s}\lambda_i(s)\omega_i(c)<\infty,$$

$$B=\lim_{n\to\infty}\sum_{s=n_0}^{n-1}\sum_{i=n_0}^{s}r_i(s)u_i(d)<\infty$$

*hold for some positive constants* $c$ *and* $d$ *if and only if any solutions* $\{(x(n),y(n))\}\in\Omega$ *of* (1.1) *belongs to the set* $K(\alpha,\beta)$.

*Proof.* Let $\{(x(n),y(n))\}$ be a solution in $\Omega$ with $\lim_{n\to\infty}x(n)=\alpha>0$ and $\lim_{n\to\infty}y(n)=\beta>0$. Then, there exists a integer $N>0$ and two positive constants, namely, $c$ and $d$ such that $d\le x(n)\le\alpha, c\le y(n)\le\beta$ for $n>N$. From system (1.1) we have

$$x(n)=x(N)+\sum_{s=N}^{n-1}f(s,x(s),y(s))\ge\sum_{s=N}^{n-1}\sum_{i=N}^{s}\lambda_i(s)\omega_i(y(s))\ge\sum_{s=N}^{n-1}\sum_{i=N}^{s}\lambda_i(s)\omega_i(c)$$

and

$$y(n)=y(N)+\sum_{s=N}^{n-1}g(s,y(s),x(s))\ge\sum_{s=N}^{n-1}\sum_{i=N}^{s}r_i(s)u_i(x(s))\ge\sum_{s=N}^{n-1}\sum_{i=N}^{s}r_i(s)u_i(d).$$

Thus,

$$\alpha\ge x(n)\ge\sum_{s=N}^{n-1}\sum_{i=N}^{s}\lambda_i(s)\omega_i(c),\quad \beta\ge y(n)\ge\sum_{s=N}^{n-1}\sum_{i=N}^{s}r_i(s)u_i(d).$$

Hence, we obtain

$$A=\lim_{n\to\infty}\sum_{s=n_0}^{n-1}\sum_{i=n_0}^{s}\lambda_i(s)\omega_i(c)<\infty,$$

and

$$B=\lim_{n\to\infty}\sum_{s=n_0}^{n-1}\sum_{i=n_0}^{s}r_i(s)u_i(d)<\infty.$$

Conversely, suppose that $A < \infty$ and $B < \infty$. First notice that for $n \geq n_0$ the first equation of (1.1) can be written as

$$x(n) = x_0 + \sum_{s=n_0}^{n-1} f(s, x(s), y(s)) \geq \sum_{s=n_0}^{n-1} \sum_{i=n_0}^{s} \lambda_i(s)\omega_i(y(s)).$$

In a similar fashion, we obtain from the second equation of (1.1),

$$y(n) = y_0 + \sum_{s=n_0}^{n-1} g(s, y(s), x(s)) \geq \sum_{s=n_0}^{n-1} \sum_{i=n_0}^{s} r_i(s)u_i(x(s)).$$

Next, we can choose an integer $N$ large enough so that

$$\sum_{s=N}^{\infty} \sum_{i=N}^{s} \lambda_i(s)\omega_i(c) < \frac{c}{2}, \quad \sum_{s=N}^{\infty} \sum_{i=N}^{s} r_i(s)u_i(d) < \frac{d}{2}. \tag{2.7}$$

Let $\mathbb{X}$ be the Banach space of all bounded real valued sequences $\{(x(n), y(n))\}_{n=N}^{\infty}$ with the norm $\|(x, y)\| = \max\{\sup_{n\geq N} |x(n)|, \sup_{n\geq N} |y(n)|\}$, and with the usual pointwise ordering $\leq$. Define a subset $\Omega$ of $\mathbb{X}$ by

$$\Omega = \left\{(x(n), y(n)) \in \mathbb{X} : \frac{c}{2} \leq x(n) \leq c, \frac{d}{2} \leq y(n) \leq d, n \geq N\right\}.$$

It is clear that any subset $\mathbb{B}$ of $\Omega$, we have $\inf \mathbb{B} \in \Omega$ and $\sup \mathbb{B} \in \Omega$. Define the operator $E : \Omega \to \mathbb{X}$ by

$$E\begin{pmatrix} x(n) \\ y(n) \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} - \begin{pmatrix} \sum_{s=N}^{\infty} \sum_{i=N}^{s} \lambda_i(s)\omega_i(y(s)) \\ \sum_{s=N}^{\infty} \sum_{i=N}^{s} r_i(s)u_i(x(s)) \end{pmatrix}, (x(n), y(n)) \in \Omega.$$

First, we claim that $E$ maps $\Omega$ into $\Omega$. To see this we let $x(n) \in \Omega$. Then

$$c \geq (Ex)(n) = c - \sum_{s=N}^{\infty} \sum_{i=N}^{s} \lambda_i(s)\omega_i(y(s)) \geq c - \sum_{s=N}^{\infty} \sum_{i=N}^{s} \lambda_i(s)\omega_i(\frac{d}{2}) \geq \frac{c}{2}.$$

Showing that, for $y \in \Omega, \frac{d}{2} \leq (Ey)(n) \leq d$ is similar and hence we omit it. It is clear from the definition of the operator $E$ that $E$ is decreasing. Thus, by Knaster's fixed point theorem [4], we conclude that there exists $(x, y) \in \Omega$ such that $(x, y) = E(x, y)$, i.e.,

$$x(n) = c - \sum_{s=N}^{\infty} \sum_{i=N}^{s} \lambda_i(s)\omega_i(y(s)) \text{ and } y(n) = d - \sum_{s=N}^{\infty} \sum_{i=N}^{s} r_i(s)u_i(x(s)),\ n \geq N.$$

Thus we obtain $\lim_{n\to\infty} x(n) = c_0$ and $\lim_{n\to\infty} y(n) = d_0$. In view of the comparison principle

$$c - \sum_{s=N}^{n-1}\sum_{i=N}^{s} \lambda_i(s)\omega_i(y(s)) \geq c - \sum_{s=N}^{n-1} f(s, x(s), y(s)),$$

$$d - \sum_{s=N}^{n-1}\sum_{i=N}^{s} r_i(s)u_i(x(s)) \geq d - \sum_{s=N}^{n-1} g(s, y(s), x(s)).$$

Hence, we obtain

$$\lim_{n\to\infty} \sum_{s=N}^{n-1} f(s, x(s), y(s)) = \alpha_1 \text{ and } \lim_{n\to\infty} \sum_{s=N}^{n-1} g(s, y(s), x(s)) = \beta_1.$$

Therefore, $\{(x(n), y(n))\}$ is a positive solution of (1.1) which belongs to $K(\alpha, \beta)$. This completes the proof. □


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