Asymptotic Behavior of Solutions of a Class of Nonlinear Difference Systems

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Abstract
In this paper we study the asymptotic behavior of the two-dimensional nonlinear difference system
\[
\begin{align*}
\Delta x(n) &= f(n, x(n), y(n)) \\
\Delta y(n) &= g(n, y(n), x(n)),
\end{align*}
\]
and necessary as well as sufficient conditions are established.

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1 Introduction

The problem of asymptotic behavior is one of the most important topic in the qualitative study of nonlinear scalars and systems of difference equations and has been the subject
of many investigations. Recently, Li [5] and Agarwal, Li and Pang [1] studied a class of two-dimensional nonlinear difference systems of the form

\[
\begin{align*}
\Delta x(n) &= a(n)f(y(n)), \\
\Delta y(n) &= b(n)g(x(n)).
\end{align*}
\]

They provided a classification scheme for positive solutions of the above system and gave conditions for the existence of solutions with designated asymptotic properties. Moreover, Li and Raffoul [6] studied the classification and existence of positive solutions of the Volterra nonlinear difference system

\[
\begin{align*}
\Delta x(n) &= h(n)x(n) + \sum_{i=0}^{n} a(n,i)f(y(i)), \\
\Delta y(n) &= p(n)y(n) + \sum_{i=0}^{n} b(n,i)g(x(i)), n > 0.
\end{align*}
\]

In this paper, we shall consider the general nonlinear difference system

\[
\begin{align*}
\Delta x(n) &= f(n, x(n), y(n)), \\
\Delta y(n) &= g(n, y(n), x(n)), n \geq n_0, 
\end{align*}
\]

(1.1)

where \( f, g : \mathbb{N}(n_0) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \mathbb{N}(n_0) = \{ n_0, n_0 + 1, n_0 + 2, \cdots \} \) (n_0 a nonnegative integer), \( x(n_0) = x_0, y(n_0) = y_0 \), and \( \Delta \) denotes the forward difference operator, that is, \( \Delta x(n) = x(n + 1) - x(n) \) for a sequence \( x(n) \). Next, we state the following definition.

**Definition 1.1.** We say that (1.1) has an asymptotic equilibrium if

(i) there exist \( \xi \in \mathbb{R} \) and \( r > 0 \) such that any solution \( x(n, n_0, x_0) \) of (1.1) with \( |x_0| < r \) satisfies

\[
x(n) = \xi + o(1) \quad \text{as} \quad n \to \infty; \tag{1.2}
\]

(ii) corresponding to each \( \xi \in \mathbb{R} \), there is a solution of (1.1) satisfying (1.2).

\section{Main Results}

First, we need the following comparison principle for difference inequalities.

**Lemma 2.1 (See [3]).** Let \( \psi(n, r) \) be a nonnegative nondecreasing function in \( r \) for any fixed \( n \in \mathbb{N}(n_0) \). Suppose that for any \( n \geq n_0 \), nonnegative functions \( u(n) \) and \( v(n) \) defined on \( \mathbb{N}(n_0) \) satisfy the inequality

\[
v(n) - \sum_{s=n_0}^{n-1} \psi(s, v(s)) < u(n) - \sum_{s=n_0}^{n-1} \psi(s, u(s)).
\]

If \( v(n_0) < u(n_0) \), then \( v(n) < u(n) \) for all \( n \geq n_0 \).
Theorem 2.2. Assume that for \( n \in \mathbb{N}(n_0), x, y \in \mathbb{R} \), and the functions \( f \) and \( g \) satisfy
\[
|f(n, x, y)| \leq \psi(n, |x|), \quad |g(n, y, x)| \leq \psi(n, |y|),
\]
(2.1)
where \( \psi : \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}^+ \) is nondecreasing with respect to the second variable for each \( n \in \mathbb{N}(n_0) \). Also, suppose that each solution \( u(n) \) of the difference equation
\[
\Delta u(n) = \psi(n, u(n)), \quad u(n_0) = u_0, \quad n \geq n_0
\]
(2.2)
is bounded on \( \mathbb{N}(n_0) \). Then any solution \( \{(x(n), y(n))\} \) of (1.1) with \( \sqrt{|x|^2 + |y|^2} < r \) for each \( n \geq n_0 \) satisfies the asymptotic property (1.2).

Proof. Let \( \{(x(n), y(n))\} \) be any solution of (1.1). Then we have
\[
|x(n)| \leq |x(n_0)| + \sum_{s=n_0}^{n-1} |f(s, x(s), y(s))| \leq |x_0| + \sum_{s=n_0}^{n-1} \psi(s, |x(s)|)
\]
and
\[
|y(n)| \leq |y(n_0)| + \sum_{s=n_0}^{n-1} |g(s, y(s), x(s))| \leq |y_0| + \sum_{s=n_0}^{n-1} \psi(s, |y(s)|).
\]
If \( u_0 \geq |x_0|, \quad u_0 \geq |y_0| \), then
\[
|x(n)| - \sum_{s=n_0}^{n-1} \psi(s, |x(s)|) \leq |x_0| \leq u(n) - \sum_{s=n_0}^{n-1} \psi(s, u(s))
\]
and
\[
|y(n)| - \sum_{s=n_0}^{n-1} \psi(s, |y(s)|) \leq |y_0| \leq u(n) - \sum_{s=n_0}^{n-1} \psi(s, u(s)).
\]
By Lemma 2.1, we obtain
\[
|x(n)| < u(n), \quad |y(n)| < u(n), \quad n \geq n_0.
\]
(2.3)
Since every solution \( u(n) \) of (2.2) is bounded on \( \mathbb{N}(n_0) \), it follows from (2.3) that the solution \( \{(x(n), y(n))\} \) of (1.1) is also bounded on \( \mathbb{N}(n_0) \). Furthermore, for any \( n > m \geq n_0 \), we have
\[
|x(n) - x(m)| \leq \sum_{s=m}^{n-1} |f(s, x(s), y(s))| \leq \sum_{s=m}^{n-1} \psi(s, |x(s)|)
\]
\[
\leq \sum_{s=m}^{n-1} \psi(s, u(s)) = u(n) - u(m),
\]
(2.4)
\[
|y(n) - y(m)| \leq \sum_{s=m}^{n-1} |g(s, y(s), x(s))| \leq \sum_{s=m}^{n-1} \psi(s, |y(s)|)
\]
\[
\leq \sum_{s=m}^{n-1} \psi(s, u(s)) = u(n) - u(m).
\]
Since every solution $u(n)$ of (2.2) is nondecreasing and bounded on $\mathbb{N}(n_0)$, this implies that, given any $\epsilon > 0$, we can choose a $T > 0$ sufficiently large so that $0 \leq u(n) - u(m) < \epsilon$ for all $n \geq T$. It then follows from (2.4) that $|x(n) - x(m)| < \epsilon$, and $|y(n) - y(m)| < \epsilon$ for all $n \geq m \geq T$, which shows that the solution $\{(x(n), y(n))\}$ of (1.1) converges to $\{(\xi_1, \xi_2)\}$ as $n \to \infty$. This completes the proof.

**Theorem 2.3.** Let the assumptions of Theorem 2.2 hold. In addition, assume that $f(n, x, y)$ and $g(n, y, x)$ are continuous in $x$ and $y$ for any fixed $n \in \mathbb{N}(n_0)$ and

$$
\sup_{i \geq 0} |x_i(n_0)| = \hat{u}_0 \leq \rho, \quad \sup_{j \geq 0} |y_j(n_0)| = \hat{v}_0 \leq \rho.
$$

Then for each $\rho > 0$ there is an $n_0$ large enough such that for every $\xi_i \in \mathbb{R}$ with $|\xi_i| \leq \rho$, there exists a solution $\{(x(n), y(n))\}$ of (1.1) tending to $\{(\xi_1, \xi_2)\}$ as $n \to \infty$.

**Proof.** Let $\xi_i \in \mathbb{R}$ with $2|\xi_i| < \rho$. Define

$$
\mathbb{B}_\rho = \{(x, y) = (x(n), y(n)) ||x|| \leq \rho, ||y|| \leq \rho\},
$$

where

$$
||x|| = \sup_{n \geq n_0} |x(n)| \text{ and } ||y|| = \sup_{n \geq n_0} |y(n)|.
$$

Also, we define the operator $\mathbb{S}$ by

$$
\mathbb{S} \left( \begin{array}{c} x(n) \\ y(n) \end{array} \right) = \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right) - \left( \begin{array}{c} \sum_{s=n_0}^{\infty} f(s, x(s), y(s)) \\ \sum_{s=n_0}^{\infty} g(s, y(s), x(s)) \end{array} \right), \quad n \geq n_0.
$$

(2.5)

Then we can show that the following properties (i)–(iii) hold, which are needed in the application of S. S. Cheng and W. T. Patula [2].

(i) There exists an $n_0$ such that $\mathbb{S}$ maps $\mathbb{B}_\rho$ into itself. In fact, from the proof of Theorem 2.2, we see

$$
\sum_{s=n_0}^{n-1} \psi(s, u(s)) \leq u(n) - |x_0|.
$$

This means $\sum_{s=n_0}^{\infty} \psi(s, u(s)) \leq \infty$, which is possible since $u(n)$ is convergent. Therefore, we take $n_0$ large enough so that

$$
\sum_{s=n_0}^{\infty} \psi(s, u(s)) \leq \frac{\rho}{2}.
$$
Then \((x, y) \in B_\rho\) implies that for \(n \geq n_0\),

\[
\left( \begin{array}{c} |Sx(n)| \\ |Sy(n)| \end{array} \right) \leq \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right) + \left( \begin{array}{c} \sum_{s=n_0}^\infty \psi(s, |x(s)|) \\ \sum_{s=n_0}^\infty \psi(s, |y(s)|) \end{array} \right) \leq \left( \begin{array}{c} \rho/2 \\ \rho/2 \end{array} \right) + \left( \begin{array}{c} \sum_{s=n_0}^\infty \psi(s, u(s)) \\ \sum_{s=n_0}^\infty \psi(s, u(s)) \end{array} \right) \leq \left( \begin{array}{c} \rho \\ \rho \end{array} \right).
\tag{2.6}
\]

(ii) The operator \(S\) is continuous. Let \((x, y) \in B_\rho\), and \(\{x_i\}_{i=0}^\infty, \{y_i\}_{i=0}^\infty\) be arbitrary sequences of elements of \(B_\rho\) such that \(\lim_{i \to \infty} \|x_i - x\| = 0\) and \(\lim_{i \to \infty} \|y_i - y\| = 0\). Since the solutions \(\pi(n)\) and \(u(n)\) of (2.2) are convergent for any \(\epsilon > 0\), we can choose \(n_1 \in \mathbb{N}(n_0)\) large enough that

\[
\sum_{s=n_1}^\infty \psi(s, \pi(s)) + \sum_{s=n_1}^\infty \psi(s, u(s)) < \frac{\epsilon}{2}.
\]

We obtain

\[
|Sx_i(n) - Sx(n)| \leq \sum_{s=n_0}^\infty |f(s, x_i(s), y_i(s)) - f(s, x(s), y(s))|
\leq \sum_{s=n_0}^{n_1-1} |f(s, x_i(s), y_i(s)) - f(s, x(s), y(s))| + \sum_{s=n_1}^\infty \psi(s, |x_i(s)|) + \sum_{s=n_1}^\infty \psi(s, |x(s)|)
\leq \sum_{s=n_0}^{n_1-1} |f(s, x_i(s), y_i(s)) - f(s, x(s), y(s))| + \sum_{s=n_1}^\infty \psi(s, \pi(s)) + \sum_{s=n_1}^\infty \psi(s, u(s))
\leq \sum_{s=n_0}^{n_1-1} |f(s, x_i(s), y_i(s)) - f(s, x(s), y(s))| + \frac{\epsilon}{2},
\]

from which, as a result of the continuity of \(f\), it follows that

\[
\lim_{i \to \infty} \left[ \sum_{s=n_0}^{n_1-1} |f(s, x_i(s), y_i(s)) - f(s, x(s), y(s))| \right] = 0.
\]

Consequently,

\[
\lim_{i \to \infty} \sup_{n \geq n_1} |Sx_i(n) - Sx(n)| = 0.
\]
Showing that
\[ \lim_{i \to \infty} \sup_{n \geq n_1} |S y_i(n) - S y(n)| = 0 \]
is similar and hence we omit it. Hence, \( S \) is continuous.

(iii) \( \mathbb{S} \mathbb{B}_\rho \) is relatively compact. It suffices to show that \( \mathbb{S} \mathbb{B}_\rho \) is bounded and equiconvergent to \((\xi_1, \xi_2)\). Since \( |x(n_0)| \leq u_0 \leq \rho, \quad |y(n_0)| \leq u_0 \leq \rho \) for any \((x, y) \in \mathbb{B}_\rho\). We have
\[
|S x(n)| \leq |\xi_1| + \sum_{s=n_0}^{\infty} \psi(s, u(s)) < \infty
\]
and
\[
|S y(n)| \leq |\xi_2| + \sum_{s=n_0}^{\infty} \psi(s, u(s)) < \infty.
\]
Therefore, the set \( \mathbb{S} \mathbb{B}_\rho \) is a uniformly bounded subset of the Banach Space. Moreover, it is equiconvergent to \((\xi_1, \xi_2)\), since for every \( \epsilon > 0 \), there exists a \( n_2 = n_2(\xi_1, \xi_2) \) such that
\[
|S x(n) - \xi_1| \leq \sum_{s=n_2}^{\infty} \psi(s, u(s)) < \epsilon,
\]
and
\[
|S y(n) - \xi_2| \leq \sum_{s=n_2}^{\infty} \psi(s, u(s)) < \epsilon,
\]
for every \( n \geq n_2 \) and all \((x, y) \in \mathbb{B}_\rho\). Thus, \( \mathbb{S} \mathbb{B}_\rho \) is relatively compact.

Therefore, by S. S. Cheng and W. T. Patula’s fixed-point theorem, there exists \((x, y) \in \mathbb{B}_\rho\) such that \( S(x, y) = (x, y) \). That is, there exists a solution \( \{ (x(n), y(n)) \} \) of (2.5). Obviously, \( \{ (x(n), y(n)) \} \) is a solution of the problem (1.1). The proof is complete. \( \square \)

In the following paragraphs, we consider the asymptotic behavior of (1.1) under the following conditions (H):

\[
|f(n, x(n), y(n))| \geq \sum_{i=n_0}^{n} \lambda_i(n) \omega_i(|y(n)|),
\]
and
\[
|g(n, y(n), x(n))| \geq \sum_{i=n_0}^{n} r_i(n) u_i(|x(n)|),
\]
where \( f, g : \mathbb{N}(n_0) \times \mathbb{R} \to \mathbb{R}, f(n, u, v) > 0, g(n, u, v) > 0 \) for \( u, v > 0 \). \( \lambda_i(n) \) and \( r_i(n) \) are two sequences of nonnegative real numbers, \( \omega_i(s) \) and \( u_i(s) \) are nondecreasing and strictly positive for \( s \geq n_0 \).

Let \( \mathbb{F} \) be the set of all sequences and define
\[
\Omega = \{ \{(x(n), y(n))\} \in \mathbb{F} : x(n), y(n) > 0, \quad \text{for} \quad n > n_0 \}.
\]
and

\[ K(\alpha, \beta) = \{(x(n), y(n)) \in \Omega : \lim_{n \to \infty} x(n) = \alpha \text{ and } \lim_{n \to \infty} y(n) = \beta \}. \]

(\alpha, \beta \text{ are two positive numbers}). Now we have the following result.

**Theorem 2.4.** Suppose the conditions (H). Then

\[
A = \lim_{n \to \infty} \sum_{s=n_0}^{n-1} \sum_{i=n_0}^{s} \lambda_i(s) \omega_i(c) < \infty,
\]

\[
B = \lim_{n \to \infty} \sum_{s=n_0}^{n-1} \sum_{i=n_0}^{s} r_i(s) u_i(d) < \infty
\]

hold for some positive constants \(c\) and \(d\) if and only if any solutions \(\{(x(n), y(n))\} \in \Omega\) of (1.1) belongs to the set \(K(\alpha, \beta)\).

**Proof.** Let \(\{(x(n), y(n))\}\) be a solution in \(\Omega\) with \(\lim_{n \to \infty} x(n) = \alpha > 0\) and \(\lim_{n \to \infty} y(n) = \beta > 0\). Then, there exists an integer \(N > 0\) and two positive constants, namely, \(c\) and \(d\) such that \(d \leq x(n) \leq \alpha, c \leq y(n) \leq \beta\) for \(n > N\). From system (1.1) we have

\[
x(n) = x(N) + \sum_{s=N}^{n-1} f(s, x(s), y(s)) \geq \sum_{s=N}^{n-1} \sum_{i=N}^{s} \lambda_i(s) \omega_i(y(s)) \geq \sum_{s=N}^{n-1} \sum_{i=N}^{s} \lambda_i(s) \omega_i(c)
\]

and

\[
y(n) = y(N) + \sum_{s=N}^{n-1} g(s, y(s), x(s)) \geq \sum_{s=N}^{n-1} \sum_{i=N}^{s} r_i(s) u_i(x(s)) \geq \sum_{s=N}^{n-1} \sum_{i=N}^{s} r_i(s) u_i(d).
\]

Thus,

\[
\alpha \geq x(n) \geq \sum_{s=N}^{n-1} \sum_{i=N}^{s} \lambda_i(s) \omega_i(c), \quad \beta \geq y(n) \geq \sum_{s=N}^{n-1} \sum_{i=N}^{s} r_i(s) u_i(d).
\]

Hence, we obtain

\[
A = \lim_{n \to \infty} \sum_{s=n_0}^{n-1} \sum_{i=n_0}^{s} \lambda_i(s) \omega_i(c) < \infty,
\]

and

\[
B = \lim_{n \to \infty} \sum_{s=n_0}^{n-1} \sum_{i=n_0}^{s} r_i(s) u_i(d) < \infty.
\]
In a similar fashion, we obtain from the second equation of (1.1),

\[ y(n) = y_0 + \sum_{s=n_0}^{n-1} g(s, y(s), x(s)) \geq \sum_{s=n_0}^{n-1} \sum_{i=n_0}^{s} \lambda_i(s)\omega_i(y(s)). \]

In a similar fashion, we obtain from the second equation of (1.1),

\[ y(n) = y_0 + \sum_{s=n_0}^{n-1} g(s, y(s), x(s)) \geq \sum_{s=n_0}^{n-1} \sum_{i=n_0}^{s} \lambda_i(s)\omega_i(y(s)). \]

Next, we can choose an integer \( N \) large enough so that

\[ \sum_{s=N}^{\infty} \sum_{i=N}^{s} \lambda_i(s)\omega_i(c) < \frac{c}{2}, \quad \sum_{s=N}^{\infty} \sum_{i=N}^{s} r_i(s)u_i(d) < \frac{d}{2}. \tag{2.7} \]

Let \( \mathbb{X} \) be the Banach space of all bounded real valued sequences \( \{(x(n), y(n))\}_{n=N}^{\infty} \) with the norm \( \|(x, y)\| = \max\{\sup_{n \geq N} |x(n)|, \sup_{n \geq N} |y(n)|\} \), and with the usual pointwise ordering \( \leq \). Define a subset \( \Omega \) of \( \mathbb{X} \) by

\[ \Omega = \left\{ (x(n), y(n)) \in \mathbb{X} : \frac{c}{2} \leq x(n) \leq c, \frac{d}{2} \leq y(n) \leq d, n \geq N \right\}. \]

It is clear that any subset \( \mathbb{B} \) of \( \Omega \), we have \( \inf \mathbb{B} \in \Omega \) and \( \sup \mathbb{B} \in \Omega \). Define the operator \( E : \Omega \to \mathbb{X} \) by

\[ E(x(n), y(n)) = \begin{pmatrix} c \\ d \end{pmatrix} - \begin{pmatrix} \sum_{s=N}^{\infty} \sum_{i=N}^{s} \lambda_i(s)\omega_i(y(s)) \\ \sum_{s=N}^{\infty} \sum_{i=N}^{s} r_i(s)u_i(x(s)) \end{pmatrix}, (x(n), y(n)) \in \Omega. \]

First, we claim that \( E \) maps \( \Omega \) into \( \Omega \). To see this we let \( x(n) \in \Omega \). Then

\[ c \geq (Ex)(n) = c - \sum_{s=N}^{\infty} \sum_{i=N}^{s} \lambda_i(s)\omega_i(y(s)) \geq c - \sum_{s=N}^{\infty} \sum_{i=N}^{s} \lambda_i(s)\omega_i(d) \geq \frac{c}{2}. \]

Showing that, for \( y \in \Omega, \frac{d}{2} \leq (Ey)(n) \leq d \) is similar and hence we omit it. It is clear from the definition of the operator \( E \) that \( E \) is decreasing. Thus, by Knaster’s fixed point theorem [4], we conclude that there exists \( (x, y) \in \Omega \) such that \( (x, y) = E(x, y) \), i.e.,

\[ x(n) = c - \sum_{s=N}^{\infty} \sum_{i=N}^{s} \lambda_i(s)\omega_i(y(s)) \text{ and } y(n) = d - \sum_{s=N}^{\infty} \sum_{i=N}^{s} r_i(s)u_i(x(s)), \ n \geq N. \]
Thus we obtain \( \lim_{n \to \infty} x(n) = c_0 \) and \( \lim_{n \to \infty} y(n) = d_0 \). In view of the comparison principle

\[
c - \sum_{s=N}^{n-1} \sum_{i=N}^{s} \lambda(s) \omega_i(y(s)) \geq c - \sum_{s=N}^{n-1} f(s, x(s), y(s)),
\]

\[
d - \sum_{s=N}^{n-1} \sum_{i=N}^{s} r_i(s) u_i(x(s)) \geq d - \sum_{s=N}^{n-1} g(s, y(s), x(s)).
\]

Hence, we obtain

\[
\lim_{n \to \infty} \sum_{s=N}^{n-1} f(s, x(s), y(s)) = \alpha_1 \quad \text{and} \quad \lim_{n \to \infty} \sum_{s=N}^{n-1} g(s, y(s), x(s)) = \beta_1.
\]

Therefore, \( \{(x(n), y(n))\} \) is a positive solution of (1.1) which belongs to \( K(\alpha, \beta) \). This completes the proof.

\[\square\]

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