

# Boundary Value Problems for First-Order Dynamic Equations

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## Abstract

This paper concerns boundary value problems for first-order dynamic equations with deviating arguments on time scales. We formulate sufficient conditions under which such problems have extremal solutions and a unique solution too. Dynamic inequalities with deviating arguments are also discussed. Some examples illustrate the results.

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## 1 Introduction

Throughout this paper, we denote by  $\mathbb{T}$  any time scale (nonempty closed subset of the real numbers  $\mathbb{R}$ ). We assume that  $0, T \in \mathbb{T}$  and denote by  $J = [0, T]$  a subset of  $\mathbb{T}$  such that  $[0, T] = \{t \in \mathbb{T} : 0 \leq t \leq T\}$ . By  $\sigma$  we denote the forward jump operator  $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ . The graininess function  $\mu : \mathbb{T} \rightarrow \mathbb{R}_+$  is defined by  $\mu(t) = \sigma(t) - t$  with  $\mathbb{R}_+ = [0, \infty)$ . Let  $C(J, \mathbb{R})$  denote the set of continuous functions  $u : J \rightarrow \mathbb{R}$ .

In this paper, we investigate the following first-order dynamic equation on time scales of the form

$$\begin{cases} x^\Delta(t) = f(t, x(t), x(\alpha(t))) \equiv (\mathcal{F}x)(t), & t \in [0, T], \\ 0 = g(x(0), x(T)), \end{cases} \quad (1.1)$$

where  $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ ,  $\alpha \in C(J, J)$ ,  $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ .

We apply a monotone iterative method together with a Banach's fixed point theorem to prove that nonlinear dynamic problems of type (1.1) have extremal solutions. Using a corresponding result for a dynamic inequality we are able to show that our problem has a unique solution. The monotone iterative method was earlier applied also to dynamic problems on time scales, see for example [1, 2, 4–8]. According to our knowledge it is a first paper when this method is applied to dynamic equations with deviating arguments. The results are new. This paper extends the application of this method to such problems. We give also some remarks and examples showing the applications to differential equations.

## 2 Dynamic Inequalities

In this section we present some linear dynamic inequalities which are needed in Section 4.

**Lemma 2.1.** *Assume that*

( $H_1$ ) *there exist functions  $n \in C(J, \mathbb{R}_+)$ ,  $\alpha \in C(J, J)$ ,  $\alpha(t) \leq t$  and  $\alpha(J) \neq J$ .*

*In addition, we assume that*

$$\rho_1 \equiv \int_0^T n(t) \Delta t \leq 1.$$

*Let*

$$\begin{cases} x^\Delta(t) \leq -n(t)x(\alpha(t)), & t \in J, \\ x(0) \leq 0. \end{cases} \quad (2.1)$$

*Then  $x(t) \leq 0$ ,  $t \in J$ .*

*Proof.* We need to prove that  $x(t) \leq 0$ ,  $t \in J$ . Suppose that the inequality  $x(t) \leq 0$ ,  $t \in J$  is not true. Then, we can find  $t_0 \in (0, T]$  such that  $x(t_0) > 0$ . Put

$$x(t_1) = \min_{[0, t_0]} x(t) \leq 0.$$

Integrating the dynamic inequality in (2.1) from  $t_1$  to  $t_0$ , we obtain

$$\begin{aligned} x(t_0) - x(t_1) &\leq - \int_{t_1}^{t_0} n(t)x(\alpha(t)) \Delta t \\ &\leq -x(t_1) \int_0^T n(t) \Delta t \leq -x(t_1). \end{aligned}$$

It contradicts the assumption that  $x(t_0) > 0$ . This proves that  $x(t) \leq 0$  on  $J$  and the proof is complete.  $\square$

**Remark 2.2.** Let  $n(t) = 0$ ,  $t \in J$ . Then  $\rho_1 = 0 < 1$ . In this case,  $x$  is nonincreasing, so  $x(t) \leq x(0) \leq 0$ ,  $t \in J$ .

**Example 2.3.** Let  $\mathbb{T} = \mathbb{N}$ ,  $J = \{0 \leq j \leq L, j \in \mathbb{N}\}$ . Assume that  $n(i) \in \mathbb{R}_+$  for  $i \in P = \{0, 1, \dots, L-1\}$  and  $\sum_{i=0}^{L-1} n(i) \leq 1$ . Let  $0 \leq \alpha(i) \leq i$ ,  $i \in P$  and  $\alpha(P) \neq P$  (for example  $\alpha(i) = k_i \in \mathbb{N}$ ,  $k_i \leq i$ ,  $i \in P$  and  $k_0 = k_1 = 0$ ). Assume that

$$\begin{cases} \Delta x(i) \leq -n(i)x(\alpha(i)), & i \in P, \\ x(0) \leq 0 \end{cases}$$

with  $\Delta x(i) = x(i+1) - x(i)$ . Then,  $x(i) \leq 0$ ,  $i \in J$ , by Lemma 2.1.

**Lemma 2.4.** Assume  $(H_1)$  and

$(H_2)$  there exists a continuous function  $m : J \rightarrow \mathbb{R}$  such that  $-m \in \mathcal{R}_+$ , i.e.,

$$\sup_{t \in J} [\mu(t)m(t)] < 1. \quad (2.2)$$

In addition, we assume that

$$\rho_2 \equiv \int_0^T \mathcal{N}(t) \Delta t \leq 1 \quad \text{with} \quad \mathcal{N}(t) = n(t) e_{(-m)}(\alpha(t), \sigma(t)). \quad (2.3)$$

Let

$$\begin{cases} x^\Delta(t) \leq -m(t)x(t) - n(t)x(\alpha(t)), & t \in J, \\ x(0) \leq 0. \end{cases} \quad (2.4)$$

Then  $x(t) \leq 0$ ,  $t \in J$ .

*Proof.* Let  $p(t) = e_{\ominus(-m)}(t, 0)x(t) = \frac{x(t)}{e_{(-m)}(t, 0)}$ . Note that,  $e_{(-m)}(t, 0) > 0$ , by assumption  $(H_2)$ , see [2, Theorem 2.48(i)]. Indeed,  $\ominus(m)(t) = -\frac{m(t)}{1 + \mu(t)m(t)}$ . Computing  $p^\Delta$ , we have

$$\begin{aligned} p^\Delta(t) &= e_{\ominus(-m)}(\sigma(t), 0)x^\Delta(t) + \ominus(-m)(t)e_{\ominus(-m)}(t, 0)x(t) \\ &= e_{\ominus(-m)}(\sigma(t), 0)x^\Delta(t) + \frac{m(t)}{1 - \mu(t)m(t)}e_{\ominus(-m)}(t, 0)x(t) \\ &= e_{\ominus(-m)}(\sigma(t), 0)x^\Delta(t) + m(t)e_{\ominus(-m)}(\sigma(t), 0)x(t) \\ &= e_{\ominus(-m)}(\sigma(t), 0) [m(t)x(t) + x^\Delta(t)] \\ &\leq -e_{\ominus(-m)}(\sigma(t), 0)n(t)x(\alpha(t)) \\ &= -\mathcal{N}(t)p(\alpha(t)), \end{aligned}$$

because

$$\frac{e_{\ominus(-m)}(\sigma(t), 0)}{e_{\ominus(-m)}(\alpha(t), 0)} = e_{(-m)}(0, \sigma(t))e_{(-m)}(\alpha(t), 0) = e_{(-m)}(\alpha(t), \sigma(t)),$$

by [2, Theorem 2.36]. Then problem (2.4) takes the form

$$\begin{cases} p^\Delta(t) \leq -\mathcal{N}(t)p(\alpha(t)), & t \in J, \\ p(0) \leq 0. \end{cases}$$

It yields  $p(t) \leq 0$  on  $J$ , by Lemma 2.1. This shows that  $x(t) \leq 0$  on  $J$ . The proof is complete.  $\square$

*Remark 2.5.* If  $m(t) \equiv 0$ , then  $e_m(s, t) \equiv 1$ , by [2, Theorem 2.36(i)]. In this case Lemma 2.4 reduces to Lemma 2.1.

*Remark 2.6.* Let  $\mathbb{T} = \mathbb{R}$ . Then  $\mu(t) = 0$ ,  $\sigma(t) = t$ , condition (2.2) holds and

$$e_{(-m)}(\alpha(t), \sigma(t)) = \exp\left(\int_{\alpha(t)}^t m(s)ds\right).$$

In this case,  $\rho_2$  from condition (2.3) has the form

$$\rho_2 = \int_0^T n(t)e^{\int_{\alpha(t)}^t m(s)ds} dt$$

and we have [3, Lemma 2.2].

*Remark 2.7.* Let  $\mathbb{T} = \mathbb{R}$  and  $m(t) \geq 0$  on  $J$ . Then

$$e_{(-m)}(\alpha(t), \sigma(t)) = e^{\int_{\alpha(t)}^t m(s)ds} \leq e^{\int_0^t m(s)ds}.$$

If we assume that

$$\rho_3 \equiv \int_0^T n(t)e^{\int_0^t m(s)ds} dt \leq 1,$$

then condition (2.3) holds. Note that  $\rho_3$  does not depend on  $\alpha$ . Moreover, if  $m(t) = m > 0$ ,  $n(t) = n > 0$ , then  $\rho_3$  takes the form

$$\rho_3 = \frac{n}{m} (e^{mT} - 1).$$

### 3 Dynamic Equations

Now we consider the linear dynamic equation of the form

$$\begin{cases} x^\Delta(t) = -m(t)x(t) - n(t)x(\alpha(t)) + h(t), & t \in J, \\ x(0) = x_0 \in \mathbb{R}. \end{cases} \quad (3.1)$$

**Theorem 3.1.** *Let assumptions  $(H_1)$ ,  $(H_2)$  hold and let  $h \in C(J, \mathbb{R})$ . Then problem (3.1) has a unique solution.*

*Proof.* We first show that solving (3.1) is equivalent to solving a fixed point problem. Let  $x$  be any solution of problem (3.1). We use the product rule and [2, Theorem 2.36(ii)], so

$$\begin{aligned} [xe_{\ominus(-m)}(\cdot, 0)]^\Delta(t) &= x^\Delta(t)e_{\ominus(-m)}(\sigma(t), 0) + x(t) \ominus (-m)e_{\ominus(-m)}(t, 0) \\ &= [x^\Delta(t) + m(t)x(t)]e_{\ominus(-m)}(\sigma(t), 0). \end{aligned}$$

Hence

$$\begin{aligned} x(t)e_{\ominus(-m)}(t, 0) &= x(0)e_{\ominus(-m)}(0, 0) \\ &\quad + \int_0^t [x^\Delta(s) + m(s)x(s)]e_{\ominus(-m)}(\sigma(s), 0)\Delta s \\ &= x_0e_{(-m)}(0, 0) + \int_0^t [-n(s)x(\alpha(s)) + h(s)]e_{\ominus(-m)}(\sigma(s), 0)\Delta s. \end{aligned}$$

Now,

$$\begin{aligned} x(t) &= x_0e_{(-m)}(t, 0) + \int_0^t [-n(s)x(\alpha(s)) + h(s)]e_{(-m)}(t, \sigma(s))\Delta s \\ &\equiv (\mathcal{A}_h x)(t), \end{aligned}$$

since

$$\frac{e_{\ominus(-m)}(\sigma(s), 0)}{e_{\ominus(-m)}(t, 0)} = e_{(-m)}(0, \sigma(s))e_{(-m)}(t, 0) = e_{(-m)}(t, \sigma(s));$$

see [2, Theorem 2.36]. Similarly it is easy to see that if  $x \in C_{rd}(J, \mathbb{R})$  is any solution of  $x = \mathcal{A}_h x$ , then  $x$  is a solution of problem (3.1).

Now we use Banach's fixed point theorem. Let

$$X = \{x \in C_{rd}(J, \mathbb{R}) \quad \text{with} \quad \|x\| = \max_{t \in J} e_{(-\lambda_0)}(t, 0)|x(t)|\}$$

with a positive constant  $\lambda_0$  such that  $\lambda_0 \sup_{t \in J} \mu(t) < 1$ ,  $\lambda_0 \geq n_0$  and

$$n_0 = \max_{t \in J} \max_{0 \leq s \leq t} n(s)e_{(-m)}(t, \sigma(s)).$$

Note  $X$  is a Banach space. For  $x, y \in X$  we have

$$\begin{aligned} \|\mathcal{A}_h x - \mathcal{A}_h y\| &\leq \max_{t \in J} e_{(-\lambda_0)}(t, 0) \int_0^t n(s)|x(\alpha(s)) - y(\alpha(s))|e_{(-m)}(t, \sigma(s))\Delta s \\ &\leq \|x - y\|n_0 \max_{t \in J} e_{(-\lambda_0)}(t, 0) \int_0^t e_{\ominus(-\lambda_0)}(\alpha(s), 0)\Delta s \\ &= \|x - y\|H(\lambda_0), \end{aligned}$$

where

$$H(\lambda_0) = n_0 \max_{t \in J} e_{(-\lambda_0)}(t, 0) \int_0^t e_{\ominus(-\lambda_0)}(\alpha(s), 0) \Delta s.$$

Note that

$$1 + \ominus(-\lambda_0)(t)\mu(t) = 1 + \frac{\lambda_0\mu(t)}{1 - \lambda_0\mu(t)} = \frac{1}{1 - \lambda_0\mu(t)} > 0$$

in view of assumption (H<sub>2</sub>). It shows that  $\ominus(-\lambda_0) \in \mathcal{R}_+$ , so

$$e_{\ominus(-\lambda_0)}(s, 0) > 0, \quad s \in J,$$

by [2, Theorem 2.48(i)]. We see that

$$[e_{\ominus(-\lambda_0)}(\cdot, 0)]^\Delta(s) = \ominus(-\lambda_0)e_{\ominus(-\lambda_0)}(s, 0) > 0.$$

It shows that  $e_{\ominus(-\lambda_0)}$  is nondecreasing, so by [2, Theorem 1.76], we have

$$e_{\ominus(-\lambda_0)}(\alpha(s), 0) \leq e_{\ominus(-\lambda_0)}(s, 0), \quad s \in J$$

and now

$$H(\lambda_0) \leq n_0 \max_{t \in J} e_{(-\lambda_0)}(t, 0) \int_0^t e_{\ominus(-\lambda_0)}(s, 0) \Delta s.$$

Indeed,

$$\begin{aligned} \int_0^t e_{\ominus(-\lambda_0)}(s, 0) \Delta s &= \int_0^t \frac{1}{\ominus(-\lambda_0)(s)} [e_{\ominus(-\lambda_0)}(\cdot, 0)]^\Delta(s) \Delta s \\ &\leq \frac{1}{\lambda_0} [e_{(-\lambda_0)}(t, 0) - 1], \end{aligned}$$

because  $0 < 1 - \lambda_0\mu(s) \leq 1$  and

$$\ominus(-\lambda_0)(s) = \frac{\lambda_0}{1 - \lambda_0\mu(s)} \geq \lambda_0.$$

It yields

$$\begin{aligned} H(\lambda_0) &\leq \frac{n_0}{\lambda_0} \max_{t \in J} e_{(-\lambda_0)}(t, 0) [e_{(-\lambda_0)}(0, t) - 1] \\ &= \frac{n_0}{\lambda_0} \max_{t \in J} [1 - e_{(-\lambda_0)}(t, 0)] \\ &\leq 1 - e_{(-\lambda_0)}(T, 0) \equiv \xi < 1. \end{aligned}$$

As a result

$$\|\mathcal{A}_h x - \mathcal{A}_h y\| \leq \|x - y\| \xi.$$

In view of Banach's fixed point theorem, problem (3.1) has a unique solution. This ends the proof.  $\square$

*Remark 3.2.* Let  $\mathbb{T} = \mathbb{R}$ . Then  $\mu(t) = 0$ ,  $t \in J$ , so  $\lambda_0 \sup_{t \in J} \mu(t) = 0 < 1$  and

$$e_{(-\lambda_0)}(t, 0) = e^{-\lambda_0 t}, \quad \lambda_0 \geq n_0 \quad \text{with} \quad \|x\| = \max_{t \in J} e^{-\lambda_0 t} |x(t)|.$$

## 4 Existence of Solutions for Problem (1.1)

First we formulate the following result.

**Theorem 4.1 (See [1, 8]).** *Let  $X$  be a partially ordered Banach space. Assume that there exist  $x_0, y_0 \in X$ ,  $x_0 \leq y_0$ ,  $D = [x_0, y_0]$ ,  $A : D \rightarrow X$  such that*

- (i)  *$A$  is a continuous and increasing operator,*
- (ii)  *$x_0$  and  $y_0$  are the lower and upper solutions of  $x = Ax$ , respectively,*
- (iii)  *$A(D) \subset X$  is relatively compact.*

*Then  $A$  has a minimal fixed point  $x^*$  and a maximal fixed point  $y^*$  in  $[x_0, y_0]$ . Moreover  $x_n \rightarrow x^*$  and  $y_n \rightarrow y^*$ , where  $x_n = Ax_{n-1}$ ,  $y_n = Ay_{n-1}$ ,  $n = 1, 2, \dots$ , and*

$$x_0 \leq x_1 \leq \dots \leq x_n \leq \dots \leq y_n \leq \dots \leq y_1 \leq y_0.$$

In the next theorem we give sufficient conditions so that problem (1.1) has extremal solutions.

**Theorem 4.2.** *Let  $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ ,  $k \in C(J \times J, \mathbb{R})$ . Assume that there exist differentiable functions  $x_0, y_0 : J \rightarrow \mathbb{R}$ ,  $x_0(t) \leq y_0(t)$ ,  $t \in J$ , and such that they are lower and upper solutions of problem (1.1), respectively i.e.,*

$$\begin{aligned} x_0^\Delta(t) &\leq (\mathcal{F}x_0)(t), \quad t \in J, \quad g(x_0(0), x_0(T)) \leq 0, \\ y_0^\Delta(t) &\geq (\mathcal{F}y_0)(t), \quad t \in J, \quad g(y_0(0), y_0(T)) \geq 0. \end{aligned}$$

*Moreover, we assume that all assumptions of Lemma 2.4 hold with functions  $m, n$  such that*

$$f(t, x_1, x_2) - f(t, \bar{x}_1, \bar{x}_2) \geq -m(t)(x_1 - \bar{x}_1) - n(t)(x_2 - \bar{x}_2) \quad (4.1)$$

*for  $t \in J$ ,  $x_0(t) \leq \bar{x}_1 \leq x_1 \leq y_0(t)$ ,  $x_0(\alpha(t)) \leq \bar{x}_2 \leq x_2 \leq y_0(\alpha(t))$ . In addition, we assume that  $g$  is nonincreasing in the second variable and there exists a constant  $a > 0$  such that*

$$g(u, v) - g(\bar{u}, v) \geq -a(\bar{u} - u) \quad \text{if } x_0(0) \leq u \leq \bar{u} \leq y_0(0). \quad (4.2)$$

*Then problem (1.1) has minimal and maximal solutions in the region  $D = \{u \in C(J, \mathbb{R}) : x_0(t) \leq u(t) \leq y_0(t), t \in J\}$ .*

*Proof.* Let  $G(h)$  be nondecreasing with respect to  $h$ . Choose  $h_1, h_2 \in C(J, \mathbb{R})$  such that  $h_1(t) \leq h_2(t)$  on  $J$ . Let  $x_1, x_2$  denote the solutions of problem (3.1) with  $h_1, h_2$  instead of  $h$ , and with  $G(h_1), G(h_2)$  instead of  $x_0$ , respectively. Since problem (3.1) has a unique solution for each  $h \in C(J, \mathbb{R})$ , then  $x_1, x_2$  are well defined, so

$$\begin{cases} x_1^\Delta(t) &= -(\mathcal{L}x_1)(t) + h_1(t), \quad t \in J, \\ x_1(0) &= G(h_1) \in \mathbb{R}, \end{cases}$$

and

$$\begin{cases} x_2^\Delta(t) &= -(\mathcal{L}x_2)(t) + h_2(t), \quad t \in J, \\ x_2(0) &= G(h_2) \in \mathbb{R}, \end{cases}$$

where

$$(\mathcal{L}x)(t) = m(t)x(t) + n(t)x(\alpha(t)).$$

Put  $x(t) = x_1(t) - x_2(t)$ . Then

$$\begin{cases} x^\Delta(t) &= -(\mathcal{L}x)(t) + h_1(t) - h_2(t) \leq -(\mathcal{L}x)(t), \quad t \in J, \\ x(0) &= G(h_1) - G(h_2) \leq 0. \end{cases}$$

In view of Lemma 2.4,  $x_1(t) \leq x_2(t)$  on  $J$ , so the operator  $\mathcal{A}_h$  is increasing. It is also continuous. Take  $u \in D$ . Put

$$Fu = \mathcal{F}u + \mathcal{L}u, \quad G(u) = -\frac{1}{a}g(u(0), u(T)) + u(0),$$

where operator  $\mathcal{F}$  is defined as in problem (1.1). Take  $u_1, u_2 \in D$  and let  $u_1(t) \leq u_2(t)$ ,  $t \in J$ . Then, in view of conditions (4.1) and (4.2), we have

$$\begin{aligned} (Fu_1)(t) - (Fu_2)(t) &= f(t, u_1(t), u_1(\alpha(t))) - f(t, u_2(t), u_2(\alpha(t))) \\ &\quad + (\mathcal{L}(u_1 - u_2))(t) \\ &\leq (\mathcal{L}(u_2 - u_1))(t) - (\mathcal{L}(u_2 - u_1))(t) = 0, \\ G(u_1) - G(u_2) &= \frac{1}{a}[g(u_2(0), u_2(T)) - g(u_1(0), u_1(T))] + u_1(0) - u_2(0) \\ &\leq \frac{1}{a}[g(u_2(0), u_1(T)) - g(u_1(0), u_1(T))] + u_1(0) - u_2(0) \\ &\leq 0. \end{aligned}$$

It proves that  $F$  and  $G$  are increasing in  $D$ . We define the operator  $A = \mathcal{A}_F$ . Let  $x_1 = Ax_0$ ,  $x_2 = Ay_0$  so

$$\begin{cases} x_1^\Delta(t) &= -(\mathcal{L}x_1)(t) + (Fx_0)(t), \\ x_1(0) &= G(x_0), \end{cases}$$

and

$$\begin{cases} x_2^\Delta(t) &= -(\mathcal{L}x_2)(t) + (Fy_0)(t), \\ x_2(0) &= G(y_0). \end{cases}$$

Put  $x(t) = x_0(t) - x_1(t)$ . Using the definition of the lower solution  $x_0$ , we have

$$\begin{aligned} x^\Delta(t) &\leq (\mathcal{F}x_0)(t) + (\mathcal{L}x_1)(t) - (Fx_0)(t) = -(\mathcal{L}x)(t), \quad t \in J, \\ x(0) &= x_0(0) + \frac{1}{a}g(x_0(0), x_0(T)) - x_0(0) \leq 0. \end{aligned}$$



It yields,  $x_0(t) \leq x_1(t) = (Ax_0)(t)$ , by Lemma 2.4. Similarly, we can show  $(Ay_0)(t) = x_2(t) \leq y_0(t)$  on  $J$ . Now, using again Lemma 2.4 with  $x(t) = x_1(t) - x_2(t)$  and properties of  $F$  and  $G$ , we see that  $x_1(t) \leq x_2(t)$  on  $J$ , so the operator  $A$  is increasing. The same argument as in [8, Theorem 3.2] guarantees that  $A(D)$  is relatively compact. The result now follows from Theorem 4.1. This ends the proof.  $\square$

*Remark 4.3.* Let  $\mathbb{T} = \mathbb{R}$ . In this case Theorem 4.2 reduces to one which is a special case of paper [3].

## 5 Unique Solution for Problem (1.1)

To show the next result we need the following lemma.

**Lemma 5.1.** *Assume that  $b \in C(J, \mathbb{R})$ ,  $b \in \mathcal{R}_+$ , i.e.,  $1 + \sup_{t \in J} \mu(t)b(t) > 0$ . Let*

$$\begin{cases} x^\Delta(t) \geq b(t)x(t), & t \in J, \\ x(T) \leq cx(0), & c \geq 0. \end{cases} \quad (5.1)$$

*In addition, we assume that*

$$e_b(T, 0) < c. \quad (5.2)$$

*Then  $x(t) \geq 0$ ,  $t \in J$ .*

*Proof.* First, note that  $e_b(t, 0) > 0$ ,  $t \in J$ , by [2, Theorem 2.48(i)]. Now, we replace the inequality in (5.1) by the following relation:

$$x^\Delta(t) = b(t)x(t) + B(t), \quad t \in J, \quad B \in C(J, \mathbb{R}_+).$$

Hence,

$$x(t) = e_b(t, 0) \left[ x(0) + \int_0^t e_b(0, \sigma(s))B(s)\Delta s \right],$$

by [2, Theorem 2.77]. It yields

$$x(t) \geq e_b(t, 0)x(0), \quad t \in J. \quad (5.3)$$

Using to this the boundary condition  $x(T) \leq cx(0)$ , we see that

$$x(0) [c - e_b(T, 0)] \geq 0.$$

In view of condition (5.2), we obtain  $x(0) \geq 0$ . It shows that  $x(t) \geq 0$ ,  $t \in J$ , by (5.3). The proof is complete.  $\square$

**Theorem 5.2.** *Let all assumptions of Theorem 4.2 be satisfied. In addition, we assume*

(H<sub>3</sub>)  $f$  is nonincreasing in the last argument, there exists a function  $b \in C(J, \mathbb{R})$  such that  $m(t) + b(t) \geq 0$ ,  $t \in J$ ,

$$f(t, u, v) - f(t, \bar{u}, v) \geq -b(t)[\bar{u} - u] \text{ if } x_0(t) \leq u \leq \bar{u} \leq y_0(t),$$

(H<sub>4</sub>) there exists constants  $M_1, M_2$  such that  $a \geq M_1 > 0$ ,  $M_2 \geq 0$  and

$$g(u, v) - g(\bar{u}, \bar{v}) \leq -M_1(\bar{u} - u) + M_2(\bar{v} - v)$$

$$\text{if } x_0(0) \leq u \leq \bar{u} \leq y_0(0), x_0(T) \leq v \leq \bar{v} \leq y_0(T),$$

(H<sub>5</sub>)  $M_2 e_b(T, 0) < M_1$ .

Then problem (1.1) has, in the region  $D$ , a unique solution.

*Proof.* Theorem 4.2 guarantees that problem (1.1) has extremal solutions in  $D$ . By  $x$  we denote the minimal solution of (1.1), and by  $y$  the maximal solution of (1.1). Indeed,  $x \leq y$ . To show that  $x = y$ , we put  $p = x - y$ , so  $p(t) \leq 0$ ,  $t \in J$ . Then

$$\begin{aligned} p^\Delta(t) &= f(t, x(t), x(\alpha(t))) - f(t, y(t), y(\alpha(t))) \\ &\geq f(t, x(t), y(\alpha(t))) - f(t, y(t), y(\alpha(t))) \geq b(t)p(t), \\ 0 &= g(x(0), x(T)) - g(y(0), y(T)) \leq M_1 p(0) - M_2 p(T). \end{aligned}$$

It means that

$$\begin{cases} p^\Delta(t) \geq b(t)p(t), & t \in J, \\ p(T) \leq \frac{M_1}{M_2} p(0). \end{cases}$$

In view of Lemma 5.1,  $x(t) \geq y(t)$ ,  $t \in J$ . It proves that  $x = y$ , so problem (1.1) has a unique solution. This ends the proof.  $\square$

**Example 5.3 (See [5]).** Let  $\mathbb{T} = \mathbb{R}$ . Consider the problem

$$\begin{cases} x'(t) = \beta(\sin t)x(t) - 2\beta(\sin t)x(\alpha(t)) - \beta \sin t \equiv (\mathcal{F}x)(t), & t \in J, \\ 0 = x(0) - e^{-1}x(\pi), \end{cases} \quad (5.4)$$

where  $J = [0, \pi]$ ,  $0 \leq \beta \leq \frac{1}{4}$ ,  $\alpha \in C(J, J)$ ,  $\alpha(t) \leq t$  on  $J$ . Put  $x_0(t) = -1$ ,  $y_0(t) = 0$ ,  $t \in J$ . Then

$$\begin{aligned} (\mathcal{F}x_0)(t) &= 0 = x_0'(t), & (\mathcal{F}y_0)(t) &= -\beta \sin t \leq 0 = y_0'(t), \\ g(x_0(0), x_0(\pi)) &= g(-1, -1) < 0, & g(y_0(0), y_0(\pi)) &= g(0, 0) = 0. \end{aligned}$$

It proves that  $x_0, y_0$  are lower and upper solutions of problem (5.4), respectively. Note that  $m(t) = 0$ ,  $n(t) = 2\beta \sin t$ ,  $a = 1$  and

$$\int_0^T n(t) e^{\int_{\alpha(t)}^t m(s) ds} dt = 4\beta \leq 1,$$

so assumption (H<sub>2</sub>) holds. Assumption (H<sub>4</sub>) holds with  $M_1 = 1$ ,  $M_2 = e^{-1}$ . Moreover  $b(t) = \beta \sin t$ , and

$$M_2 e^{\int_0^T b(s) ds} = e^{2\beta-1} < 1.$$

By Theorem 5.2, problem (5.4) has, in the region  $D$ , a unique solution.

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