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Symplectic Structure of Jacobi Systems on Time Scales

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Abstract

In this paper we study the structure of the Jacobi system for optimal control problems on time scales. Under natural and minimal invertibility assumptions on the coefficients we prove that the Jacobi system is a time scale symplectic system and not necessarily a Hamiltonian system. These new invertibility conditions are weaker than those considered in the current literature. This shows that the theory of time scale symplectic systems, rather than the theory of linear Hamiltonian systems, is fundamental for optimal control problems. Our results in this paper are new even for the Jacobi equations arising in the time scale calculus of variation and, in particular, for the discrete time calculus of variations and optimal control problems. We also show that nonlinear time scale Hamiltonian systems possess symplectic structure, that is, the Jacobian of the evolution mapping satisfies a time scale symplectic system.

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1 Introduction

Let $[a,b]_{\mathbb{T}}$ be a bounded time scale, i.e., a bounded nonempty closed subset of $\mathbb{R}$, see [11]. Consider the nonlinear time scale optimal control problem

$$\text{minimize} \quad \mathcal{F}(x,u) := K\big(x(a),x(b)\big) + \int_a^b F(t,x^\sigma,u)\,\Delta t, \tag{C$^\sigma$}$$

over feasible pairs (x,u), i.e., the state $x:[a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is piecewise rd-continuously Δ-differentiable ($\mathrm{C}^1_{\mathrm{prd}}$), the control $u:[a,\rho(b)]_{\mathbb{T}} \to \mathbb{R}^m$, $m \le n$, is piecewise rd-continuous ($\mathrm{C}_{\mathrm{prd}}$), and (x,u) satisfies the equation of motion

$$x^\Delta = f(t,x^\sigma,u), \quad t \in [a,\rho(b)]_{\mathbb{T}}, \tag{1.1}$$

and the state endpoints constraints

$$\varphi\big(x(a),x(b)\big) = 0. \tag{1.2}$$

Here we assume that the data satisfy

$$\begin{aligned} &K:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}, && F:[a,\rho(b)]_{\mathbb{T}}\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R},\\ &\varphi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^r,\ r\le 2n, && f:[a,\rho(b)]_{\mathbb{T}}\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n, \end{aligned} \tag{1.3}$$

and certain differentiability assumptions so that we can work with the first and second variations of the functional $\mathcal{F}$, see the details in [35]. The forward shift in the notation (C^σ) refers to presence of the forward shift in the state variable x in the argument of the objective functional $\mathcal{F}$ and in the equation of motion (1.1). Later we will consider a control problem without this shift in x.

The subject of this paper is to study the structure of the *Jacobi system* arising from the control problem (C^σ). The Jacobi system is here defined as the system obtained from the weak Pontryagin maximum principle [35, Theorem 6.1] applied to the second variation (the accessory problem) of the functional $\mathcal{F}$. More precisely, the Jacobi system for problem (C^σ) takes the form

$$\eta^\Delta = \mathcal{A}\eta^\sigma + \mathcal{B}v, \quad q^\Delta = -\mathcal{A}^T q + P\eta^\sigma + Qv, \quad -\mathcal{B}^T q + Q^T\eta^\sigma + Rv = 0, \tag{J$^\sigma$}$$

where η and v are the variations of the state and control variables x and u, and where the coefficients are defined through the data of the problem (C^σ), see [36, Section 4.1]. The main question in this paper is: Under what conditions on the coefficients the Jacobi system (J^σ) has a *symplectic structure*? That is when it corresponds to a time scale symplectic system

$$\eta^\Delta = \mathbb{A}\,\eta + \mathbb{B}\,q, \quad q^\Delta = \mathbb{C}\,\eta + \mathbb{D}\,q, \tag{$\mathbb{S}$}$$

whose coefficients are $n\times n$ matrices satisfying

$$\mathbb{S}^T\mathcal{J} + \mathcal{J}\,\mathbb{S} + \mu\,\mathbb{S}^T\mathcal{J}\,\mathbb{S} = 0, \quad \mathbb{S} := \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{pmatrix}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{1.4}$$

Systems of the form ($\mathbb{S}$) were introduced in [13]. Since then many interesting results about them can be found in the literature, see for example [3] and the recent papers [30, 32, 34, 37], which suggest that the theory of time scale symplectic systems *may be* important for the study second order optimality conditions for nonlinear control problems. In addition, it is well known (see Proposition 2.5) that if the matrices $I-\mu\mathcal{A}$, R, and $T := I + \mu\tilde{\mathcal{A}}\mathcal{B}R^{-1}Q^T$ are invertible with $\tilde{\mathcal{A}} := (I-\mu\mathcal{A})^{-1}$, then the Jacobi system (J$^\sigma$) can be written as the *linear Hamiltonian system*

$$\eta^\Delta = A\eta^\sigma + Bq, \quad q^\Delta = C\eta^\sigma - A^T q, \tag{H$^\sigma$}$$

where B and C are symmetric and $I-\mu A$ is invertible, see [36, Section 4.1] and also [24, Section 4], [44, Section 6], [39, pg. 304]. Furthermore, as it was observed in [23, Remark 4] (see Proposition 2.6), every linear Hamiltonian system (H$^\sigma$) with $I-\mu A$ invertible can be written as the time scale symplectic system ($\mathbb{S}$), in which the matrix $I+\mu\mathbb{A}$ is invertible. This relationship between the systems (H$^\sigma$) and ($\mathbb{S}$) is mutual whenever the matrix $I-\mu A$ or $I+\mu\mathbb{A}$ is invertible, see [13, Remark 4]. Therefore, whenever the matrices $I-\mu\mathcal{A}$, R, and T are invertible, the Jacobi system (J$^\sigma$) produces a time scale symplectic system ($\mathbb{S}$), see [43] or Proposition 2.8 for the details.

The invertibility of $I-\mu\mathcal{A}$ will be needed in all calculations related to the Jacobi system (J$^\sigma$), and hence it is assumed to hold. In this paper we discuss the requirement that R is invertible. More precisely, we shall show that if R is invertible, then we have T invertible if and only if the matrix $S := R + \mu\, Q^T\tilde{\mathcal{A}}\mathcal{B}$ is invertible (Corollary 3.3). Therefore, the invertibility of R and S (or T) yields that (J$^\sigma$) is in fact a Hamiltonian system (H$^\sigma$) and hence, it is a symplectic system ($\mathbb{S}$).

At this stage it appears that the usage of the *symplectic systems theory* for the control setting is *superfluous*, because the systems (J$^\sigma$) and (H$^\sigma$) and ($\mathbb{S}$) are equivalent under R and T invertible, and hence the simpler form (H$^\sigma$) is more attractive than that of ($\mathbb{S}$). In the *discrete* calculus of variations and control problems it is assumed that R_k is invertible in order to transfer from the Jacobi system (J$^\sigma$) to a linear Hamiltonian system (H$^\sigma$), which is known to enjoy rich properties, see [2, 4–6, 24, 25, 27, 29, 38]. As is already noted in [29, pg. 863], in the *discrete* calculus of variations setting one can study the definiteness of the corresponding quadratic form in terms of the Jacobi equation (J$^\sigma$), i.e.,

$$\Delta\big(R_k\Delta\eta_k + Q_k^T\eta_{k+1}\big) = P_k\eta_{k+1} + Q_k\Delta\eta_k, \tag{1.5}$$

without the assumption that R_k is invertible. This involves the theory of three-term recurrence relations, see [6, 25, 27]. Therefore in this setting the invertibility of R_k is an unnatural assumption.

Inspired by this situation and by the fact that the strengthened Legendre condition

$$R(t^\pm) \geq \alpha I, \quad \text{for all dense points } t \in [\sigma(a), \rho(b)]_{\mathbb{T}}, \tag{1.6}$$

for some $\alpha > 0$, does not involve $R(t)$ invertible for isolated points (which is the setting of discrete optimal control), it is an important question, when R is not invertible, whether (J^σ) can be written directly as a symplectic system ($\mathbb{S}$) by avoiding passing through the Hamiltonian system (H^σ) that imposes the invertibility of R.

The aim of this paper is to show a *direct way* to go from the Jacobi system (J^σ) to the symplectic system ($\mathbb{S}$) without assuming R invertible, but instead we assume that the matrix S is invertible (see Theorem 3.1). This appears to be much more natural, since only the invertibility of S is needed to get a time scale symplectic system. As a consequence of this important conclusion we deduce that

the theory of symplectic systems is crucial to the control problems

and that Hamiltonian systems of the form (H^σ) can be disregarded. A direct impact of this fact is viewed in the calculus of variations setting, where $\mathcal{A} = 0$ and $\mathcal{B} = I$ (with $m = n$) and where the corresponding Jacobi system (J^σ) is

$$[R\eta^\Delta + Q^T\eta^\sigma]^\Delta = P\eta^\sigma + Q\eta^\Delta. \qquad (J^\sigma_{\text{cov}})$$

In this case, if $S := R + \mu\, Q^T$ is invertible, then the systems (J^σ_{cov}) and ($\mathbb{S}$) are equivalent (without being Hamiltonian), with (J^σ_{cov}) and (H^σ) equivalent if and only if R is invertible. Therefore, all the results in [31, 33] regarding quadratic functionals associated with the Jacobi system (J^σ_{cov}) that required R invertible (and which were obtained through the time scale symplectic systems theory) now hold without this assumption (see Remark 5.5).

The fact that Jacobi systems for control problems on time scales are symplectic systems (in particular, discrete Jacobi systems are symplectic) is important for their numerical analysis. Symplectic numerical methods are known to be the right ones for solving Hamiltonian systems, see [18–21, 42]. These methods can now be possibly used for solving the Jacobi systems arising from control problems such as (J^σ).

With each of the Jacobi system (J^σ), the Hamiltonian system (H^σ), and symplectic system ($\mathbb{S}$), we consider the associated quadratic form. These quadratic forms are obtained from the second variation of the nonlinear time scale control problem (C^σ). The second aim of this paper is show that the transition between the systems (J^σ), (H^σ), and ($\mathbb{S}$) yields the same transition between the corresponding quadratic forms.

In [35, Section 3] we discuss different formulations of time scale optimal control problems. In particular, we consider a problem *without shift* in the state variable, that is, with the argument (t, x, u) in the objective functional and in the equation of motion. The same type of results stated above will be shown to hold for the Jacobi and time scale symplectic systems corresponding to these problems (see Theorem 4.8).

The final results in this paper concern *nonlinear Hamiltonian systems*. We show that these nonlinear Hamiltonian systems also possess a symplectic structure, namely the Jacobian of the evolution mapping satisfies the time scale symplectic system ($\mathbb{S}$). This is a generalization of the corresponding discrete time result in [40, Theorem 2.1].

Although we assume that all quantities in this paper are real-valued, the results remain unchanged provided we replace the transpose of a vector or matrix by the conjugate transpose, and replace the word “symmetric” by “Hermitian”.

The paper is organized as follows. For a better understanding of and comparison with the new results of this paper, we present in Section 2 the details of the known transformations between the systems (J^σ), (H^σ), and ($\mathbb{S}$) and between their corresponding quadratic forms. In Section 3 we show how to transform the Jacobi system (J^σ) directly to the symplectic system ($\mathbb{S}$). In Section 4 we discuss the parallel results for the Jacobi system arising from the nonlinear control problem without the shift in x. In Section 5 we specify these new transformations to the calculus of variations setting (with and without shift in the state variable) and compare these new results with the known transformations in the literature.

2 Traditional Approach to Jacobi Systems

In this section we examine in detail the traditional approach to Jacobi systems (J^σ), (H^σ), and ($\mathbb{S}$). That is, we consider the system (J^σ) as a symplectic system ($\mathbb{S}$) via (H^σ). We fix the following notation as for the dimensions of the quantities appearing in the Jacobi system (J^σ).

Notation 2.1 *(Jacobi system* (J^σ)*).* The matrices $\mathcal{A}$, $\mathcal{B}$, P, Q, R and the vectors η, v, q in (J^σ) have the following properties: $\mathcal{A}, P \in \mathbb{R}^{n\times n}$, $\mathcal{B}, Q \in \mathbb{R}^{n\times m}$, $R \in \mathbb{R}^{m\times m}$, and these matrices are defined on $[a, \rho(b)]_{\mathbb{T}}$ and P and R are symmetric; $\eta, q \in \mathbb{R}^n$ are defined on $[a, b]_{\mathbb{T}}$ and $v \in \mathbb{R}^m$ is defined on $[a, \rho(b)]_{\mathbb{T}}$. Furthermore, we assume that the matrix $I - \mu\mathcal{A}$ is invertible and define the matrices $\tilde{\mathcal{A}} \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{m\times m}$ by

$$\tilde{\mathcal{A}} := (I - \mu\mathcal{A})^{-1}, \quad S := R + \mu\, Q^T\tilde{\mathcal{A}}\mathcal{B}. \tag{2.1}$$

In addition, whenever the matrix R is invertible and whenever the matrix S is invertible, we define the matrices $T, V \in \mathbb{R}^{n\times n}$ by

$$T := I + \mu\tilde{\mathcal{A}}\mathcal{B}R^{-1}Q^T, \quad V := I - \mu\tilde{\mathcal{A}}\mathcal{B}S^{-1}Q^T. \tag{2.2}$$

Note that $\tilde{\mathcal{A}}\mathcal{A} = \mathcal{A}\tilde{\mathcal{A}}$. Similarly, for the coefficients of the Hamiltonian system (H^σ) we adopt the following.

Notation 2.2 *(Hamiltonian system* (H^σ)*).* The matrices A, B, C and the vectors η, q in (H^σ) have the following properties: $A, B, C \in \mathbb{R}^{n\times n}$ are defined on $[a, \rho(b)]_{\mathbb{T}}$ and B and C are symmetric; $\eta, q \in \mathbb{R}^n$ are defined on $[a, b]_{\mathbb{T}}$. Furthermore, we assume that the matrix $I - \mu A$ is invertible and define the matrix $\tilde{A} \in \mathbb{R}^{n\times n}$ by

$$\tilde{A} := (I - \mu A)^{-1}.$$

Note that $\tilde{A}A = A\tilde{A}$. Finally, for the symplectic system ($\mathbb{S}$) we will use the following.

Notation 2.3 *(Symplectic system* ($\mathbb{S}$)*).* The matrices $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ and the vectors η, q in ($\mathbb{S}$) have the following properties: $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D} \in \mathbb{R}^{n\times n}$ are defined on $[a, \rho(b)]_{\mathbb{T}}$ and satisfy (1.4); $\eta, q \in \mathbb{R}^n$ are defined on $[a, b]_{\mathbb{T}}$.

The invertibility of $I - \mu\mathcal{A}$ in Notation 2.1 corresponds to the regressivity of the first two equations in the Jacobi system (J$^\sigma$), which is needed for the unique solvability of dynamic equations on time scales, see e.g. [11, Theorem 2.62]. Similarly, the invertibility of $I - \mu A$ in Notation 2.2 corresponds to the regressivity of the Hamiltonian system (H$^\sigma$). The smoothness assumption on the coefficients, such that the piecewise rd-continuity, is not really essential for the results in this paper. Observe that the matrix $\tilde{\mathcal{A}}$ resp. $\tilde{A}$ is piecewise rd-continuous if $\mathcal{A}$ resp. A is piecewise rd-continuous, just because the limits of $I - \mu\mathcal{A}$ resp. $I - \mu A$ are equal to I at all dense points (i.e., the limits are bounded away from zero so that the inverse $(I - \mu\mathcal{A})^{-1}$ resp. $(I - \mu A)^{-1}$ has finite limits at all dense points). The same argument reveals that if S is piecewise rd-continuous (i.e., if $R, Q, \mathcal{A}, \mathcal{B}$ have this property), then the matrix S^{-1} is piecewise rd-continuous if and only if the eigenvalues of $R(t^\pm)$ are bounded away from zero. This is satisfied for example if the strengthened Legendre condition (1.6) holds. Moreover, under (1.6), it follows that the matrices T^{-1} and V^{-1} are also piecewise rd-continuous. This shows the importance of the time scale strengthened Legendre condition (1.6) for the *control problems* as was the case for the time scale calculus of variations problems, see [31, 33, 41]. Therefore, we just proved the following statement.

Corollary 2.4. *Assume that $\mathcal{A}$, $\mathcal{B}$, P, Q, R are piecewise rd-continuous on $[a, \rho(b)]_{\mathbb{T}}$ and satisfy the conditions in Notation 2.1 and the strengthened Legendre condition* (1.6). *Then the matrices $\tilde{\mathcal{A}}$, S^{-1}, T^{-1}, V^{-1} are also piecewise rd-continuous on $[a, \rho(b)]_{\mathbb{T}}$, whenever they exist.*

The identity in (1.4) reduces to the conditions

$$\left.\begin{array}{l} \mathbb{C}^T(I + \mu\mathbb{A}) \;\text{ and }\; (I + \mu\mathbb{D}^T)\,\mathbb{B} \;\text{ are symmetric,} \\ \text{and }\; \mathbb{A}^T + \mathbb{D} + \mu\,(\mathbb{A}^T\mathbb{D} - \mathbb{C}^T\,\mathbb{B}) = 0 \end{array}\right\} \tag{2.3}$$

in terms of the coefficients of the system ($\mathbb{S}$). We will usually use the above conditions in order to verify that a system satisfies (1.4). Note that the identity in (1.4) implies that $I + \mu\,\mathbb{S}$ is a symplectic matrix, i.e., $(I + \mu\,\mathbb{S}^T)\,\mathcal{J}\,(I + \mu\,\mathbb{S}) = \mathcal{J}$. Hence, the matrix $I + \mu\,\mathbb{S}$ is invertible, which means in the time scale terminology that the matrix $\mathbb{S}$ is regressive. Therefore, if the coefficients $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ are piecewise rd-continuous (and satisfy (1.4)), then the initial value problems associated with system ($\mathbb{S}$) have unique solutions defined on the whole interval $[a, b]_{\mathbb{T}}$ starting at any initial point $t_0 \in [a, b]_{\mathbb{T}}$ with any initial value. Equivalent conditions to the ones in (2.3) are derived in [12, Remark 10.1], namely

$$\left.\begin{array}{l} (I + \mu\mathbb{A})\,\mathbb{B}^T \;\text{ and }\; (I + \mu\mathbb{D})\,\mathbb{C}^T \;\text{ are symmetric,} \\ \text{and }\; \mathbb{A}^T + \mathbb{D} + \mu\,(\mathbb{D}\,\mathbb{A}^T - \mathbb{C}\,\mathbb{B}^T) = 0. \end{array}\right\} \tag{2.4}$$

These conditions imply that $I + \mu\,\mathbb{S}^T$ is a symplectic matrix, which is known to be equivalent to $I + \mu\,\mathbb{S}$ being symplectic.

For easy comparison with the main results of this paper we recall now the known transformations between the systems (J$^\sigma$), (H$^\sigma$), and ($\mathbb{S}$). First we present a statement when the Jacobi system (J$^\sigma$) becomes the Hamiltonian system (H$^\sigma$).

Proposition 2.5 (Jacobi (J$^\sigma$) **to Hamiltonian** (H$^\sigma$)**).** *Assume that $\mathcal{A}$, $\mathcal{B}$, P, Q, R, T satisfy the conditions in Notation 2.1 with R and T invertible. Then the Jacobi system* (J$^\sigma$) *is the Hamiltonian system* (H$^\sigma$)*, whose coefficients*

$$A = \mathcal{A} - \mathcal{B}R^{-1}Q^T, \quad B = \mathcal{B}R^{-1}\mathcal{B}^T, \quad C = P - QR^{-1}Q^T \tag{2.5}$$

with $\tilde{A} = T^{-1}\tilde{\mathcal{A}}$ satisfy the conditions in Notation 2.2.

Proof. Since the matrix R is assumed to be invertible, we can solve the third equation in (J$^\sigma$) for v, i.e., $v = R^{-1}(\mathcal{B}^T q - Q^T \eta^\sigma)$, and insert this formula into the first and second equation in (J$^\sigma$). See also [36, Formula (40)]. □

Next we display a statement when the Hamiltonian system (H$^\sigma$) becomes the symplectic system ($\mathbb{S}$).

Proposition 2.6 (Hamiltonian (H$^\sigma$) **to symplectic** ($\mathbb{S}$)**).** *Assume that A, B, C satisfy the conditions in Notation 2.2. Then the Hamiltonian system* (H$^\sigma$) *is the time scale symplectic system* ($\mathbb{S}$)*, whose coefficients*

$$\mathbb{A} = \tilde{A}A, \quad \mathbb{B} = \tilde{A}B, \quad \mathbb{C} = C\tilde{A}, \quad \mathbb{D} = \mu C\tilde{A}B - A^T \tag{2.6}$$

with $I + \mu\mathbb{A} = \tilde{A}$ satisfy the conditions in Notation 2.3.

Proof. See [23, Remark 4(ii)] or [13, Example 3]. □

For completeness we present a converse to Proposition 2.6, i.e., a statement when a time scale symplectic system ($\mathbb{S}$) is actually a Hamiltonian system (H$^\sigma$).

Proposition 2.7 (Symplectic ($\mathbb{S}$) **to Hamiltonian** (H$^\sigma$)**).** *Assume that $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ satisfy the conditions in Notation 2.3 with $I + \mu\mathbb{A}$ invertible. Then the time scale symplectic system* ($\mathbb{S}$) *is the Hamiltonian system* (H$^\sigma$)*, whose coefficients*

$$A = (I + \mu\mathbb{A})^{-1}\mathbb{A}, \quad B = (I + \mu\mathbb{A})^{-1}\,\mathbb{B}, \quad C = \mathbb{C}\,(I + \mu\mathbb{A})^{-1} \tag{2.7}$$

with $\tilde{A} = I + \mu\mathbb{A}$ satisfy the conditions in Notation 2.2.

Proof. See [13, Remark 4] or [11, Theorem 7.17]. Note that the coefficient identities in (2.4) are used in these calculations. □

A combination of the previous results yields the following.

Proposition 2.8 (Jacobi (J$^\sigma$) **to symplectic** ($\mathbb{S}$)**).** *Assume that $\mathcal{A}$, $\mathcal{B}$, P, Q, R, T satisfy the conditions in Notation 2.1 with R and T invertible. Then the Jacobi system* (J$^\sigma$) *is the symplectic system* ($\mathbb{S}$)*, whose coefficients*

$$\left.\begin{aligned}
&\mathbb{A} = T^{-1}\tilde{\mathcal{A}}(\mathcal{A} - \mathcal{B}R^{-1}Q^T),\\
&\mathbb{B} = T^{-1}\tilde{\mathcal{A}}\mathcal{B}R^{-1}\mathcal{B}^T,\\
&\mathbb{C} = (P - QR^{-1}Q^T)\,T^{-1}\tilde{\mathcal{A}},\\
&\mathbb{D} = \mu(P - QR^{-1}Q^T)\,T^{-1}\tilde{\mathcal{A}}\mathcal{B}R^{-1}\mathcal{B}^T - \mathcal{A}^T + QR^{-1}\mathcal{B}^T
\end{aligned}\right\} \tag{2.8}$$

with $I + \mu\mathbb{A} = T^{-1}\tilde{\mathcal{A}}$ satisfy the conditions in Notation 2.3.

Proof. It is a simple combination of Propositions 2.5 and 2.6. □

Remark 2.9. We will see in Corollary 3.3 that the matrix T^{-1} appearing at several places in Propositions 2.5 and 2.8 is equal to the matrix V.

Next we consider the quadratic forms associated with the Jacobi systems (J^σ), (H^σ), and ($\mathbb{S}$). Let P, Q, R satisfy the conditions in Notation 2.1. We say that a pair of functions (η, v) with $\eta(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^m$ is $(\mathcal{A}, \mathcal{B})$-admissible if $\eta \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a,b]_{\mathbb{T}}$, $v \in \mathrm{C}_{\mathrm{prd}}$ on $[a,\rho(b)]_{\mathbb{T}}$, and $\eta^\Delta = \mathcal{A}\eta^\sigma + \mathcal{B}v$. For such $(\mathcal{A}, \mathcal{B})$-admissible pairs (η, v) we define the quadratic form

$$\omega(\eta, v) := (\eta^\sigma)^T P \eta^\sigma + 2\,(\eta^\sigma)^T Q v + v^T R v.$$

Similarly, if A, B, C satisfy the conditions in Notation 2.2, then a pair of functions (η, q) with $\eta(t), q(t) \in \mathbb{R}^n$ is (A, B)-admissible if $\eta \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a,b]_{\mathbb{T}}$, $q \in \mathrm{C}_{\mathrm{prd}}$ on $[a,\rho(b)]_{\mathbb{T}}$, and $\eta^\Delta = A\eta^\sigma + Bq$. For such (A, B)-admissible pairs (η, q) we define the quadratic form

$$\Omega(\eta, q) := (\eta^\sigma)^T C \eta^\sigma + q^T B q.$$

Finally, if $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ satisfy the identity in (1.4), then a pair of functions (η, q) with $\eta, q \in \mathbb{R}^n$ is called $(\mathbb{A}, \mathbb{B})$-admissible if $\eta \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a,b]_{\mathbb{T}}$, $q \in \mathrm{C}_{\mathrm{prd}}$ on $[a,\rho(b)]_{\mathbb{T}}$, and $\eta^\Delta = \mathbb{A}\eta + \mathbb{B}q$. For such $(\mathbb{A}, \mathbb{B})$-admissible pairs (η, q) we define the quadratic form

$$\mathbb{Q}(\eta, q) := \eta^T \mathbb{C}^T (I + \mu\mathbb{A})\,\eta + 2\mu\,\eta^T \mathbb{C}^T \mathbb{B}\,q + q^T (I + \mu\mathbb{D}^T)\,\mathbb{B}\,q. \tag{2.9}$$

The systems (J^σ), (H^σ), ($\mathbb{S}$) are the Euler–Lagrange systems for the quadratic functionals having respectively $\omega(\eta, v)$, $\Omega(\eta, q)$, $\mathbb{Q}(\eta, q)$ as their integrand, which can be verified by the application of the weak Pontryagin maximum principle [35, Theorem 6.1] to these quadratic functionals, see also [36, Section 4]. The following results show that the relationship between these quadratic forms is exactly the same as the relationship between their corresponding Jacobi systems.

Proposition 2.10 (Quadratic forms for (J^σ) and (H^σ)). *Assume that $\mathcal{A}$, $\mathcal{B}$, P, Q, R satisfy the conditions in Notation 2.1 with R invertible. Let A, B, C be given by* (2.5). *If (η, q) is (A, B)-admissible, then the pair (η, v) with $v := R^{-1}(\mathcal{B}^T q - Q^T \eta^\sigma)$ is $(\mathcal{A}, \mathcal{B})$-admissible and $\omega(\eta, v) = \Omega(\eta, q)$. Conversely, if (η, v) is $(\mathcal{A}, \mathcal{B})$-admissible and if $\mathcal{B}^T\mathcal{B}$ is invertible, then the pair (η, q) with $q := \mathcal{B}(\mathcal{B}^T\mathcal{B})^{-1}(Rv + Q^T\eta^\sigma)$ is (A, B)-admissible and $\Omega(\eta, q) = \omega(\eta, v)$.*

Proof. The proof is analogous to the discrete case in [24, Lemma 2]. □

Proposition 2.11 (Quadratic forms for (H^σ) and ($\mathbb{S}$)). *Assume that*

(i) *either A, B, C satisfy the conditions in Notation 2.2 and $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ are given by* (2.6),

(ii) *or* $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ *satisfy the conditions in Notation 2.3 with* $I+\mu\mathbb{A}$ *invertible and* A, B, C *are given by* (2.7).

Then (η, q) *is* (A, B)*-admissible if and only if it is* $(\mathbb{A}, \mathbb{B})$*-admissible, and in this case* $\Omega(\eta, q) = \mathbb{Q}(\eta, q)$.

Proof. This is a direct calculation. □

Proposition 2.12 (Quadratic forms for (J^σ) **and** ($\mathbb{S}$)**).** *Assume that* $\mathcal{A}$, $\mathcal{B}$, P, Q, R, T *satisfy the conditions in Notation 2.1 with* R *and* T *invertible. Let* $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ *be given by* (2.8)*. If* (η, q) *is* $(\mathbb{A}, \mathbb{B})$*-admissible, then the pair* (η, v) *with* $v := R^{-1}(\mathcal{B}^T q - Q^T \eta^\sigma)$ *is* $(\mathcal{A}, \mathcal{B})$*-admissible and* $\omega(\eta, v) = \mathbb{Q}(\eta, q)$*. Conversely, if* (η, v) *is* $(\mathcal{A}, \mathcal{B})$*-admissible and if* $\mathcal{B}^T\mathcal{B}$ *is invertible, then the pair* (η, q) *with* $q := \mathcal{B}(\mathcal{B}^T\mathcal{B})^{-1}(Rv + Q^T\eta^\sigma)$ *is* $(\mathbb{A}, \mathbb{B})$*-admissible and* $\mathbb{Q}(\eta, q) = \omega(\eta, v)$.

Proof. This is a combination of Propositions 2.10 and 2.11(i). □

3 Main Results

In this section we establish one of the main results of this paper, namely how to transform the Jacobi system (J^σ) directly to the symplectic system ($\mathbb{S}$) without assuming R and T invertible (as in Proposition 2.8), but instead we assume that the matrix S is invertible. This yields that the matrix $I+\mu\mathbb{A}$ in the resulting system ($\mathbb{S}$) is not necessarily invertible, so that the system (J^σ) is not necessarily equivalent to the system (H^σ) as it explained in Propositions 2.6 and 2.7.

Theorem 3.1 (Jacobi (J^σ) **to symplectic** ($\mathbb{S}$)**).** *Assume that* $\mathcal{A}$, $\mathcal{B}$, P, Q, R, S, V *satisfy the conditions in Notation 2.1 with* S *invertible. Then the Jacobi system* (J^σ) *is the symplectic system* ($\mathbb{S}$)*, whose coefficients*

$$\left.\begin{array}{ll} \mathbb{A} = (\mathcal{A} - \tilde{\mathcal{A}}\mathcal{B}S^{-1}Q^T)\,\tilde{\mathcal{A}}, & \mathbb{C} = (PV - QS^{-1}Q^T)\,\tilde{\mathcal{A}}, \\ \mathbb{B} = \tilde{\mathcal{A}}\mathcal{B}S^{-1}\mathcal{B}^T, & \mathbb{D} = (Q + \mu P\tilde{\mathcal{A}}\mathcal{B})\,S^{-1}\mathcal{B}^T - \mathcal{A}^T \end{array}\right\} \tag{3.1}$$

with $I+\mu\mathbb{A} = V\tilde{\mathcal{A}}$ *satisfy the conditions in Notation 2.3. Thus, the resulting symplectic system* ($\mathbb{S}$) *is Hamiltonian if and only if the matrix* V *(or equivalently* R*) is invertible.*

The proof of Theorem 3.1 is displayed later in this section, since it needs deriving of some matrix inversion identities involving the matrix S. One of these identities is from [22], see also [8, Corollary 2.8.8], and it says the following. Given any matrices A, B, C, D such that the products below are defined, then the invertibility of A, D, and $D - CA^{-1}B$ implies the invertibility of $A - BD^{-1}C$ with the formula for its inverse

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B\,(D - CA^{-1}B)^{-1}CA^{-1}. \tag{3.2}$$

The next result shows the exact relation between the two sets of invertibility assumptions (R and T invertible in Proposition 2.8, and S invertible in Theorem 3.1). In particular, we deduce that S invertible is weaker than R and T invertible.

Lemma 3.2. *Assume that $\mathcal{A}$, $\mathcal{B}$, P, Q, R, S, T, V satisfy the conditions in Notation 2.1. Then the following statements are equivalent:*

(i) R *and* T *are invertible,*

(ii) R *and* S *are invertible,*

(iii) S *and* V *are invertible.*

Proof. As we shall see, all these invertibility conditions will be proven via the inversion formula (3.2). First we will prove the following implications:

$$R \text{ and } T \text{ invertible} \implies S \text{ invertible}, \tag{3.3}$$
$$R \text{ and } S \text{ invertible} \implies V \text{ and } T \text{ invertible}, \tag{3.4}$$
$$S \text{ and } V \text{ invertible} \implies R \text{ invertible}. \tag{3.5}$$

For (3.3), we set $A := R$, $B := -\mu Q^T$, $C := \mathcal{B}$, $D := \tilde{\mathcal{A}}^{-1}$. Then the matrices A, D, and $D - CA^{-1}B = \tilde{\mathcal{A}}^{-1} + \mu \mathcal{B} R^{-1} Q^T = \tilde{\mathcal{A}}^{-1} T$ are invertible, so that the inversion formula (3.2) yields the invertibility of the matrix $A - BD^{-1}C = R + \mu Q^T \tilde{\mathcal{A}} \mathcal{B} = S$ with the inverse

$$S^{-1} = R^{-1} - \mu R^{-1} Q^T T^{-1} \tilde{\mathcal{A}} \mathcal{B} R^{-1} = R^{-1} (R - \mu Q^T T^{-1} \tilde{\mathcal{A}} \mathcal{B})\, R^{-1}. \tag{3.6}$$

Note that from (3.6) we also gain for free the invertibility of the matrix $R - \mu Q^T T^{-1} \tilde{\mathcal{A}} \mathcal{B}$, but this fact will not be needed. To prove (3.4), we set $A := I$, $B := \mu \tilde{\mathcal{A}} \mathcal{B}$, $C := Q^T$, $D := S$. Then the matrices A, D, and $D - CA^{-1}B = R$ are invertible, so that formula (3.2) yields the invertibility of the matrix $A - BD^{-1}C = I - \mu \tilde{\mathcal{A}} \mathcal{B} S^{-1} Q^T = V$ with the inverse

$$V^{-1} = I + \mu \tilde{\mathcal{A}} \mathcal{B} R^{-1} Q^T = T. \tag{3.7}$$

This also implies that T is invertible. To prove (3.5), we set $A := S$, $B := Q^T$, $C := \mu \tilde{\mathcal{A}} \mathcal{B}$, $D := I$. Then the matrices A, D, and $D - CA^{-1}B = I - \mu \tilde{\mathcal{A}} \mathcal{B} S^{-1} Q^T = V$ are invertible, so that the inversion formula (3.2) yields the invertibility of $A - BD^{-1}C = S - \mu Q^T \tilde{\mathcal{A}} \mathcal{B} = R$ with the inverse

$$R^{-1} = S^{-1} + \mu S^{-1} Q^T V^{-1} \tilde{\mathcal{A}} \mathcal{B} S^{-1} = S^{-1} (S + \mu Q^T V^{-1} \tilde{\mathcal{A}} \mathcal{B})\, S^{-1}. \tag{3.8}$$

Note that from (3.8) we also get the invertibility of $S + \mu Q^T V^{-1} \tilde{\mathcal{A}} \mathcal{B}$, but this fact will not be needed.

Now suppose that condition (i) holds. Then (3.3) yields that S is invertible, so that condition (ii) holds. Next, assuming (ii), we have from (3.4) that V is invertible, i.e., condition (iii) holds. Finally, if we assume (iii), then (3.5) implies R invertible, and then in turn (3.4) yields the invertibility of T. Therefore, condition (i) holds and the proof is complete. □

Corollary 3.3. *Assume that $\mathcal{A}$, $\mathcal{B}$, P, Q, R, S, T, V satisfy the conditions in Notation 2.1.*

(i) *Assume that R is invertible. Then T is invertible if and only if S is invertible. In this case the matrix V is also invertible and $V^{-1} = T$.*

(ii) *Assume that S is invertible. Then V is invertible if and only if R is invertible. In this case the matrix T is also invertible and $T^{-1} = V$.*

Proof. This follows directly from Lemma 3.2 and from formula (3.7). □

Remark 3.4. Note that the invertibility of the matrix S alone (without assuming the invertibility of R) in general implies neither the invertibility of V nor the invertibility of R (hence the nonexistence of T). However, if R and S (and then by Lemma 3.2 also T and V) are invertible, then the systems (J$^\sigma$), (H$^\sigma$), and ($\mathbb{S}$) are equivalent with the coefficients of ($\mathbb{S}$) given by (3.1) or (2.8). Straightforward calculations (e.g. subtract the corresponding coefficients and simplify to zero) show that these two sets of coefficients (3.1) and (2.8) in this case coincide.

Remark 3.5. As a byproduct of Theorem 3.1 and Proposition 2.7 we gain that, under S and V (and then by Lemma 3.2 also R and T) invertible, then the Jacobi system (J$^\sigma$) and the Hamiltonian system (H$^\sigma$) are equivalent with the coefficients

$$\left.\begin{aligned} &A = \tilde{A}^{-1}V^{-1}(\mathcal{A} - \tilde{\mathcal{A}}\mathcal{B}S^{-1}Q^T)\,\tilde{\mathcal{A}}, \qquad C = P - QS^{-1}Q^TV^{-1}, \\ &B = \tilde{A}^{-1}V^{-1}\tilde{\mathcal{A}}\mathcal{B}S^{-1}\mathcal{B}^T, \end{aligned}\right\} \tag{3.9}$$

with $\tilde{A} = V\tilde{\mathcal{A}}$ satisfying the conditions in Notation 2.2. However, the coefficients of the system (H$^\sigma$) in (3.9) and (2.5) are the same as can be verified by direct calculations using the identities $\mathcal{A}\tilde{\mathcal{A}} = \tilde{\mathcal{A}}\mathcal{A}$, $V^{-1} = T$, and $\mu Q^T\tilde{\mathcal{A}}\mathcal{B} = S - R$.

Remark 3.6. In the continuous time setting $\mu \equiv 0$ and hence, $I - \mu\mathcal{A} = I$ is always invertible (and $\tilde{\mathcal{A}} = I$). In this case the invertibility of $S = R$ is the natural assumption in order to transform the corresponding Jacobi system into the symplectic system (= Hamiltonian system in this case). On the other hand, in the discrete case $\mu \equiv 1$ and hence,

$$I - \mathcal{A}_k \ \text{ and } \ S_k := R_k + Q_k^T(I - \mathcal{A}_k)^{-1}\mathcal{B}_k \ \text{ invertible}$$

are the needed assumptions in order to transform the corresponding discrete Jacobi system into a discrete symplectic system.

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let (η, v, q) solve the Jacobi system (J$^\sigma$). By using the formula $\eta^\sigma = \eta + \mu\eta^\Delta$ we obtain from the first equation of system (J$^\sigma$) that $\eta^\sigma = \tilde{\mathcal{A}}\eta + \mu\tilde{\mathcal{A}}\mathcal{B}v$. Replacing η^σ in each of the three equations in (J$^\sigma$) we get

$$\left.\begin{aligned} &\eta^\Delta = \tilde{\mathcal{A}}\mathcal{A}\eta + \tilde{\mathcal{A}}\mathcal{B}v, \\ &q^\Delta = -\mathcal{A}^Tq + P\tilde{\mathcal{A}}\eta + (\mu P\tilde{\mathcal{A}}\mathcal{B} + Q)\,v, \\ &Sv = \mathcal{B}^Tq - Q^T\tilde{\mathcal{A}}\eta. \end{aligned}\right\} \tag{3.10}$$

Since we assume that S is invertible, we can solve for v in the last equation to obtain $v = S^{-1}(\mathcal{B}^T q - Q^T \tilde{\mathcal{A}}\eta)$. If we now insert this formula for v into the first two equations in (3.10), we get the system ($\mathbb{S}$) with the coefficients $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ given by (3.1). It remains to show that these coefficients indeed define a time scale symplectic system, that is, the conditions in (2.3) hold true. From the definition of the matrix S we have

$$\mu Q^T \tilde{\mathcal{A}}\mathcal{B} = S - R, \quad \mu \mathcal{B}^T \tilde{\mathcal{A}}^T Q = S^T - R. \tag{3.11}$$

Using the definition of V and S, it follows that the matrices

$$\begin{aligned}
\mathbb{C}^T (I + \mu\mathbb{A}) &= \tilde{\mathcal{A}}^T (V^T P - Q S^{T-1} Q^T)\, V \tilde{\mathcal{A}} \\
&\overset{(3.11)}{=} \tilde{\mathcal{A}}^T V^T P V \tilde{\mathcal{A}} - \tilde{\mathcal{A}}^T Q S^{T-1} R S^{-1} Q^T \tilde{\mathcal{A}}, \\
(I + \mu\mathbb{D}^T)\, \mathbb{B} &= [I + \mu \mathcal{B} S^{T-1} (Q^T + \mu \mathcal{B}^T \tilde{\mathcal{A}}^T P) - \mu\mathcal{A}]\, \tilde{\mathcal{A}} \mathcal{B} S^{-1} \mathcal{B}^T \\
&\overset{(3.11)}{=} \mathcal{B} S^{T-1} (S + S^T - R + \mu^2 \mathcal{B}^T \tilde{\mathcal{A}}^T P \tilde{\mathcal{A}} \mathcal{B})\, S^{-1} \mathcal{B}^T
\end{aligned}$$

are symmetric and

$$\begin{aligned}
&\mathbb{A}^T + \mathbb{D} + \mu\, (\mathbb{A}^T \mathbb{D} - \mathbb{C}^T\, \mathbb{B}) \\
&\quad = \mathbb{A}^T + (I + \mu\mathbb{A}^T)\, \mathbb{D} - \mu\, \mathbb{C}^T \mathbb{B} \\
&\quad = \tilde{\mathcal{A}}^T (\mathcal{A}^T - Q S^{T-1} \mathcal{B}^T \tilde{\mathcal{A}}^T) + \tilde{\mathcal{A}}^T V\, [(Q + \mu P \tilde{\mathcal{A}} \mathcal{B})\, S^{-1} \mathcal{B}^T - \mathcal{A}^T] \\
&\qquad\quad - \mu\, \tilde{\mathcal{A}}^T (V^T P - Q S^{T-1} Q^T)\, \tilde{\mathcal{A}} \mathcal{B} S^{-1} \mathcal{B}^T \\
&\quad \overset{(3.11)}{=} -\tilde{\mathcal{A}}^T Q S^{T-1} \mathcal{B}^T + \tilde{\mathcal{A}}^T Q S^{-1} \mathcal{B}^T - \tilde{\mathcal{A}}^T Q S^{T-1} (S^T - R)\, S^{-1} \mathcal{B}^T \\
&\qquad\quad + \tilde{\mathcal{A}}^T Q S^{T-1} (S - R)\, S^{-1} \mathcal{B}^T = 0.
\end{aligned}$$

Therefore, the coefficients in (3.1) define a time scale symplectic system ($\mathbb{S}$). □

Next we consider the corresponding quadratic forms.

Proposition 3.7 (Quadratic forms for (J^σ) **and** ($\mathbb{S}$)**).** *Assume that $\mathcal{A}$, $\mathcal{B}$, P, Q, R, S, V satisfy the conditions in Notation 2.1 with S invertible. Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ be given by* (3.1)*. If (η, q) is $(\mathbb{A}, \mathbb{B})$-admissible, then the pair (η, v) with $v := S^{-1}(\mathcal{B}^T q - Q^T \tilde{\mathcal{A}}\eta)$ is $(\mathcal{A}, \mathcal{B})$-admissible and $\omega(\eta, v) = \mathbb{Q}(\eta, q)$. Conversely, if (η, v) is $(\mathcal{A}, \mathcal{B})$-admissible and if $\mathcal{B}^T \mathcal{B}$ is invertible, then the pair (η, q) with $q := \mathcal{B}(\mathcal{B}^T \mathcal{B})^{-1}(Sv + Q^T \tilde{\mathcal{A}}\eta)$ is $(\mathbb{A}, \mathbb{B})$-admissible and $\mathbb{Q}(\eta, q) = \omega(\eta, v)$.*

Proof. Let S be invertible. Assume first that (η, q) is $(\mathbb{A}, \mathbb{B})$-admissible and define v as in the statement. Then

$$\eta^\sigma = (I + \mu\mathbb{A})\, \eta + \mu\mathbb{B} q = \tilde{\mathcal{A}}\, V \eta + \mu \tilde{\mathcal{A}} \mathcal{B} S^{-1} \mathcal{B}^T q, \tag{3.12}$$

and the identity $\mathcal{A}\tilde{\mathcal{A}} = \tilde{\mathcal{A}}\mathcal{A}$ yields

$$\begin{aligned}
\mathcal{A}\eta^\sigma + \mathcal{B} v - (\mathbb{A}\eta + \mathbb{B} q) &\overset{(3.12)}{=} \mathcal{A}\, (\tilde{\mathcal{A}}\, V\eta + \mu \tilde{\mathcal{A}} \mathcal{B} S^{-1} \mathcal{B}^T q) + \mathcal{B} S^{-1} (\mathcal{B}^T q - Q^T \tilde{\mathcal{A}} \eta) \\
&\qquad - \tilde{\mathcal{A}}\, (\mathcal{A} - \mathcal{B} S^{-1} Q^T \tilde{\mathcal{A}})\, \eta - \tilde{\mathcal{A}} \mathcal{B} S^{-1} \mathcal{B}^T q = 0.
\end{aligned}$$

Hence, (η, v) is $(\mathcal{A}, \mathcal{B})$-admissible. Moreover, by using identity (3.11) we obtain after some calculations $\omega(\eta, v) - \mathbb{Q}(\eta, q) = 0$. Conversely, assume that (η, v) is $(\mathcal{A}, \mathcal{B})$-admissible, $\mathcal{B}^T\mathcal{B}$ is invertible, and define q as in the statement. Then the identities $\eta^\sigma = \tilde{\mathcal{A}}(\eta + \mu\mathcal{B}v)$ and $\mathcal{A}\tilde{\mathcal{A}} = \tilde{\mathcal{A}}\mathcal{A}$ yield

$$\begin{aligned}\mathcal{A}\eta^\sigma + \mathcal{B}v - (\mathbb{A}\eta + \mathbb{B}q) = \mathcal{A}\tilde{\mathcal{A}}\,(\eta + \mu\mathcal{B}v) + \mathcal{B}v - \tilde{\mathcal{A}}\,(\mathcal{A} - \mathcal{B}S^{-1}Q^T\tilde{\mathcal{A}})\,\eta \\ - \tilde{\mathcal{A}}\mathcal{B}S^{-1}\mathcal{B}^T\mathcal{B}(\mathcal{B}^T\mathcal{B})^{-1}(Sv + Q^T\tilde{\mathcal{A}}\eta) = 0.\end{aligned}$$

Hence, (η, q) is $(\mathbb{A}, \mathbb{B})$-admissible and with the help of identity (3.11) we obtain similarly as in the previous case that $\omega(\eta, v) - \mathbb{Q}(\eta, q) = 0$. □

Remark 3.8. The first part of Proposition 3.7 is of particular interest for the second order necessary optimality conditions for the time scale nonlinear control problem (C$^\sigma$). More precisely, in [35, Theorem 7.2] we proved that the nonnegativity of the second variation of the functional $\mathcal{F}$ in (C$^\sigma$) is a necessary condition for a weak local minimum in problem (C$^\sigma$). Since this second variation contains the functional $\int_a^b \omega(\eta, v)(t)\,\Delta t$ over $(\mathcal{A}, \mathcal{B})$-admissible pairs (η, v), our Proposition 3.7 implies the nonnegativity of a functional involving $\int_a^b \mathbb{Q}(\eta, q)(t)\,\Delta t$ over $(\mathbb{A}, \mathbb{B})$-admissible pairs (η, q). Therefore, under S invertible, any conditions which imply or which are equivalent to the nonnegativity of the latter functional are also necessary optimality conditions for the original nonlinear control problem (C$^\sigma$). For example, these are the conditions in terms of the time scale Riccati matrix equation in [34].

4 Control Problems without Shift

In this section we consider the Jacobi system for the nonlinear time scale optimal control problem

$$\text{minimize} \quad \mathcal{G}(x, u) := K\big(x(a), x(b)\big) + \int_a^b G(t, x, u)\,\Delta t, \tag{C}$$

over feasible pairs (x, u) satisfying the equation of motion

$$x^\Delta = g(t, x, u), \quad t \in [a, \rho(b)]_{\mathbb{T}}, \tag{4.1}$$

and the state endpoints constraints (1.2). The data in the problem (C) satisfy similar assumptions as the data in the problem (C$^\sigma$) in (1.3). The corresponding Jacobi system then has the form

$$\eta^\Delta = \underline{A}\eta + \underline{B}v, \quad q^\Delta = -\underline{A}^T q^\sigma + \underline{P}\eta + \underline{Q}v, \quad -\underline{B}^T q^\sigma + \underline{Q}^T\eta + \underline{R}v = 0. \tag{\underline{J}}$$

System ($\underline{\text{J}}$) was derived in [36, Section 4.2]. Moreover, it is shown in the same reference that under the corresponding invertibility assumptions (including the invertibility of $\underline{R}$, see Proposition 4.4) the system ($\underline{\mathrm{J}}$) is the Hamiltonian system

$$\eta^\Delta = \underline{A}\eta + \underline{B}q^\sigma, \quad q^\Delta = \underline{C}\eta - \underline{A}^T q^\sigma, \tag{$\underline{\mathrm{H}}$}$$

where $\underline{B}$ and $\underline{C}$ are symmetric and $I + \mu\underline{A}$ is invertible. Hence in turn (see Proposition 4.7), system ($\underline{\mathrm{J}}$) is a time scale symplectic system ($\mathbb{S}$). Similarly as for the Jacobi system (J^σ) we will show that one can go directly from the Jacobi system ($\underline{\mathrm{J}}$) to the symplectic system ($\underline{\mathbb{S}}$), which is in this section written as

$$\eta^\Delta = \underline{\mathbb{A}}\,\eta + \underline{\mathbb{B}}\,q, \quad q^\Delta = \underline{\mathbb{C}}\,\eta + \underline{\mathbb{D}}\,q, \tag{$\underline{\mathbb{S}}$}$$

proving the superior importance of time scale symplectic systems over the Hamiltonian systems ($\underline{\mathrm{H}}$). This result is also new for the corresponding time scale (in particular discrete) calculus of variations setting, as it is discussed in the next section, see also [41, Section 2].

The methods of proofs in this section are similar to those in Section 2, and thus we will skip them here. First we introduce the appropriate notation. To avoid confusion with the coefficients in the previous section, we will underline in this section the corresponding coefficients (and keep their meaning at the same time).

Notation 4.1 *(Jacobi system ($\underline{\mathrm{J}}$)).* The matrices $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$, $\underline{P}$, $\underline{Q}$, $\underline{R}$ and the vectors η, v, q in ($\underline{\mathrm{J}}$) have the same properties as their corresponding counterparts in Notation 2.1 (in particular, $\underline{P}$ and $\underline{R}$ are symmetric). Furthermore, we assume that the matrix $I + \mu\underline{\mathcal{A}}$ is invertible and define the matrices $\tilde{\underline{\mathcal{A}}} \in \mathbb{R}^{n\times n}$ and $\underline{S} \in \mathbb{R}^{m\times m}$ by

$$\tilde{\underline{\mathcal{A}}} := (I + \mu\underline{\mathcal{A}})^{-1}, \quad \underline{S} := \underline{R} - \mu\,\underline{\mathcal{B}}^T\tilde{\underline{\mathcal{A}}}^T\underline{Q}. \tag{4.2}$$

In addition, whenever the matrix $\underline{R}$ is invertible and whenever the matrix $\underline{S}$ is invertible, we define the matrices $\underline{T}, \underline{V} \in \mathbb{R}^{n\times n}$ by

$$\underline{T} := I - \mu\underline{Q}\,\underline{R}^{-1}\underline{\mathcal{B}}^T\tilde{\underline{\mathcal{A}}}^T, \quad \underline{V} := I + \mu\underline{Q}\,\underline{S}^{-1}\underline{\mathcal{B}}^T\tilde{\underline{\mathcal{A}}}^T. \tag{4.3}$$

Notation 4.2 *(Hamiltonian system ($\underline{\mathrm{H}}$)).* The matrices $\underline{A}$, $\underline{B}$, $\underline{C}$ and the vectors η, q in ($\underline{\mathrm{H}}$) have the same properties as their corresponding counterparts in Notation 2.2 (in particular, $\underline{B}$ and $\underline{C}$ are symmetric). Furthermore, we assume that the matrix $I + \mu\underline{A}$ is invertible and define the matrix $\tilde{\underline{A}} \in \mathbb{R}^{n\times n}$ by

$$\tilde{\underline{A}} := (I + \mu\underline{A}^T)^{-1}.$$

Notation 4.3 *(Symplectic system ($\underline{\mathbb{S}}$)).* The matrices $\underline{\mathbb{A}}$, $\underline{\mathbb{B}}$, $\underline{\mathbb{C}}$, $\underline{\mathbb{D}}$ and the vectors η, q in ($\underline{\mathbb{S}}$) have the same properties as their corresponding counterparts in Notation 2.3.

Next we recall the known transformations between the systems ($\underline{\mathrm{J}}$), ($\underline{\mathrm{H}}$), ($\underline{\mathbb{S}}$).

Proposition 4.4 (Jacobi ($\underline{J}$) to Hamiltonian ($\underline{H}$)). *Assume that $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$, $\underline{P}$, $\underline{Q}$, $\underline{R}$, $\underline{T}$ satisfy the conditions in Notation 4.1 with $\underline{R}$ and $\underline{T}$ invertible. Then the Jacobi system ($\underline{J}$) is the Hamiltonian system ($\underline{H}$), whose coefficients*

$$\underline{A} = \underline{\mathcal{A}} - \underline{\mathcal{B}}\underline{R}^{-1}\underline{Q}^T, \quad \underline{B} = \underline{\mathcal{B}}\underline{R}^{-1}\underline{\mathcal{B}}^T, \quad \underline{C} = \underline{P} - \underline{Q}\underline{R}^{-1}\underline{Q}^T \tag{4.4}$$

with $\underline{\tilde{A}} = \underline{\tilde{\mathcal{A}}}^T\underline{T}^{-1}$ satisfy the conditions in Notation 4.2.

Proof. See [36, Formula (45)]. □

Proposition 4.5 (Hamiltonian ($\underline{H}$) to symplectic ($\underline{\mathbb{S}}$)). *Assume that $\underline{A}$, $\underline{B}$, $\underline{C}$ satisfy the conditions in Notation 4.2. Then the Hamiltonian system ($\underline{H}$) is the time scale symplectic system ($\underline{\mathbb{S}}$), whose coefficients*

$$\underline{\mathbb{A}} = \mu\,\underline{B\tilde{A}C} + \underline{A}, \quad \underline{\mathbb{B}} = \underline{B\tilde{A}}, \quad \underline{\mathbb{C}} = \underline{\tilde{A}C}, \quad \underline{\mathbb{D}} = -\underline{A}^T\underline{\tilde{A}} \tag{4.5}$$

with $I + \mu\underline{\mathbb{D}} = \underline{\tilde{A}}$ satisfy the conditions in Notation 4.3.

Proof. See [36, Proposition 4.2] and also the discrete case in [26, Remark 4]. □

Proposition 4.6 (Symplectic ($\underline{\mathbb{S}}$) to Hamiltonian ($\underline{H}$)). *Assume that $\underline{\mathbb{A}}$, $\underline{\mathbb{B}}$, $\underline{\mathbb{C}}$, $\underline{\mathbb{D}}$ satisfy the conditions in Notation 4.3 with $I + \mu\underline{\mathbb{D}}$ invertible. Then the time scale symplectic system ($\underline{\mathbb{S}}$) is the Hamiltonian system ($\underline{H}$), whose coefficients*

$$\underline{A} = -(I + \mu\underline{\mathbb{D}}^T)^{-1}\underline{\mathbb{D}}^T, \quad \underline{B} = \underline{\mathbb{B}}\,(I + \mu\underline{\mathbb{D}})^{-1}, \quad \underline{C} = (I + \mu\underline{\mathbb{D}})^{-1}\underline{\mathbb{C}} \tag{4.6}$$

with $\underline{\tilde{A}} = I + \mu\underline{\mathbb{D}}$ satisfy the conditions in Notation 4.2.

Proof. Using the invertibility of $I + \mu\underline{\mathbb{D}}$ we solve the second equation of ($\underline{\mathbb{S}}$) for q and insert it back to ($\underline{\mathbb{S}}$). Note that the coefficient identities in (2.3) are used in these calculations. □

Proposition 4.7 (Jacobi ($\underline{J}$) to symplectic ($\underline{\mathbb{S}}$)). *Assume that $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$, $\underline{P}$, $\underline{Q}$, $\underline{R}$, $\underline{T}$ satisfy the conditions in Notation 4.1 with $\underline{R}$ and $\underline{T}$ invertible. Then the Jacobi system ($\underline{J}$) is the symplectic system ($\underline{\mathbb{S}}$), whose coefficients*

$$\left.\begin{aligned}
\underline{\mathbb{A}} &= \mu\underline{\mathcal{B}}\underline{R}^{-1}\underline{\mathcal{B}}^T\underline{\tilde{\mathcal{A}}}^T\underline{T}^{-1}(\underline{P}-\underline{Q}\underline{R}^{-1}\underline{Q}^T) + \underline{\mathcal{A}}-\underline{\mathcal{B}}\underline{R}^{-1}\underline{Q}^T,\\
\underline{\mathbb{B}} &= \underline{\mathcal{B}}\underline{R}^{-1}\underline{\mathcal{B}}^T\underline{\tilde{\mathcal{A}}}^T\underline{T}^{-1},\\
\underline{\mathbb{C}} &= \underline{\tilde{\mathcal{A}}}^T\underline{T}^{-1}(\underline{P}-\underline{Q}\underline{R}^{-1}\underline{Q}^T),\\
\underline{\mathbb{D}} &= (\underline{Q}\underline{R}^{-1}\underline{\mathcal{B}}^T-\underline{\mathcal{A}}^T)\,\underline{\tilde{\mathcal{A}}}^T\underline{T}^{-1}
\end{aligned}\right\} \tag{4.7}$$

with $I + \mu\underline{\mathbb{D}} = \underline{\tilde{\mathcal{A}}}^T\underline{T}^{-1}$ satisfy the conditions in Notation 4.3.

Proof. It is a simple combination of Propositions 4.4 and 4.5. □

Next we present the main result of this section, which shows how to transform the Jacobi system ($\underline{\mathrm{J}}$) directly to the symplectic system ($\underline{\mathbb{S}}$) by avoiding the Hamiltonian system ($\underline{\mathrm{H}}$).

Theorem 4.8 (Jacobi ($\underline{\mathrm{J}}$) to symplectic ($\underline{\mathbb{S}}$)). *Assume that $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$, $\underline{P}$, $\underline{Q}$, $\underline{R}$, $\underline{S}$, $\underline{V}$ satisfy the conditions in Notation 4.1 with $\underline{S}$ invertible. Then the Jacobi system* ($\underline{\mathrm{J}}$) *is the symplectic system* ($\underline{\mathbb{S}}$), *whose coefficients*

$$\left.\begin{aligned} &\underline{\mathbb{A}} = \underline{\mathcal{A}} + \underline{\mathcal{B}}\underline{S}^{-1}(\mu\underline{\mathcal{B}}^T\tilde{\underline{\mathcal{A}}}^T\underline{P} - \underline{Q}^T), && \underline{\mathbb{C}} = \tilde{\underline{\mathcal{A}}}^T(\underline{V}\,\underline{P} - \underline{Q}\,\underline{S}^{-1}\underline{Q}^T), \\ &\underline{\mathbb{B}} = \underline{\mathcal{B}}\underline{S}^{-1}\underline{\mathcal{B}}^T\tilde{\underline{\mathcal{A}}}^T, && \underline{\mathbb{D}} = \tilde{\underline{\mathcal{A}}}^T(\underline{Q}\,\underline{S}^{-1}\underline{\mathcal{B}}^T\tilde{\underline{\mathcal{A}}}^T - \underline{\mathcal{A}}^T) \end{aligned}\right\} \tag{4.8}$$

with $I+\mu\underline{\mathbb{D}} = \tilde{\underline{\mathcal{A}}}^T\underline{V}$ satisfy the conditions in Notation 4.3. Thus, the resulting symplectic system ($\underline{\mathbb{S}}$) *is Hamiltonian if and only if the matrix $\underline{V}$ (or equivalently $\underline{R}$) is invertible.*

Proof. The proof is analogous to the proof of Theorem 3.1 and it consists of verifying the conditions in (2.3) for the above coefficients $\underline{\mathbb{A}}, \underline{\mathbb{B}}, \underline{\mathbb{C}}, \underline{\mathbb{D}}$. □

The following lemma shows the exact relationship between the invertibility of $\underline{R}$, $\underline{T}$, $\underline{S}$, $\underline{V}$.

Lemma 4.9. *Assume that $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$, $\underline{P}$, $\underline{Q}$, $\underline{R}$, $\underline{S}$, $\underline{T}$, $\underline{V}$ satisfy the conditions in Notation 4.1. Then the following statements are equivalent:*

(i) *$\underline{R}$ and $\underline{T}$ are invertible,*

(ii) *$\underline{R}$ and $\underline{S}$ are invertible,*

(iii) *$\underline{S}$ and $\underline{V}$ are invertible.*

Proof. The proof is similar to the proof of Lemma 3.2. By using the matrix inversion formula (3.2) we prove the implications

$$\underline{R} \text{ and } \underline{T} \text{ invertible} \implies \underline{S} \text{ invertible}, \tag{4.9}$$
$$\underline{R} \text{ and } \underline{S} \text{ invertible} \implies \underline{V} \text{ and } \underline{T} \text{ invertible}, \tag{4.10}$$
$$\underline{S} \text{ and } \underline{V} \text{ invertible} \implies \underline{R} \text{ invertible}. \tag{4.11}$$

with the inverses in the conclusion of (4.9), (4.10), (4.11) given by

$$\underline{S}^{-1} = \underline{R}^{-1}(\underline{R} + \mu\underline{\mathcal{B}}^T\tilde{\underline{\mathcal{A}}}^T\underline{T}^{-1}\underline{Q})\,\underline{R}^{-1}, \quad \underline{V}^{-1} = \underline{T}, \quad \underline{R}^{-1} = \underline{S}^{-1}(\underline{S} - \mu\underline{\mathcal{B}}^T\tilde{\underline{\mathcal{A}}}^T\underline{V}^{-1}\underline{Q})\,\underline{S}^{-1},$$

respectively. The details are here omitted. □

Corollary 4.10. *Assume that $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$, $\underline{P}$, $\underline{Q}$, $\underline{R}$, $\underline{S}$, $\underline{T}$, $\underline{V}$ satisfy the conditions in Notation 4.1.*

(i) *Assume that $\underline{R}$ is invertible. Then $\underline{T}$ is invertible if and only if $\underline{S}$ is invertible. In this case the matrix $\underline{V}$ is also invertible and $\underline{V}^{-1} = \underline{T}$.*

(ii) *Assume that $\underline{S}$ is invertible. Then $\underline{V}$ is invertible if and only if $\underline{R}$ is invertible. In this case the matrix $\underline{T}$ is also invertible and $\underline{T}^{-1} = \underline{V}$.*

Remark 4.11. Similarly as in Remark 3.5, we have from Theorem 4.8 and Proposition 4.6 that, under $\underline{S}$ and $\underline{V}$ invertible (and then by Lemma 4.9 also $\underline{R}$ and $\underline{T}$ are invertible), the Jacobi system ($\underline{\text{J}}$) and the Hamiltonian system ($\underline{\text{H}}$) are equivalent with the coefficients

$$\left.\begin{array}{l} \underline{A} = -\underline{\tilde{\mathcal{A}}}^{T-1} \underline{V}^{T-1} (\underline{\tilde{\mathcal{A}}} \underline{\mathcal{B}} \underline{S}^{T-1} \underline{Q}^T - \underline{\mathcal{A}})\, \underline{\tilde{\mathcal{A}}}, \qquad \underline{C} = \underline{P} - \underline{V}^{-1} \underline{Q} \underline{S}^{-1} \underline{Q}^T \\ \underline{B} = \underline{\mathcal{B}} \underline{S}^{-1} \underline{\mathcal{B}}^T \underline{\tilde{\mathcal{A}}}^T \underline{V}^{-1} \underline{\tilde{\mathcal{A}}}^{T-1} \end{array}\right\} \tag{4.12}$$

with $\underline{\tilde{A}} = \underline{\tilde{\mathcal{A}}}^T \underline{V}$ satisfying the conditions in Notation 2.2. These coefficients of the system ($\underline{\text{H}}$) in (4.12) and (4.4) are however the same.

The quadratic forms associated with systems ($\underline{\text{J}}$) and ($\underline{\text{H}}$) are defined by

$$\underline{\omega}(\eta, v) := \eta^T \underline{P}\, \eta + 2\, \eta^T \underline{Q}\, v + v^T \underline{R}\, v, \tag{4.13}$$

$$\underline{\Omega}(\eta, q) := \eta^T \underline{C}\, \eta + (q^\sigma)^T \underline{B}\, q^\sigma, \tag{4.14}$$

while the quadratic form $\underline{\mathbb{Q}}(\eta, q)$ for system ($\underline{\mathbb{S}}$) is defined analogously to the quadratic form $\mathbb{Q}(\eta, q)$ in (2.9) using the coefficients $\underline{\mathbb{A}}$, $\underline{\mathbb{B}}$, $\underline{\mathbb{C}}$, $\underline{\mathbb{D}}$. The pairs (η, v) in (4.13), resp. (η, q) in (4.14), resp. in $\underline{\mathbb{Q}}(\eta, q)$, are called $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$-admissible, resp. $(\underline{A}, \underline{B})$-admissible, resp. $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible, provided $\eta^\Delta = \underline{\mathcal{A}}\eta + \underline{\mathcal{B}}v$, resp. $\eta^\Delta = \underline{A}\eta + \underline{\mathcal{B}}q^\sigma$, resp. $\eta^\Delta = \underline{\mathbb{A}}\eta + \underline{\mathbb{B}}q$. The relationship between these quadratic forms, similarly as it was done in Propositions 2.10, 2.11, and 2.12, can now be easily established. We shall not give the details of these obvious propositions. However, we state explicitly the parallel result to Proposition 3.7, since it is important for second order necessary optimality conditions for the control problem (C), see Remark 3.8 and also [35, Theorem 9.7].

Proposition 4.12 (Quadratic forms for ($\underline{\text{J}}$) and ($\underline{\mathbb{S}}$)). *Assume that $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$, $\underline{P}$, $\underline{Q}$, $\underline{R}$, $\underline{S}$ satisfy the conditions in Notation 4.1 with $\underline{S}$ invertible. Let $\underline{\mathbb{A}}$, $\underline{\mathbb{B}}$, $\underline{\mathbb{C}}$, $\underline{\mathbb{D}}$ be given by* (4.8). *If (η, q) is $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible, then the pair (η, v) with $v := \underline{S}^{-1} [\underline{\mathcal{B}}^T \underline{\tilde{\mathcal{A}}}^T q + (\mu \underline{\mathcal{B}}^T \underline{\tilde{\mathcal{A}}}^T \underline{P} - \underline{Q}^T)\, \eta\,]$ is $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$-admissible and $\underline{\omega}(\eta, v) = \underline{\mathbb{Q}}(\eta, q)$. Conversely, if (η, v) is $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$-admissible and if $\underline{\mathcal{B}}^T \underline{\mathcal{B}}$ is invertible, then the pair (η, q) with $q := \underline{\tilde{\mathcal{A}}}^{T-1} \underline{\mathcal{B}} (\underline{\mathcal{B}}^T \underline{\mathcal{B}})^{-1} [\underline{S} v - (\mu \underline{\mathcal{B}}^T \underline{\tilde{\mathcal{A}}}^T \underline{P} - \underline{Q}^T)\, \eta\,]$ is $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible and $\underline{\mathbb{Q}}(\eta, q) = \underline{\omega}(\eta, v)$.*

Proof. The proof is similar to the proof of Proposition 3.7 and it is based on direct calculations. The details are here omitted. □

Remark 4.13. The first part of Proposition 4.12 leads via [35, Theorem 9.7] to second order necessary optimality conditions for the control problem (C) in terms of the nonnegativity of a quadratic functional involving $\int_a^b \underline{\mathbb{Q}}(\eta,q)(t)\,\Delta t$ over $(\underline{\mathbb{A}},\underline{\mathbb{B}})$-admissible pairs (η,q). In this respect the Riccati equation conditions from [34] which are equivalent to the nonnegativity of this quadratic functional represent second order necessary optimality conditions for problem (C). Note that the results in [34] can be applied to both problems (C) and (C$^\sigma$), as we saw in Remark 3.8.

Next we will examine the direct relationship between the Jacobi systems (J̲) and (J$^\sigma$). We will see that the Jacobi system (J̲) can be transformed into the Jacobi system (J$^\sigma$) (and vice versa). Therefore, all the results which were obtained for the system (J$^\sigma$), resp. (J̲), can now be transformed to the system (J̲), resp. (J$^\sigma$), via the transformations below.

Proposition 4.14 (Jacobi (J̲) to Jacobi (J$^\sigma$)**).** *Assume that the matrices $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$, $\underline{P}$, $\underline{Q}$, $\underline{R}$, $\underline{S}$ satisfy the conditions in Notation 4.1, in particular, $\underline{P}$ and $\underline{R}$ are symmetric and $I+\mu\underline{\mathcal{A}}$ is invertible. Then the Jacobi system* (J̲) *is the Jacobi system* (J$^\sigma$)*, whose coefficients*

$$\left.\begin{aligned} &\mathcal{A} := \tilde{\underline{\mathcal{A}}}\underline{\mathcal{A}}, \quad \mathcal{B} := \tilde{\underline{\mathcal{A}}}\underline{\mathcal{B}}, \quad P := \tilde{\underline{\mathcal{A}}}^T \underline{P}\tilde{\underline{\mathcal{A}}}, \quad Q := \tilde{\underline{\mathcal{A}}}^T(\underline{Q} - \mu\underline{P}\tilde{\underline{\mathcal{A}}}\underline{\mathcal{B}}), \\ &R := \underline{R} - \mu \underline{Q}^T\tilde{\underline{\mathcal{A}}}\underline{\mathcal{B}} - \mu\underline{\mathcal{B}}^T\tilde{\underline{\mathcal{A}}}^T\underline{Q} + \mu^2\underline{\mathcal{B}}^T\tilde{\underline{\mathcal{A}}}^T\underline{P}\tilde{\underline{\mathcal{A}}}\underline{\mathcal{B}} \end{aligned}\right\} \tag{4.15}$$

with $\tilde{\mathcal{A}} = \tilde{\underline{\mathcal{A}}}^{-1}$ and $S = \underline{S}$ satisfy the conditions in Notation 2.1.

Proof. By using the formulas $\mu\eta^\Delta = \eta^\sigma - \eta$ and $\mu q^\Delta = q^\sigma - q$, we obtain from the first two equations in (J̲) the identities

$$\eta = \tilde{\underline{\mathcal{A}}}\,(\eta^\sigma - \mu\underline{\mathcal{B}}v), \quad q^\sigma = \tilde{\underline{\mathcal{A}}}^T q + \mu\tilde{\underline{\mathcal{A}}}^T\underline{P}\tilde{\underline{\mathcal{A}}}\eta^\sigma + \mu\tilde{\underline{\mathcal{A}}}^T(\underline{Q} - \mu\underline{P}\tilde{\underline{\mathcal{A}}}\underline{\mathcal{B}})\,v.$$

Upon inserting these formulas for η and q^σ into system (J̲) we will obtain the system (J$^\sigma$) with the coefficients $\mathcal{A},\mathcal{B},P,Q,R$ given by (4.15). Finally, for the matrices $\tilde{\mathcal{A}}$ and S given in (2.1) we have

$$\tilde{\mathcal{A}} = (I-\mu\tilde{\mathcal{A}})^{-1} = \tilde{\underline{\mathcal{A}}}^{-1}, \quad S = R + \mu Q^T\tilde{\mathcal{A}}\mathcal{B} = \underline{R} - \mu\underline{\mathcal{B}}^T\tilde{\underline{\mathcal{A}}}^T\underline{Q} = \underline{S}$$

as in (4.2). This completes the proof. □

Proposition 4.15 (Jacobi (J$^\sigma$) **to Jacobi (J̲)).** *Assume that the matrices $\mathcal{A}$, $\mathcal{B}$, P, Q, R, S satisfy the conditions in Notation 2.1, in particular, P and R are symmetric and $I-\mu\mathcal{A}$ is invertible. Then the Jacobi system* (J$^\sigma$) *is the Jacobi system* (J̲)*, whose coefficients*

$$\left.\begin{aligned} &\underline{\mathcal{A}} := \tilde{\mathcal{A}}\mathcal{A}, \quad \underline{\mathcal{B}} := \tilde{\mathcal{A}}\mathcal{B}, \quad \underline{P} := \tilde{\mathcal{A}}^T P\tilde{\mathcal{A}}, \quad \underline{Q} := \tilde{\mathcal{A}}^T(\mu P\tilde{\mathcal{A}}\mathcal{B} + Q), \\ &\underline{R} := R + \mu Q^T\tilde{\mathcal{A}}\mathcal{B} + \mu\mathcal{B}^T\tilde{\mathcal{A}}^T Q + \mu^2\mathcal{B}^T\tilde{\mathcal{A}}^T P\tilde{\mathcal{A}}\mathcal{B} \end{aligned}\right\} \tag{4.16}$$

with $\tilde{\underline{\mathcal{A}}} = \tilde{\mathcal{A}}^{-1}$ and $\underline{S} = S$ satisfy the conditions in Notation 4.1.

Proof. The proof is similar to the proof of Proposition 4.14, the details are omitted. □

Remark 4.16. The transformations in Propositions 4.14 and 4.15 show that the symplectic structure is deeply rooted in the Jacobi systems (J^σ) and ($\underline{\mathrm{J}}$). Indeed, the result of Theorem 4.8 (and Proposition 4.12) can be obtained from Theorem 3.1 (and Proposition 3.7) applied to the data in (4.15). Conversely, the result of Theorem 3.1 (and Proposition 3.7) can be obtained from Theorem 4.8 (and Proposition 4.12) applied to the data in (4.16). However, this equivalence of results is lost when we start considering the Jacobi systems (J^σ) and ($\underline{\mathrm{J}}$) as their corresponding Hamiltonian systems (H^σ) and ($\underline{\mathrm{H}}$). For example, the matrices T (and V) obtained from their definitions in (2.2) through the coefficients (4.15) become much more complicated than the matrices $\underline{T}$ (and $\underline{V}$) used in (4.3), because the matrix T involves the inverse of R from (4.15). Therefore, Proposition 4.4 cannot be obtained from Proposition 2.5 just because the forms of the systems (H^σ) and ($\underline{\mathrm{H}}$) are different. For this reason a direct approach to the result in Lemma 4.9 is essential.

Remark 4.17. In [40] it is shown that the Jacobian matrix of discrete nonlinear Hamiltonian systems is symplectic. The same result can be derived for nonlinear Hamiltonian systems on time scales

$$x^\Delta = \mathcal{H}_p(t, x^\sigma, p), \quad p^\Delta = -\mathcal{H}_x(t, x^\sigma, p), \quad t \in [a, \rho(b)]_{\mathbb{T}}.$$

More precisely, assuming that the function $\mathcal{H}(t, \cdot, \cdot)$ has continuous second order partial derivatives with respect to x and p and the matrix $I - \mu\mathcal{H}_{px}$ is invertible for all $(t, x, u) \in [a, \rho(b)]_{\mathbb{T}} \times \mathcal{D}$ for some domain $\mathcal{D} \subseteq \mathbb{R}^{2n}$, then the associated phase flow $(F, G)(t, \cdot, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defines a symplectic transformation, i.e.,

$$\begin{pmatrix} F_x & F_p \\ G_x & G_p \end{pmatrix}^\Delta = \mathbb{S} \begin{pmatrix} F_x & F_p \\ G_x & G_p \end{pmatrix}, \tag{4.17}$$

with

$$\mathbb{S} := \begin{pmatrix} \mathcal{H}_{px}\,(I - \mu\mathcal{H}_{px})^{-1} & (I - \mu\mathcal{H}_{px})^{-1}\mathcal{H}_{pp} \\ -\mathcal{H}_{xx}\,(I - \mu\mathcal{H}_{px})^{-1} & -\mu\mathcal{H}_{xx}\,(I - \mu\mathcal{H}_{px})^{-1}\mathcal{H}_{pp} - \mathcal{H}_{xp} \end{pmatrix}, \tag{4.18}$$

where the matrix $\mathbb{S}$ satisfies (1.4). This is a generalization of the discrete time result in [40, Theorem 2.1]. Since the proof of the above time scale result is similar to the corresponding proof in [40] (using $\mathcal{H}_{px} = \mathcal{H}_{xp}^T$ and the symmetry of $\mathcal{H}_{xx}$ and $\mathcal{H}_{pp}$), it is here omitted. Note that system (4.17) with $\mathbb{S}$ given by (4.18) is in fact a Hamiltonian system (H^σ) from Section 1 with $A := \mathcal{H}_{px}$, $B := \mathcal{H}_{pp}$, $C := \mathcal{H}_{xx}$, and $\tilde{A} = (I - \mu\mathcal{H}_{px})^{-1}$.

Similarly we can consider the nonlinear Hamiltonian system

$$x^\Delta = \underline{\mathcal{H}}_p(t, x, p^\sigma), \quad p^\Delta = -\underline{\mathcal{H}}_x(t, x, p^\sigma), \quad t \in [a, \rho(b)]_{\mathbb{T}},$$

with $I+\mu\,\underline{\mathcal{H}}_{xp}$ invertible for all $(t,x,u)\in[a,\rho(b)]_{\mathbb{T}}\times\mathcal{D}$. Then the corresponding phase flow satisfies system (4.17) with $\mathbb{S}:=\underline{\mathbb{S}}$, where

$$\underline{\mathbb{S}}:=\begin{pmatrix}\underline{\mathcal{H}}_{px}-\mu\,\underline{\mathcal{H}}_{pp}\,(I+\mu\,\underline{\mathcal{H}}_{xp})^{-1}\underline{\mathcal{H}}_{xx} & \underline{\mathcal{H}}_{pp}\,(I+\mu\,\underline{\mathcal{H}}_{xp})^{-1}\\ -(I+\mu\,\underline{\mathcal{H}}_{xp})^{-1}\underline{\mathcal{H}}_{xx} & -\underline{\mathcal{H}}_{xp}\,(I+\mu\,\underline{\mathcal{H}}_{xp})^{-1}\end{pmatrix} \tag{4.19}$$

satisfies (1.4). The system in (4.17) with $\mathbb{S}:=\underline{\mathbb{S}}$ given by (4.19) is then a Hamiltonian system (H̲) from the beginning of this section with $\underline{A}:=\underline{\mathcal{H}}_{px}$, $\underline{B}:=\underline{\mathcal{H}}_{pp}$, $\underline{C}:=\underline{\mathcal{H}}_{xx}$, and $\underline{\tilde{A}}=(I+\mu\mathcal{H}_{xp}^{T})^{-1}$.

5 Jacobi Systems for Calculus of Variations Problems

Problems in time scale calculus of variations have attracted a lot of interest recently, see e.g. [7, 9, 28, 33]. These are the problems of the form (C^σ) with $f(t,x^\sigma,u)=u$ in the equation of motion (1.1), or of the form (C) with $g(t,x,u)=u$ in the equation of motion (4.1). Therefore, in both cases we have $x^\Delta=u$ (which implies that $m=n$) and the matrices $\mathcal{A}=0=\underline{\mathcal{A}}$, $\mathcal{B}=I=\underline{\mathcal{B}}$, and $\tilde{\mathcal{A}}=I=\underline{\tilde{\mathcal{A}}}$ are constant. In this case we can solve the third equation in systems (J^σ) and (J̲) for v and insert it back to the first two equations. This way we obtain the Jacobi equation ($J^\sigma_{\rm cov}$) displayed in Section 1 and the Jacobi equation ($\underline{J}_{\rm cov}$) presented below. The results from Sections 3 and 4 specified to the calculus of variations setting then yield the following new results. Note that these results are new even in the discrete time calculus of variations case.

Notation 5.1 *(Jacobi system* ($J^\sigma_{\rm cov}$)*).* The matrices P, Q, R and the vector η in ($J^\sigma_{\rm cov}$) have the following properties: $P,Q,R\in\mathbb{R}^{n\times n}$ and these matrices are defined on $[a,\rho(b)]_{\mathbb{T}}$ and P and R are symmetric; $\eta\in\mathbb{R}^n$ is defined on $[a,b]_{\mathbb{T}}$. Furthermore, we define the matrix $S\in\mathbb{R}^{n\times n}$ by

$$S:=R+\mu\,Q^T.$$

Corollary 5.2 (Jacobi ($J^\sigma_{\rm cov}$) to symplectic ($\mathbb{S}$)). *Assume that P, Q, R, S satisfy the conditions in Notation 5.1 with S invertible. Then the Jacobi equation ($J^\sigma_{\rm cov}$) is the symplectic system ($\mathbb{S}$), whose coefficients*

$$\left.\begin{aligned}&\mathbb{A}=-S^{-1}Q^T, && \mathbb{C}=PS^{-1}R-QS^{-1}Q^T,\\ &\mathbb{B}=S^{-1}, && \mathbb{D}=(Q+\mu P)\,S^{-1}\end{aligned}\right\} \tag{5.1}$$

with $I+\mu\mathbb{A}=S^{-1}R$ satisfy the conditions in Notation 2.3. Thus, the resulting symplectic system ($\mathbb{S}$) is Hamiltonian if and only if the matrix R is invertible.

Proof. This is a special case of Theorem 3.1. □

The symplectic system ($\mathbb{S}$) arising from the Jacobi equation ($J^\sigma_{\rm cov}$) has the coefficient $\mathbb{B}=S^{-1}$ invertible. In the following we shall see that this is a characteristic property of time scale symplectic systems ($\mathbb{S}$) which are equivalent to Jacobi equations ($J^\sigma_{\rm cov}$).

Corollary 5.3 (Symplectic $(\mathbb{S})$ **to Jacobi** $(\mathrm{J}^{\sigma}_{\mathrm{cov}})$**).** *Assume that* $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ *satisfy the conditions in Notation 2.3 with* $\mathbb{B}$ *invertible. Then the time scale symplectic system* $(\mathbb{S})$ *is the Jacobi equation* $(\mathrm{J}^{\sigma}_{\mathrm{cov}})$, *whose coefficients*

$$P = \mathbb{C} - \mathbb{D}\,\mathbb{B}^{-1}\mathbb{A}, \quad Q = -\mathbb{A}^T\,\mathbb{B}^{T-1}, \quad R = \mathbb{B}^{-1}(I + \mu\mathbb{A}) \tag{5.2}$$

with $S = \mathbb{B}^{-1}$ *satisfy the conditions in Notation 5.1.*

Proof. The symmetry of P and R follows from conditions (2.3). □

The quadratic form associated with the Jacobi equation $(\mathrm{J}^{\sigma}_{\mathrm{cov}})$ is

$$\omega(\eta) := (\eta^{\sigma})^T P \eta^{\sigma} + 2\,(\eta^{\sigma})^T Q \eta^{\Delta} + (\eta^{\Delta})^T R \eta^{\Delta}.$$

The relation of $\omega(\eta)$ with the quadratic form $\mathbb{Q}(\eta, q)$ defined in (2.9) is shown in the next statement.

Proposition 5.4 (Quadratic forms for $(\mathrm{J}^{\sigma}_{\mathrm{cov}})$ **and** $(\mathbb{S})$**).** *Assume that*

(i) *either* P, Q, R, S *satisfy the conditions in Notation 5.1 with* S *invertible and* $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ *are given by* (5.1),

(ii) *or* $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ *satisfy the conditions in Notation 2.3 with* $\mathbb{B}$ *invertible and* P, Q, R *are given by* (5.2).

If (η, q) *is* $(\mathbb{A}, \mathbb{B})$*-admissible, then* $\omega(\eta) = \mathbb{Q}(\eta, q)$. *Conversely, for any* $\eta \in \mathrm{C}^1_{\mathrm{prd}}$ *on* $[a, b]_{\mathbb{T}}$ *the pair* (η, q) *with* $q := S\eta^{\Delta} + Q^T\eta = \mathbb{B}^{-1}(\eta^{\Delta} - \mathbb{A}\eta)$ *is* $(\mathbb{A}, \mathbb{B})$*-admissible and* $\mathbb{Q}(\eta, q) = \omega(\eta)$.

Proof. Part (i) is a special case of Proposition 3.7, while part (ii) can be shown by direct calculations. □

The above proposition suggests that the quadratic forms $\omega(\eta)$ and $\mathbb{Q}(\eta, q)$ are completely equivalent whenever the two sets of coefficients P, Q, R and $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ are connected by (5.1) or (5.2). The application of this observation to positivity and coercivity of quadratic functionals involving $\omega(\eta)$ and to second order sufficient optimality conditions for calculus of variations problems on time scales is discussed in the next remark.

Remark 5.5. (i) In [33, Theorem 4.1], the coercivity of the accessory problem

$$\mathcal{I}(\eta) = \begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix}^T \Gamma \begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix} + \int_a^b \omega(\eta)(t)\,\Delta t$$

was proven to be equivalent to its positivity through [33, Propositions 4.3, 4.5], assuming R and S invertible. Since these propositions are special cases of known results for time scale symplectic systems, i.e., [32, Theorem 6.2] and [30, Lemma 7.2], then Proposition 5.4 implies that [33, Theorem 4.1] holds true *without assuming* R *invertible*.

(ii) As a consequence of part (i) of this remark and of [28, Theorem 2], it follows that the assumption R invertible in [33, Theorem 6.1] can now be eliminated.

Remark 5.6. In the *discrete time case*, the discrete Jacobi equation (1.5) was intensively studied in the past. In particular, in [6, Example 3.17] it is shown that under R_k and $S_k = R_k + Q_k^T$ invertible the discrete Jacobi equation (1.5) can be written as the corresponding discrete linear Hamiltonian system (H^σ), and then as a discrete symplectic system. A direct consequence of Corollary 5.2 we obtain that the only assumption in order to write equation (1.5) as a discrete symplectic system is the invertibility of the matrix S_k. Compare also with [6, Section 1.2], [4, Section 4], [2, Section 4], [1, Section 6]. The key message resulting from this paper is that discrete Jacobi equations (1.5) should be studied within the framework of *discrete symplectic systems* (assuming S_k is invertible). We note that the current literature on discrete symplectic systems is very rich, see [10, 14–17].

In the second part of this section we will study the Jacobi equation

$$[(\underline{R} - \mu\underline{Q})\,\eta^\Delta + (\underline{Q}^T - \mu\underline{P})\,\eta]^\Delta = \underline{P}\,\eta + \underline{Q}\,\eta^\Delta \tag{$\underline{\mathrm{J}}_{\mathrm{cov}}$}$$

and its corresponding symplectic system formulation. Equation ($\underline{\mathrm{J}}_{\mathrm{cov}}$) is obtained from the Jacobi system ($\underline{\mathrm{J}}$) by taking $\underline{\mathcal{A}} = 0$ and $\underline{\mathcal{B}} = I$, or it can be derived directly from the corresponding Euler–Lagrange equation in [41, pg. 228].

Notation 5.7 *(Jacobi system ($\underline{\mathrm{J}}_{\mathrm{cov}}$)).* The matrices $\underline{P}$, $\underline{Q}$, $\underline{R}$ and the vector η in ($\underline{\mathrm{J}}_{\mathrm{cov}}$) have the following properties: $\underline{P}, \underline{Q}, \underline{R} \in \mathbb{R}^{n\times n}$ and these matrices are defined on $[a, \rho(b)]_{\mathbb{T}}$ and $\underline{P}$ and $\underline{R}$ are symmetric; $\eta \in \mathbb{R}^n$ is defined on $[a, b]_{\mathbb{T}}$. Furthermore, we define the matrix $\underline{S} \in \mathbb{R}^{n\times n}$ by

$$\underline{S} := \underline{R} - \mu\,\underline{Q}.$$

Corollary 5.8 (Jacobi ($\underline{\mathrm{J}}_{\mathrm{cov}}$) to symplectic ($\mathbb{S}$)). *Assume that $\underline{P}$, $\underline{Q}$, $\underline{R}$, $\underline{S}$ satisfy the conditions in Notation 5.7 with $\underline{S}$ invertible. Then the Jacobi equation* ($\underline{\mathrm{J}}_{\mathrm{cov}}$) *is the symplectic system* ($\mathbb{S}$)*, whose coefficients*

$$\left.\begin{array}{ll} \underline{\mathbb{A}} = \underline{S}^{-1}(\mu\underline{P} - \underline{Q}^T), & \underline{\mathbb{C}} = \underline{P} + \underline{Q}\,\underline{S}^{-1}(\mu\underline{P} - \underline{Q}^T), \\ \underline{\mathbb{B}} = \underline{S}^{-1}, & \underline{\mathbb{D}} = \underline{Q}\,\underline{S}^{-1} \end{array}\right\} \tag{5.3}$$

with $I + \mu\underline{\mathbb{D}} = \underline{R}\,\underline{S}^{-1}$ satisfy the conditions in Notation 4.3. Thus, the resulting symplectic system ($\mathbb{S}$) *is Hamiltonian if and only if the matrix $\underline{R}$ is invertible.*

Proof. This is a special case of Theorem 4.8. □

Similarly to Corollary 5.3, time scales symplectic systems ($\mathbb{S}$) with $\mathbb{B}$ invertible characterize Jacobi equations ($\underline{\mathrm{J}}_{\mathrm{cov}}$).

Corollary 5.9 (Symplectic ($\mathbb{S}$) to Jacobi ($\underline{\mathrm{J}}_{\mathrm{cov}}$)). *Assume that $\underline{\mathbb{A}}$, $\underline{\mathbb{B}}$, $\underline{\mathbb{C}}$, $\underline{\mathbb{D}}$ satisfy the conditions in Notation 4.3 with $\underline{\mathbb{B}}$ invertible. Then the time scale symplectic system* ($\mathbb{S}$) *is the Jacobi equation* ($\underline{\mathrm{J}}_{\mathrm{cov}}$)*, whose coefficients*

$$\underline{P} = \underline{\mathbb{C}} - \underline{\mathbb{D}}\,\underline{\mathbb{B}}^{-1}\underline{\mathbb{A}}, \quad \underline{Q} = \underline{\mathbb{D}}\,\underline{\mathbb{B}}^{-1}, \quad \underline{R} = (I + \mu\underline{\mathbb{D}})\,\underline{\mathbb{B}}^{-1} \tag{5.4}$$

with $\underline{S} = \underline{\mathbb{B}}^{-1}$ satisfy the conditions in Notation 5.7.

The quadratic form associated with the Jacobi equation ($\underline{\mathrm{J}}_{\mathrm{cov}}$) is

$$\underline{\omega}(\eta) := \eta^T \underline{P}\, \eta + 2\, \eta^T \underline{Q}\, \eta^\Delta + (\eta^\Delta)^T \underline{R}\, \eta^\Delta.$$

Its relation with the quadratic form $\underline{\mathbb{Q}}(\eta, q)$ is shown in the following statement.

Corollary 5.10 (Quadratic forms for ($\underline{\mathrm{J}}_{\mathrm{cov}}$) and ($\underline{\mathbb{S}}$)). *Assume that*

(i) *either $\underline{P}$, $\underline{Q}$, $\underline{R}$, $\underline{S}$ satisfy the conditions in Notation 5.7 with $\underline{S}$ invertible and $\underline{\mathbb{A}}$, $\underline{\mathbb{B}}$, $\underline{\mathbb{C}}$, $\underline{\mathbb{D}}$ are given by* (5.3)*,*

(ii) *or $\underline{\mathbb{A}}$, $\underline{\mathbb{B}}$, $\underline{\mathbb{C}}$, $\underline{\mathbb{D}}$ satisfy the conditions in Notation 4.3 with $\underline{\mathbb{B}}$ invertible and $\underline{P}$, $\underline{Q}$, $\underline{R}$ are given by* (5.4)*.*

If (η, q) is $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible, then $\underline{\omega}(\eta) = \underline{\mathbb{Q}}(\eta, q)$. Conversely, for any $\eta \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a, b]_{\mathbb{T}}$ the pair (η, q) with $q := \underline{S}\eta^\Delta - (\mu \underline{P} - \underline{Q}^T)\, \eta = \underline{\mathbb{B}}^{-1}(\eta^\Delta - \underline{\mathbb{A}}\eta)$ is $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible and $\underline{\mathbb{Q}}(\eta, q) = \underline{\omega}(\eta)$.

In Propositions 4.14 and 4.15 we presented transformations between the Jacobi systems (J^σ) and ($\underline{\mathrm{J}}$). If we specialize these transformations to the calculus of variations setting, then we obtain the following.

Corollary 5.11 (Jacobi ($\underline{\mathrm{J}}_{\mathrm{cov}}$) to Jacobi ($\mathrm{J}^\sigma_{\mathrm{cov}}$)). *Assume that the matrices $\underline{P}$, $\underline{Q}$, $\underline{R}$, $\underline{S}$ satisfy the conditions in Notation 5.7, in particular, $\underline{P}$ and $\underline{R}$ are symmetric and $\underline{S}$ is invertible. Then the Jacobi system ($\underline{\mathrm{J}}_{\mathrm{cov}}$) is the Jacobi system ($\mathrm{J}^\sigma_{\mathrm{cov}}$), whose coefficients*

$$P := \underline{P}, \quad Q := \underline{Q} - \mu \underline{P}, \quad R := \underline{R} - \mu \underline{Q} - \mu \underline{Q}^T + \mu^2 \underline{P} \tag{5.5}$$

with $S = \underline{S}$ satisfy the conditions in Notation 5.1.

Corollary 5.12 (Jacobi ($\mathrm{J}^\sigma_{\mathrm{cov}}$) to Jacobi ($\underline{\mathrm{J}}_{\mathrm{cov}}$)). *Assume that the matrices P, Q, R, S satisfy the conditions in Notation 5.1, in particular, P and R are symmetric and S is invertible. Then the Jacobi system ($\mathrm{J}^\sigma_{\mathrm{cov}}$) is the Jacobi system ($\underline{\mathrm{J}}_{\mathrm{cov}}$), whose coefficients*

$$\underline{P} := P, \quad \underline{Q} := \mu P + Q, \quad \underline{R} := R + \mu Q^T + \mu Q + \mu^2 P \tag{5.6}$$

with $\underline{S} = S$ satisfy the conditions in Notation 5.7.

Remark 5.13. (i) By using the transformations in Corollaries 5.11 and 5.12, the above results about the Jacobi equation ($\underline{\mathrm{J}}_{\mathrm{cov}}$) and the quadratic form $\underline{\omega}(\eta)$ can be obtained form the corresponding results on the Jacobi equation ($\mathrm{J}^\sigma_{\mathrm{cov}}$) and the quadratic form $\omega(\eta)$. In particular, the identity $\omega(\eta) = \underline{\omega}(\eta)$ holds for any function $\eta \in \mathrm{C}^1_{\mathrm{prd}}$, which can be also verified by a direct calculation.

(ii) The results in Corollaries 5.3 and 5.9 show that once the coefficient $\mathbb{B}$ in the symplectic system is invertible, then this symplectic system can be written as either the Jacobi equation ($\mathrm{J}^\sigma_{\mathrm{cov}}$) or the Jacobi equation ($\underline{\mathrm{J}}_{\mathrm{cov}}$). On the other hand, both Jacobi equations ($\mathrm{J}^\sigma_{\mathrm{cov}}$) and ($\underline{\mathrm{J}}_{\mathrm{cov}}$) can be written as time scale symplectic systems with $\mathbb{B}$ invertible. So from this point of view are Jacobi equations ($\mathrm{J}^\sigma_{\mathrm{cov}}$) and ($\underline{\mathrm{J}}_{\mathrm{cov}}$) equivalent, and every result pertaining one form can be transformed into a result for the other form.

Remark 5.14. The matrices T, V and $\underline{T}$, $\underline{V}$ defined in (2.2) and (4.3) have now the form

$$T = R^{-1}S, \quad V = S^{-1}R, \quad \underline{T} = \underline{S}\underline{R}^{-1}, \quad \underline{V} = \underline{R}\underline{S}^{-1}.$$

Therefore, the results of Lemmas 3.2 and 4.9 are in this case straightforward.

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