International Journal of Difference Equations
ISSN 0973-6069, Volume 4, Number 2, pp. 273–281 (2009)
`http://campus.mst.edu/ijde`

Periodicity for a Semi-ratio-dependent Predator-prey System with Delays on Time Scales

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Abstract

In this paper, the semi-ratio-dependent predator-prey system with nonmonotonic functional response on time scales is investigated. By using the coincidence degree theory, sufficient conditions for existence of periodic solutions are obtained.

AMS Subject Classifications: 92D25, 34C25.
Keywords: Periodic solution, semi-ratio-dependent, predator-prey system, coincidence degree, time scales.

1 Introduction

Recently, many researchers have concentrated on semi-ratio-dependent predator-prey systems with functional responses [4, 8, 10] of the form

$$\begin{cases} \dot{x}_1(t) = x_1(t)\left[r_1(t) - a_{11}(t)x_1(t-\tau(t))\right] - f(x_1(t))x_2(t), \\ \dot{x}_2(t) = x_2(t)\left[r_2(t) - \dfrac{a_{21}(t)x_2(t)}{x_1(t)}\right], \end{cases} \tag{1.1}$$

where x_1 and x_2 denotes the density of the prey and predator, respectively. $f(x_1)$ is the so-called predator functional response to prey, and the density of predator is in proportion to that of the prey. In [4], the simplified Monod–Haldane function of the form $f(x_1) = \dfrac{a_{12}(t)x_1}{m^2 + x_1^2}$ was considered. Moreover, time delay is usually important to the

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Received April 28, 2008; Accepted December 4, 2008
Communicated by Martin Bohner

dynamics of differential equations, so we consider the system

$$\begin{cases} \dot{x}_1(t) = x_1(t)\left[r_1(t) - a_{11}(t)x_1(t-\tau(t)) - \dfrac{a_{12}(t)x_2(t)}{m^2 + x_1^2(t)}\right], \\ \dot{x}_2(t) = x_2(t)\left[r_2(t) - \dfrac{a_{21}(t)x_2(t)}{x_1(t-\varrho(t))}\right]. \end{cases} \tag{1.2}$$

If $\varrho(t) = 0$, then system (1.2) was investigated in [4]. Motivated by [9], we can obtain the following discrete analogy of system (1.2), which is governed by difference equations with periodic coefficients:

$$\begin{cases} x_1(n+1) = x_1(n)\exp\left\{r_1(n) - a_{11}(n)x_1(n-\tau(n)) - \dfrac{a_{12}(n)x_2(n)}{m^2 + x_1^2(n)}\right\}, \\ x_2(n+1) = x_2(n)\exp\left\{r_2(n) - \dfrac{a_{21}(n)x_2(n)}{x_1(n-\varrho(n))}\right\}. \end{cases} \tag{1.3}$$

As we know, it is similar to explore the existence of periodic solutions for (1.2) and (1.3) in the approaches, the methods and the main results. So it is unnecessary to study the periodic solutions in separate ways. By using the theory of time scales, which was first proposed by Stefan Hilger [7], we can unify the existence of periodic solutions of population dynamics modeled by differential equations and difference equations. For this reason, we consider the system of dynamic equations on time scales

$$\begin{cases} u_1^{\Delta}(t) = r_1(t) - a_{11}(t)e^{u_1(t-\tau(t))} - \dfrac{a_{12}(t)e^{u_2(t)}}{m^2 + e^{u_1^2(t)}}, \\ u_2^{\Delta}(t) = r_2(t) - \dfrac{a_{21}(t)e^{u_2(t)}}{e^{u_1(t-\varrho(t))}}, \end{cases} \tag{1.4}$$

where $r_1(t)$, $r_2(t)$, $a_{11}(t)$, $a_{12}(t)$, and $a_{21}(t)$ are rd-continuous positive ω-periodic functions on time scales $\mathbb{T}$. Set $y_i(t) = e^{u_i(t)}$, $i = 1, 2$. If $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, then system (1.4) can be derived to (1.2) and (1.3) respectively. The primary aim of this paper is to explore the existence of periodic solutions for dynamics equations on time scales. The approach is based on the coincidence degree theory, such as [1, 2, 5].

The remainder of this paper is organized as follows. In the following section, some preliminary results about calculus on time scales and continuation theorem are stated. The existence of periodic solutions for system (1.4) is established in Section 3.

2 Preliminaries

For convenience, we first present some basic definitions and lemmas about time scales and the continuation theorem of the coincidence degree theory; more details can be found in [3, 6]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers

$\mathbb{R}$. Throughout this paper, we assume that the time scale $\mathbb{T}$ is unbounded above and below, such as $\mathbb{R}$, $\mathbb{Z}$ and $\bigcup_{k\in\mathbb{Z}}[2k, 2k+1]$. The following definitions and lemmas about time scales are from [3].

Definition 2.1. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, +\infty)$ are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

If $\sigma(t) = t$, then t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left-dense (otherwise: left-scattered).

Definition 2.2. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

In this case, $f^{\Delta}(t)$ is called the delta (or Hilger) derivative of f at t. Moreover, f is said to be delta or Hilger differentiable on $\mathbb{T}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$. Then we define

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for } r, s \in \mathbb{T}.$$

Definition 2.3. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{\mathrm{rd}}(\mathbb{T})$.

Lemma 2.4. *Every rd-continuous function has an antiderivative.*

Lemma 2.5. *If $a, b \in \mathbb{T}$, α, $\beta \in \mathbb{R}$ and $f, g \in C_{\mathrm{rd}}(\mathbb{T})$, then*

(a) $\displaystyle\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$*;*

(b) *if $f(t) \geq 0$ for all $a \leq t < b$, then $\displaystyle\int_a^b f(t)\Delta t \geq 0$;*

(c) *if $|f(t)| \leq g(t)$ on $[a, b) := \{t \in \mathbb{T} : a \leq t < b\}$, then $\displaystyle\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t$.*

Lemma 2.6 (see [1])**.** *Let t_1, $t_2 \in I_\omega$ and $t \in \mathbb{T}$. If $g : \mathbb{T} \to \mathbb{R}$ is ω-periodic, then*

$$g(t) \leq g(t_1) + \int_k^{k+\omega} |g^{\Delta}(s)|\Delta s \quad \text{and} \quad g(t) \geq g(t_2) - \int_k^{k+\omega} |g^{\Delta}(s)|\Delta s.$$

For simplicity, we use the following notations throughout this paper. Let $\mathbb{T}$ be ω-periodic, that is $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$,

$$k = \min\{\mathbb{R}^+ \cap \mathbb{T}\}, \quad I_\omega = [k, k + \omega] \cap \mathbb{T}, \quad g^L = \inf_{t\in\mathbb{T}} g(t),$$

$$g^M = \sup_{t\in\mathbb{T}} g(t), \quad \bar{g} = \frac{1}{\omega} \int_{I_\omega} g(s)\Delta s = \frac{1}{\omega} \int_k^{k+\omega} g(s)\Delta s,$$

where $g \in C_{\mathrm{rd}}(\mathbb{T})$ is an ω-periodic real function, i.e., $g(t + \omega) = g(t)$ for all $t \in \mathbb{T}$.

Now, we introduce some concepts and a useful result from [6]. Let X, Z be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \ker L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \ker L$, $\operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$, then it follows that $L| \operatorname{Dom} L \cap \ker P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$.

Next, we state Mawhin's continuation theorem, which is a main tool in the proof of our theorem.

Lemma 2.7. *Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Suppose*

(a) *for each $\lambda \in (0, 1)$, every solution u of $Lu = \lambda N u$ is such that $u \notin \partial\Omega$;*

(b) *$QNu \neq 0$ for each $u \in \partial\Omega \cap \ker L$ and the Brouwer degree* $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$.

Then the operator equation $Lu = Nu$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

3 Existence of Periodic Solutions

Theorem 3.1. *If the assumption*

$$\bar{r}_1 m^2 - \bar{a}_{12} e^{M_2} > 0$$

holds, where $M_2 = \ln \dfrac{\bar{r}_2 \bar{r}_1}{\bar{a}_{21} \bar{a}_{11}} + 2\omega(\bar{r}_1 + \bar{r}_2)$, then system (1.4) *has at least one ω-periodic solution.*

Proof. Let

$$X = Z = \left\{(u_1, u_2)^T \in C(\mathbb{T}, \mathbb{R}^2) : u_i(t + \omega) = u_i(t),\ i = 1, 2, \forall t \in \mathbb{T}\right\},$$

$$\|(u_1, u_2)^T\| = \sum_{i=1}^{2} \max_{t\in I_\omega} |u_i(t)|, \quad (u_1, u_2)^T \in X \text{ (or in } Z).$$

Then X and Z are both Banach spaces when they are endowed with the above norm $\|\cdot\|$. Let

$$N\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}N_1\\N_2\end{bmatrix}=\begin{bmatrix}r_1(t)-a_{11}(t)e^{u_1(t-\tau(t))}-\dfrac{a_{12}(t)e^{u_2(t)}}{m^2+e^{u_1^2(t)}}\\ r_2(t)-\dfrac{a_{21}(t)e^{u_2(t)}}{e^{u_1(t-\varrho(t))}}\end{bmatrix},$$

$$L\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}u_1^\Delta\\u_2^\Delta\end{bmatrix},\quad P\begin{bmatrix}u_1\\u_2\end{bmatrix}=Q\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}\dfrac{1}{\omega}\displaystyle\int_k^{k+\omega}u_1(t)\Delta t\\ \dfrac{1}{\omega}\displaystyle\int_k^{k+\omega}u_2(t)\Delta t\end{bmatrix}.$$

Obviously,

$$\begin{aligned}\ker L&=\big\{(u_1,u_2)^T\in X:(u_1(t),u_2(t))^T=(h_1,h_2)^T\in\mathbb{R}^2,t\in\mathbb{T}\big\},\\ \operatorname{Im}L&=\big\{(u_1,u_2)^T\in Z:\bar{u}_1=\bar{u}_2=0,t\in\mathbb{T}\big\},\\ \dim\ker L&=2=\operatorname{codim}\operatorname{Im}L.\end{aligned}$$

Since $\operatorname{Im}L$ is closed in Z, L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projections such that $\operatorname{Im}P=\ker L$ and $\operatorname{Im}L=\ker Q=\operatorname{Im}(I-Q)$. Furthermore, the generalized inverse (of L) $K_P:\operatorname{Im}L\to\ker P\cap\operatorname{Dom}L$ exists and is given by

$$K_P\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}\displaystyle\int_k^t u_1(s)\Delta s-\frac{1}{\omega}\int_k^{k+\omega}\int_k^t u_1(s)\Delta s\Delta t\\ \displaystyle\int_k^t u_2(s)\Delta s-\frac{1}{\omega}\int_k^{k+\omega}\int_k^t u_2(s)\Delta s\Delta t\end{bmatrix}.$$

Thus

$$QN\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}\dfrac{1}{\omega}\displaystyle\int_k^{k+\omega}\left(r_1(t)-a_{11}(t)e^{u_1(t-\tau(t))}-\frac{a_{12}(t)e^{u_2(t)}}{m^2+e^{u_1^2(t)}}\right)\Delta t\\ \dfrac{1}{\omega}\displaystyle\int_k^{k+\omega}\left(r_2(t)-\frac{a_{21}(t)e^{u_2(t)}}{e^{u_1(t-\varrho(t))}}\right)\Delta t\end{bmatrix}$$

and

$$\begin{aligned}&K_P(I-Q)N\begin{bmatrix}u_1\\u_2\end{bmatrix}\\ &=\begin{bmatrix}\displaystyle\int_k^t u_1(s)\Delta s-\frac{1}{\omega}\int_k^{k+\omega}\int_k^t u_1(s)\Delta s\Delta t-\left(t-k-\frac{1}{\omega}\int_k^{k+\omega}(t-k)\Delta t\right)\bar{u}_1\\ \displaystyle\int_k^t u_2(s)\Delta s-\frac{1}{\omega}\int_k^{k+\omega}\int_k^t u_2(s)\Delta s\Delta t-\left(t-k-\frac{1}{\omega}\int_k^{k+\omega}(t-k)\Delta t\right)\bar{u}_2\end{bmatrix}.\end{aligned}$$

Clearly, QN and $K_P(I-Q)N$ are continuous. According to the Arzela–Ascoli theorem, it is not difficult to show that $K_P(I-Q)N(\bar{\Omega})$ is compact for any open bounded set $\Omega \subset X$ and $QN(\bar{\Omega})$ is bounded. Thus, N is L-compact on $\bar{\Omega}$.

Now, we shall search an appropriate open bounded subset Ω for the application of the continuation theorem, Lemma 2.7. For the operator equation $Lu = \lambda Nu$, where $\lambda \in (0,1)$, we have

$$\begin{cases} u_1^{\Delta}(t) = \lambda\left(r_1(t) - a_{11}(t)e^{u_1(t-\tau(t))} - \dfrac{a_{12}(t)e^{u_2(t)}}{m^2 + e^{u_1^2(t)}}\right), \\ u_2^{\Delta}(t) = \lambda\left(r_2(t) - \dfrac{a_{21}(t)e^{u_2(t)}}{e^{u_1(t-\varrho(t))}}\right). \end{cases} \tag{3.1}$$

Assume that $(u_1,u_2)^T \in X$ is a solution of system (3.1) for a certain $\lambda \in (0,1)$. Integrating (3.1) on both sides from k to $k+\omega$, we obtain

$$\begin{cases} \bar{r}_1\omega = \displaystyle\int_k^{k+\omega} a_{11}(t)e^{u_1(t-\tau(t))}\Delta t + \int_k^{k+\omega} \frac{a_{12}(t)e^{u_2(t)}}{m^2 + e^{u_1^2(t)}}\Delta t, \\ \bar{r}_2\omega = \displaystyle\int_k^{k+\omega} \frac{a_{21}(t)e^{u_2(t)}}{e^{u_1(t-\varrho(t))}}\Delta t. \end{cases} \tag{3.2}$$

Since $(u_1,u_2)^T \in X$, there exist $\xi_i, \eta_i \in [k, k+\omega]$, $i = 1,2$, such that

$$u_i(\xi_i) = \min_{t\in[k,k+\omega]}\{u_i(t)\}, \quad u_i(\eta_i) = \max_{t\in[k,k+\omega]}\{u_i(t)\}, \quad i = 1,2. \tag{3.3}$$

From (3.1) and (3.2), we have

$$\int_k^{k+\omega} \left|u_1^{\Delta}(t)\right| \Delta t < \bar{r}_1\omega + \int_k^{k+\omega} a_{11}(t)e^{u_1(t-\tau(t))}\Delta t + \int_k^{k+\omega} \frac{a_{12}(t)e^{u_2(t)}}{m^2 + e^{u_1^2(t)}}\Delta t = 2\bar{r}_1\omega,$$

$$\int_k^{k+\omega} \left|u_2^{\Delta}(t)\right| \Delta t < \bar{r}_2\omega + \int_k^{k+\omega} \frac{a_{21}(t)e^{u_2(t)}}{e^{u_1(t-\varrho(t))}}\Delta t = 2\bar{r}_2\omega.$$

From the first equation of (3.2) and (3.3), we have

$$\bar{r}_1\omega > \bar{a}_{11}\omega e^{u_1(\xi_1)}$$

and

$$u_1(\xi_1) < \ln\frac{\bar{r}_1}{\bar{a}_{11}} =: l_1,$$

thus,

$$u_1(t) \le u_1(\xi_1) + \int_k^{k+\omega} |u_1^{\Delta}(t)|\Delta t < \ln\frac{\bar{r}_1}{\bar{a}_{11}} + 2\bar{r}_1\omega =: M_1.$$

On the other hand, from the second equation of (3.2) and (3.3), we have

$$\bar{r}_2\omega \ge \bar{a}_{21}\omega\frac{e^{u_2(\xi_2)}}{e^{M_1}}$$

and

$$u_2(\xi_2) \le \ln \frac{\bar{r}_2 e^{M_1}}{\bar{a}_{21}} =: l_2,$$

so,

$$u_2(t) \le u_2(\xi_2) + \int_k^{k+\omega} |u_2^{\Delta}(t)|\Delta t \le \ln \frac{\bar{r}_2 e^{M_1}}{\bar{a}_{21}} + 2\bar{r}_2\omega =: M_2.$$

By the first equation of (3.2) and (3.3),

$$\bar{r}_1\omega \le \bar{a}_{11}\omega e^{u_1(\eta_1)} + \bar{a}_{12}\omega \frac{e^{M_2}}{m^2}$$

and

$$u_1(\eta_1) \ge \ln \frac{\bar{r}_1 m^2 - \bar{a}_{12} e^{M_2}}{\bar{a}_{11} m^2} =: L_1,$$

so we have

$$u_1(t) \ge u_1(\eta_1) - \int_k^{k+\omega} |u_1^{\Delta}(t)|\Delta t \ge \ln \frac{\bar{r}_1 m^2 - \bar{a}_{12} e^{M_2}}{\bar{a}_{11} m^2} - 2\bar{r}_1\omega =: M_3.$$

From the second equation of (3.2) and (3.3), we have

$$\bar{r}_2\omega \le \bar{a}_{21}\omega e^{u_2(\eta_2) - M_3}$$

and

$$u_2(\eta_2) \ge \ln \frac{\bar{r}_2 e^{M_3}}{\bar{a}_{21}} =: L_2,$$

thus,

$$u_2(t) \ge u_2(\eta_2) - \int_k^{k+\omega} |u_2^{\Delta}(t)|\Delta t \ge \ln \frac{\bar{r}_2 e^{M_3}}{\bar{a}_{21}} - 2\bar{r}_2\omega =: M_4.$$

So, we have

$$\max_{t\in[k,k+\omega]} |u_1(t)| \le \max\{|M_1|, |M_3|\} =: R_1,$$
$$\max_{t\in[k,k+\omega]} |u_2(t)| \le \max\{|M_2|, |M_4|\} =: R_2.$$

Clearly, R_1 and R_2 are independent of λ. Let $R = R_1 + R_2 + R_0$, where R_0 is taken sufficiently large such that $R_0 \ge |l_1| + |l_2| + |L_1| + |L_2|$. Now, we consider the algebraic equations

$$\begin{cases} \bar{r}_1 - \bar{a}_{11} e^{x} - \dfrac{\bar{a}_{12} e^{y}}{m^2 + e^{x^2}} = 0, \\ \bar{r}_2 - \bar{a}_{21} e^{y-x} = 0. \end{cases} \tag{3.4}$$

Every solution $(x^*, y^*)^T$ of (3.4) satisfies $\|(x^*, y^*)^T\| < R$. Now, we define

$$\Omega = \{(u_1(t), u_2(t))^T \in X, \|(u_1(t), u_2(t))^T\| < R\}.$$

Then it is clear that Ω verifies the requirement (a) of Lemma 2.7. If

$$(u_1(t), u_2(t))^T \in \partial\Omega \cap \ker L = \partial\Omega \cap \mathbb{R}^2,$$

then $(u_1(t), u_2(t))^T$ is a constant vector in $\mathbb{R}^2$ with $\|(u_1(t), u_2(t))^T\| = |u_1| + |u_2| = R$, so we have

$$QN \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By direct computation, we can obtain $\deg(JQN, \Omega \cap \ker L, 0) = 1 \neq 0$. By now, we have verified that Ω fulfills all requirements of Lemma 2.7; therefore, system (1.4) has at least one ω-periodic solution in $\operatorname{Dom} L \cap \bar{\Omega}$. The proof is complete. □

Remark 3.2. If $\mathbb{T} = \mathbb{R}$, then system (1.2) is the special case of (1.4). So our result is more general than that of [4]. Further, If $\mathbb{T} = \mathbb{Z}$, then the existence of periodic solution for system (1.3) is established.

Remark 3.3. By Theorem 3.1, we know that system (1.4) has at least one periodic solution with the same period as the parameters under certain condition. Besides, time delays do not change the periodicity of the dynamic equations.

Acknowledgements

This paper is supported by Natural Science Foundation of the Higher Education Institutions of Anhui Province (No. KJ2009B076Z).

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