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Necessary and Sufficient Conditions for Oscillation of Certain Higher Order Partial Difference Equations

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Abstract

In this paper, some necessary and sufficient conditions for the oscillation of the higher order partial difference equation of the form

$$\Delta_n^h \Delta_m^r A_{m,n} + (-1)^{h+r+1} p A_{m-k,n-l} = 0, \qquad (m,n) \in \mathbb{N}_0^2$$

are established, where $k, l \in \mathbb{N}_0$, $h, r \in \mathbb{N}_1$, p is a nonnegative real number.

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1 Introduction

Partial difference equations are difference equations that involve functions with two or more independent variables. Recently, there are many papers that devoted to the development of qualitative theory of difference equations [5–7, 9]. Their significance is illustrated in applications involving random walk problems, molecular structure problems and numerical difference approximation problems etc.

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In this paper, we consider the higher order partial difference equations of the form

$$\Delta_n^h \Delta_m^r A_{m,n} + (-1)^{h+r+1} p A_{m-k,n-l} = 0, \tag{1.1}$$

where $m, n, k, l \in \mathbb{N}_0$, $r, h \in \mathbb{N}_1$, $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$, p is a nonnegative real number. The forward partial differences Δ_m and Δ_n are defined as usual, i.e., $\Delta_m A_{m,n} = A_{m+1,n} - A_{m,n}$ and $\Delta_n A_{m,n} = A_{m,n+1} - A_{m,n}$. The higher order partial differences for any positive integers r and h are defined as $\Delta_m^r A_{m,n} = \Delta_m(\Delta_m^{r-1} A_{m,n})$, $\Delta_m^0 A_{m,n} = A_{m,n}$, $\Delta_n^h A_{m,n} = \Delta_n(\Delta_n^{h-1} A_{m,n})$ and $\Delta_n^0 A_{m,n} = A_{m,n}$.

There are several works about qualitative theory of higher order partial difference equations. B. G. Zhang and S. T. Liu [7,8] studied the oscillatory behaviour of solutions of the partial difference equations of the forms

$$A_{m+k,n+l} + \sum_{i=1}^{s_1} \sum_{j=1}^{s_2} q_{i,j} A_{m+k-i,n+l-j} = 0$$

and

$$A_{m,n} = \sum_{i=1}^{u} p_i A_{m-k_i,n-l_i} + \sum_{j=1}^{\nu} q_j A_{m+\tau_j,n+\sigma_j}.$$

B. G. Zhang, Y. Zhou and Y. Q. Huang [10] investigated existence of positive solutions for nonlinear higher order neutral partial difference equations of the form

$$\Delta_n^h \Delta_m^r (A_{m,n} - c A_{m-k,n-l}) + (-1)^{h+r+1} p_{m,n} f(A_{m-\tau,n-\sigma}) = 0.$$

In 2007, C. F. Li and Y. Zhou [2] studied existence of bounded and unbounded nonoscillatory solutions for partial difference equations of the form

$$\Delta_n^h \Delta_m^r (A_{m,n} + c_{m,n} A_{m-k,n-l}) + p_{m,n} A_{m-\sigma_1,n-\rho_1} - q_{m,n} A_{m-\sigma_2,n-\rho_2} = 0.$$

Ch. G. Philos and Y. G. Sficas [3] studied the oscillation of the ordinary difference equations of the form

$$(-1)^{m+1} \Delta^m A_n + \sum_{k=0}^{\infty} p_k A_{n-l_k} = 0.$$

By a solution of (1.1), we mean a nontrivial double sequence $\{A_{m,n}\}$ which is defined for $m \geq -k$ and $n \geq -l$ and satisfies (1.1) for $m \geq 0$, $n \geq 0$. A solution $\{A_{m,n}\}$ of (1.1) is said to be eventually positive (or negative) if $A_{m,n} > 0$ (or $A_{m,n} < 0$) for all large m and n. It is said to be oscillatory if it is neither eventually positive nor eventually negative.

A solution $\{A_{i,j}\}$ of (1.1) is called to be proper, if there exist positive numbers M, α and β such that

$$|A_{m,n}| \leq M \alpha^m \beta^n, \tag{1.2}$$

for all large m and n.

The set $\Omega = \mathbb{N}_{-k} \times \mathbb{N}_{-l}/\mathbb{N}_1 \times \mathbb{N}_1$ is called the initial domain. A function $\phi_{i,j}$ defined on Ω is called the initial function. It is easy to construct by induction a double sequence $\{A_{i,j}\}$ which equals $\phi_{i,j}$ on Ω and satisfies (1.1) on $\mathbb{N}_0 \times \mathbb{N}_0$. It is not difficult to prove that if the initial data satisfy

$$|\phi_{m,n}| \leq M_1 \alpha^m \beta^n,\ (m,n) \in \Omega, \tag{1.3}$$

for some positive numbers M_1, α and β, then the corresponding solution is proper.

We look for the solutions of the form

$$A_{m,n} = \lambda^m \mu^n, \tag{1.4}$$

where λ and μ are complex numbers. Substituting (1.4) into (1.1), we obtain the characteristic equation

$$\Phi(\lambda,\mu) = (\lambda-1)^r (\mu-1)^h + (-1)^{h+r+1} p\lambda^{-k}\mu^{-l} = 0 \tag{1.5}$$

or

$$\Phi(\lambda,\mu) = \sum_{i=0}^{r}\sum_{j=0}^{h} (-1)^{i+j} \binom{r}{i}\binom{h}{j} \lambda^{r-i}\mu^{h-j} + (-1)^{h+r+1} p\lambda^{-k}\mu^{-l} = 0.$$

(λ,μ) is said to be positive root of equation (1.5), if it satisfies Equation (1.5); moreover, $\lambda > 0$ and $\mu > 0$.

2 Some Auxiliary Lemmas

Lemma 2.1 (See [9]). *Assume that there exist positive constants M_1, M, and N such that*

$$|A_{m,n}| \leq M_1 r_1^m r_2^n,\ m \geq M,\ n \geq N.$$

Then the z-transform of $\{A_{m,n}\}$, which is defined by $Z(A_{m,n}) = \sum_{m,n=0}^{\infty} A_{m,n} z_1^{-m} z_2^{-n}$, exists in the region $|z_1| > r_1$ and $|z_2| > r_2$.

In the following, we always assume that $A_{m,n} = 0$ for $m < 0$ and $n < 0$ in the series of

$$\sum_{m=p}^{\infty}\sum_{n=q}^{\infty} A_{m,n} z_1^{-m} z_2^{-n}.$$

By direct calculations, we can prove the following lemma.

Lemma 2.2 (See [9]). *The following formulas are true:*

(i)
$$Z(A_{m-k,n-l}) = z_1^{-k} z_2^{-l} F(z_1, z_2).$$

(ii)
$$\sum_{i=0}^{\infty} F(k+i, z_2) z_1^{-i} = z_1^k \left(F(z_1, z_2) - \sum_{m=0}^{k-1} F(m, z_2) z_1^{-m} \right),$$

where $F(k+i, z_2) = \sum_{n=0}^{\infty} A_{k+i,n} z_2^{-n}$.

(iii)
$$\sum_{i=0}^{\infty} \sum_{n=0}^{l-1} A_{k+i,n} z_1^{-i} z_2^{-n} = z_1^k \left(\sum_{m=0}^{\infty} \sum_{n=0}^{l-1} A_{m,n} z_1^{-m} z_2^{-n} - \sum_{m=0}^{k-1} \sum_{n=0}^{l-1} A_{m,n} z_1^{-m} z_2^{-n} \right).$$

(iv)
$$\sum_{m=0}^{\infty} \sum_{n=0}^{l-1} A_{m,n} z_1^{-m} z_2^{-n} = \sum_{i=0}^{l-1} F(z_1, i) z_2^{-i},$$

where $F(z_1, n) = \sum_{m=0}^{\infty} A_{m,n} z_1^{-m}$.

(v)
$$\begin{aligned} Z(A_{m+k,n+l}) =& z_1^k z_2^l \left(F(z_1, z_2) - \sum_{m=0}^{k-1} F(m, z_2) z_1^{-m} - \sum_{n=0}^{l-1} F(z_1, n) z_2^{-n} \right. \\ & \left. + \sum_{m=0}^{k-1} \sum_{n=0}^{l-1} A_{m,n} z_1^{-m} z_2^{-n} \right). \end{aligned}$$

3 Main Results

Theorem 3.1. *Every proper solution* $\{A_{m,n}\}$ *of* (1.1) *is oscillatory if and only if its characteristic equation* (1.5) *has no positive roots.*

Proof. Necessity. Otherwise, let (λ_0, μ_0) be a positive root of (1.5). Then it is easy to find that $\{A_{m,n}\}$ with $A_{m,n} = \lambda_0^m \mu_0^n$ is a positive proper solution of (1.1), a contradiction.

Sufficiency. Assume that (1.5) has no positive roots. Let $\{A_{m,n}\}$ be a positive proper solution of (1.1) with the initial data $\phi_{m,n}$ such that $|\phi_{m,n}| < c$. Then, by induction, it is easy to find that there exist $b > 0$ such that

$$|A_{m,n}| < bc^{m+n}, \ (m, n) \in \mathbb{N}_0^2. \tag{3.1}$$

Thus, by Lemma 2.1, for $|z_i| > c$, $i = 1, 2$, the z-transform of $\{A_{m,n}\}$

$$Z(A_{m,n}) = \sum_{m,n=0}^{\infty} A_{m,n} z_1^{-m} z_2^{-n} = F(z_1, z_2) \tag{3.2}$$

exists. By taking the z-transform of both sides of (1.1), we obtain

$$\Phi(z_1, z_2) F(z_1 z_2) = \Psi(z_1, z_2), \tag{3.3}$$

$|z_i| > c$, $i = 1, 2$, where

$$\Phi(z_1, z_2) = \sum_{i=0}^{r} \sum_{j=0}^{h} (-1)^{i+j} \binom{r}{i} \binom{h}{j} z_1^{r-i} z_2^{h-j} + (-1)^{h+r+1} p z_1^{-k} z_2^{-l}$$

and

$$\begin{aligned}\Psi(z_1, z_2) = &\sum_{i=0}^{r} \sum_{j=0}^{h} (-1)^{i+j} \binom{r}{i} \binom{h}{j} z_1^{r-i} z_2^{h-j} \\ &\times \left(\sum_{n=0}^{\infty} \sum_{m=0}^{r-i-1} A_{m,n} z_1^{-m} z_2^{-n} + \sum_{m=0}^{\infty} \sum_{n=0}^{h-j-1} A_{m,n} z_1^{-m} z_2^{-n} \right. \\ &\left. - \sum_{m=0}^{r-i-1} \sum_{n=0}^{h-j-1} A_{m,n} z_1^{-m} z_2^{-n} \right).\end{aligned}$$

We write (3.3) in the form

$$\Phi(1/z_1, 1/z_2) F(1/z_1, 1/z_2) = \Psi(1/z_1, 1/z_2). \tag{3.4}$$

Set

$$\omega(z_1, z_2) = F(1/z_1, 1/z_2) = \sum_{m,n=0}^{\infty} A_{m,n} z_1^m z_2^n. \tag{3.5}$$

Equation (3.5) has radius of convergence ρ_i, $i = 1, 2$. That is, (3.4) holds for $|z_i| < \rho_i$, $i = 1, 2$. Equivalently, (3.3) holds for $|z_i| > 1/\rho_i$, $i = 1, 2$. It is known that a power series with positive coefficients having radius of convergence ρ_i, $i = 1, 2$ has a singularity at $z_i = \rho_i$, $i = 1, 2$ [1]. By condition $\Phi(z_1, z_2) \neq 0$ for $(z_1, z_2) \in (0, \infty) \times (0, \infty)$. Thus $\Phi(1/\rho_1, 1/\rho_2) \neq 0$, and hence,

$$\omega(z_1, z_2) = \frac{\Psi(1/z_1, 1/z_2)}{\Phi(1/z_1, 1/z_2)}$$

is analytic in the region $|z_1 - \rho_1| < d_1$ and $|z_2 - \rho_2| < d_2$, which contradicts the singularity of $\omega(z_1, z_2)$ at $z_i = \rho_i$, $i = 1, 2$. Therefore we must have $\rho_i = \infty$, $i = 1, 2$, i.e., (3.3) holds for $|z_i| > 0, i = 1, 2$, which leads to $A_{m,n} = 0$ for all large m and n. Otherwise, the left-hand side of (3.3) does not equal the right-hand side. This contradiction finishes the proof. □

From Theorem 3.1, we can derive some sufficient and necessary conditions for oscillation of (1.1).

Theorem 3.2. *Assume that* $p > 0$. *Then every proper solution of* (1.1) *oscillates if and only if*

$$p\frac{(k+r)^{k+r}(l+h)^{l+h}}{k^k l^l r^r h^h} > 1. \tag{3.6}$$

Proof. Necessity. It is sufficient to prove that if (3.6) does not hold, then (1.1) has a positive solution.

(i) Let $r+h$ be odd. Obviously, if (3.6) does not hold, then in view of (1.5), we get

$$\Phi\left(1, l/(l+h)\right) > 0$$

and

$$\Phi\left(k/(k+r), l/(l+h)\right) = \frac{r^r h^h}{(k+r)^r (l+h)^h} \times \left[-1 + p\frac{(k+r)^{k+r}(l+h)^{l+h}}{k^k l^l r^r h^h}\right] \leq 0.$$

(ii) Let $r+h$ be even. If (3.6) does not hold, then we get

$$\Phi\left(1, l/(l+h)\right) < 0$$

and

$$\Phi\left(k/(k+r), l/(l+h)\right) = \frac{r^r h^h}{(k+r)^r (l+h)^h} \times \left[1 - p\frac{(k+r)^{k+r}(l+h)^{l+h}}{k^k l^l r^r h^h}\right] \geq 0.$$

Since $\Phi(\lambda,\mu)$ is continuous, there exist $\lambda_0 \in [k/(k+r), 1)$ and $\mu_0 = l/(l+h)$ such that $\Phi\left(\lambda_0, \mu_0\right) = 0$. By Theorem 3.1, (1.1) has a positive solution. This is a contradiction.

Sufficiency. It is sufficient to prove that under condition (3.6), the characteristic equation (1.5) has no positive roots.

(i) If $r+h$ is odd, then (1.5) has no positive roots for the case that $(\lambda-1)^r(\mu-1)^h \geq 0$. For the case that $(\lambda-1)^r(\mu-1)^h < 0$, we write $\Phi(\lambda,\mu)$ in the form

$$\Phi\left(\lambda,\mu\right) = -\left(\lambda-1\right)^r\left(\mu-1\right)^h\left[-1 + \frac{p}{-\lambda^k\mu^l\left(\lambda-1\right)^r\left(\mu-1\right)^h}\right].$$

Set

$$f(\lambda,\mu) = -\lambda^k \mu^l (\lambda-1)^r (\mu-1)^h .$$

It is easy to find that $f(\lambda,\mu)$ reaches its maximum value at $\lambda_0 = k/(k+r)$, $\mu_0 = l/(l+h)$. Hence,

$$\max_{\lambda,\mu\in(0,1)} f(\lambda,\mu) = f(\lambda_0,\mu_0) = \frac{k^k l^l r^r h^h}{(k+r)^{k+r}(l+h)^{l+h}}.$$

Thus, for $\lambda,\mu \in (0,1)$, we have

$$\Phi(\lambda,\mu) \geq -(\lambda-1)^r(\mu-1)^h \left[-1 + p\frac{(k+r)^{k+r}(l+h)^{l+h}}{k^k l^l r^r h^h}\right] > 0,$$

which implies that (1.5) has no positive roots.

(ii) If $r+h$ is even, then (1.5) has no positive roots for the case that $(\lambda-1)^r(\mu-1)^h \leq 0$. For the case that $(\lambda-1)^r(\mu-1)^h > 0$, we write $\Phi(\lambda,\mu)$ in the form

$$\Phi(\lambda,\mu) = (\lambda-1)^r(\mu-1)^h \left[1 - \frac{p}{\lambda^k\mu^l(\lambda-1)^r(\mu-1)^h}\right].$$

Set

$$f(\lambda,\mu) = \lambda^k \mu^l (\lambda-1)^r (\mu-1)^h .$$

Then it is not difficult to find that $f(\lambda,\mu)$ reaches its maximum value at $\lambda_0 = k/(k+r)$, $\mu_0 = l/(l+h)$. Hence,

$$\max_{\lambda,\mu\in(0,1)\text{ or }\lambda,\mu>1} f(\lambda,\mu) = f(\lambda_0,\mu_0) = \frac{k^k l^l r^r h^h}{(k+r)^{k+r}(l+h)^{l+h}}.$$

Thus, for $\lambda,\mu \in (0,1)$ or $\lambda,\mu > 1$, we have

$$\Phi(\lambda,\mu) \leq (\lambda-1)^r(\mu-1)^h \left[1 - p\frac{(k+r)^{k+r}(l+h)^{l+h}}{k^k l^l r^r h^h}\right] < 0,$$

which implies that (1.5) has no positive roots.

By Theorem 3.1, every proper solution of (1.1) oscillates. □

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