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Oscillatory and Asymptotic Properties of Impulsive Difference Equations with Time-varying Delays

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Abstract

In this paper, a class of impulsive difference equations with time-varying delays is considered. By establishing an impulsive delay difference inequality, some sufficient conditions ensuring all solutions either oscillate or tend to zero are obtained. An example is given to illustrate the feasibility of the obtained result.

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1 Introduction and Preliminaries

In recent years, there has been increasing interest on the oscillation / nonoscillation of impulsive delay differential/difference equations, and numerous papers have been published on this class of equations, see [1–11]. For example, Li [7] studied the oscillation of a kind of higher-order impulsive delay differential equations by using a comparison theorem with corresponding nonimpulsive differential equations. In [9, 10], Peng investigated the oscillation of a kind of neutral delay difference equations with impulses by analysis technique. However, the delay considered in [9, 10] is a constant fixed one.

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Moreover, the effect of delay on the solution is ignored. In fact, the delay does contribute to the system's stability properties. The purpose of this paper is to investigate the oscillation properties of a class of impulsive difference equations with time-varying delays. By establishing an impulsive delay difference inequality, we obtain some sufficient conditions ensuring all solutions either oscillate or tend to zero. The effects of impulses and delays on the solutions are stressed here. Lastly, an example is given to illustrate the feasibility of the obtained result.

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, and $\mathbb{Z}_+$ the set of positive integers. In this paper, we consider the impulsive difference equation with time-varying delays

$$\begin{cases} x(m) = -ax(m-1) + \sum_{j=1}^{N_0} b_j F_j(x(m-1)) \\ \quad + \sum_{j=1}^{N_0} c_j G_j(x(m-1-\tau_j(m))),\ m \neq m_k,\ m \in N[m_0,\infty), \\ x(m_k) = \beta_k x(m_k^-),\ k \in \mathbb{Z}_+ \\ x(m_0+\theta) = \phi(\theta),\ \theta \in N[-\tau-1,0], \end{cases} \tag{1.1}$$

where $N_0 \in \mathbb{Z}_+$, $\phi \in \mathbb{C}$, $\mathbb{C}$ denotes the set of all functions $\varphi : N[-\tau-1,0] \to \mathbb{R}$. The impulse times $m_k \in \mathbb{Z}_+$ satisfy $0 \leq m_0 < m_1 < \ldots < m_k < \ldots$, $\lim_{k\to\infty} m_k = +\infty$, $F_j, G_j \in C(\mathbb{R},\mathbb{R})$, $b_j, c_j \in \mathbb{R}$ are constants, $\tau_j(m)$ are delays and satisfy $\tau_j(m) \in N[0,\tau]$, τ is a nonnegative integer. For convenience, let

$$\begin{aligned} N[m,n] &\doteq \{m, m+1, \cdots, n\}, \\ N[m,n) &\doteq \{m, m+1, \cdots, n-1\}, \\ N[m,\infty) &\doteq \{m, m+1, \cdots\} \ \text{for}\ m<n \ \text{and}\ m,n \in \mathbb{Z}_+. \end{aligned}$$

Definition 1.1. The solution $x(m)$ of (1.1) is said to be *nonoscillatory* if the solution is eventually negative or eventually positive. Otherwise, the solution is said to be *oscillatory*.

2 Main Results

First, we shall establish an impulsive delay difference inequality which plays an important role in this paper.

Lemma 2.1. *Assume that there exist a real-valued sequence $u(m) : N[m_0-\tau,+\infty) \to \mathbb{R}_+$ and constants $p, q > 0$ such that*

(i) $u(m_k) \leq \gamma_k u(m_k^-)$, *where $\gamma_k > 0, k \in \mathbb{Z}_+$ are constants and satisfy*

$$q \max_{k\in\mathbb{Z}_+} \left\{ \frac{1}{\gamma_k}, 1 \right\} < 1 + p;$$

(ii) $u(m) \leq -p\, u(m-1) + q\overrightarrow{u(m-1)}$ *for any* $m \neq m_k,\ k \in \mathbb{Z}_+$, *where*

$$\overrightarrow{u(m)} = \sup_{n \in N[m-\tau, m]} u(n);$$

(iii) $\tau \leq m_k - m_{k-1}$, $k \in \mathbb{Z}_+$.

Then for $m \in N[m_0, \infty)$,

$$u(m) \leq \overrightarrow{u(m_0)} \left(\prod_{m_k \in N[m_0, m]} \gamma_k \right) e^{-\lambda(m-m_0)}, \tag{2.1}$$

where $\lambda > 0$ *satisfies the inequality*

$$qe^{\lambda\tau} \max_{k \in \mathbb{Z}_+} \left\{ \frac{1}{\gamma_k}, 1 \right\} \leq p + e^{-\lambda}. \tag{2.2}$$

Proof. Condition (i) implies that there exists some $\lambda > 0$ such that the inequality (2.2) holds. Next, we shall prove that (2.1) holds for all $m \in N[m_0, \infty)$. In order to do this, set

$$\Omega(m) = \begin{cases} u(m)e^{\lambda(m-m_0)}, & m \in N[m_0, \infty), \\ u(m), & m \in N[m_0 - \tau, m_0]. \end{cases}$$

Then it is obvious that $\Omega(m) \geq u(m)$ for all $m \in N[m_0 - \tau, \infty)$. Hence, we only need to show, for $m \in N[m_0, \infty)$

$$\Omega(m) \leq \overrightarrow{u(m_0)} \left(\prod_{m_k \in N[m_0, m]} \gamma_k \right). \tag{2.3}$$

First, we claim that

$$\Omega(m) \leq \overrightarrow{u(m_0)},\ m \in N[m_0, m_1) \cup m_1^-. \tag{2.4}$$

If this is not true, then there exists $m^* \in N[m_0, m_1) \cup m_1^-$ such that

$$\Omega(m^*) > \overrightarrow{u(m_0)} \text{ and } \Omega(m) \leq \overrightarrow{u(m_0)},\ \ m \in N[m_0, m^*). \tag{2.5}$$

Note that $\Omega(m) = u(m)$ for $m \in N[m_0 - \tau, m_0]$. Then we obtain

$$\Omega(m) \leq \overrightarrow{u(m_0)},\ m \in N[m_0 - \tau, m^*).$$

In view of the definition of Ω and noting (2.5), we have

$$\begin{aligned}
\Omega(m^*) &= u(m^*)e^{\lambda(m^*-m_0)} \\
&\leq \left[-pu(m^*-1)+q\overrightarrow{u(m^*-1)}\right]e^{\lambda(m^*-m_0)} \\
&\leq \Big[-p\Omega(m^*-1)e^{-\lambda(m^*-1-m_0)} \\
&\quad +q\overrightarrow{\Omega(m^*-1)}e^{-\lambda(m^*-1-\tau-m_0)}\Big]e^{\lambda(m^*-m_0)} \\
&\leq \left[-p\Omega(m^*-1)+q\overrightarrow{\Omega(m^*-1)}e^{\lambda\tau}\right]e^{\lambda} \\
&\leq \left[-p\overrightarrow{u(m_0)}+q\overrightarrow{u(m_0)}e^{\lambda\tau}\right]e^{\lambda} \\
&\leq \overrightarrow{u(m_0)}\left[-p+qe^{\lambda\tau}\max_{k\in\mathbb{Z}_+}\left\{\frac{1}{\gamma_k},1\right\}\right]e^{\lambda} \\
&\leq \overrightarrow{u(m_0)},
\end{aligned}$$

which contradicts the first inequality of (2.5). So (2.4) holds.

Suppose that (2.3) holds for all $m \in N[m_0, m_l) \cup m_l^-$, $l \in \mathbb{Z}_+$. Then

$$\Omega(m) \leq \gamma_0\gamma_1\cdots\gamma_{l-1}\overrightarrow{u(m_0)},\ m \in N[m_{l-1}, m_l), \tag{2.6}$$

$$\Omega(m_l^-) \leq \gamma_0\gamma_1\cdots\gamma_{l-1}\overrightarrow{u(m_0)},$$

where $\gamma_0 = 1$. Then we have

$$\begin{aligned}
\Omega(m_l) &= u(m_l)e^{\lambda(m_l-m_0)} \leq \gamma_l u(m_l^-)e^{\lambda(m_l-m_0)} \\
&= \gamma_l\Omega(m_l^-) \leq \gamma_0\gamma_1\cdots\gamma_{l-1}\gamma_l\overrightarrow{u(m_0)}.
\end{aligned} \tag{2.7}$$

Next, we prove that (2.3) holds for $m \in N[m_l, m_{l+1})$, i.e.,

$$\Omega(m) \leq \gamma_0\gamma_1\cdots\gamma_{l-1}\gamma_l\overrightarrow{u(m_0)},\ m \in N[m_l, m_{l+1}). \tag{2.8}$$

For the sake of contradiction, suppose (2.8) is not true. Thus we can define

$$m^\star = \inf\left\{m|\ \Omega(m) > \gamma_0\gamma_1\cdots\gamma_{l-1}\gamma_l\overrightarrow{u(m_0)},\ m \in N[m_l, m_{l+1})\right\}.$$

Since (2.7) holds, $m^\star \in N(m_l, m_{l+1})$,

$$\Omega(m^\star) > \gamma_0\gamma_1\cdots\gamma_{l-1}\gamma_l\overrightarrow{u(m_0)},$$

and

$$\Omega(m) \leq \gamma_0\gamma_1\cdots\gamma_{l-1}\gamma_l\overrightarrow{u(m_0)},\ m \in N[m_l, m^\star).$$

Hence, considering (2.6) and condition (iii), we obtain

$$
\begin{aligned}
\Omega(m^\star) &= u(m^\star)e^{\lambda(m^\star - m_0)} \\
&\leq \left[-pu(m^\star - 1) + q\overrightarrow{u(m^\star - 1)}\right] e^{\lambda(m^\star - m_0)} \\
&\leq \left[-p\Phi(m^\star - 1) + q\overrightarrow{\Omega(u(m^\star - 1))}e^{\lambda\tau}\right] e^{\lambda} \\
&\leq \Big[-p\gamma_0\gamma_1\cdots\gamma_{l-1}\gamma_l\overrightarrow{u(m_0)} \\
&\quad +q\max\left\{\gamma_0\gamma_1\cdots\gamma_{l-1}\overrightarrow{u(m_0)},\ \gamma_0\gamma_1\cdots\gamma_{l-1}\gamma_l\overrightarrow{u(m_0)}\right\} e^{\lambda\tau}\Big] e^{\lambda} \\
&\leq \gamma_0\gamma_1\cdots\gamma_{l-1}\gamma_l\overrightarrow{u(m_0)} \left[-p + q\max\left\{\frac{1}{\gamma_l},\ 1\right\} e^{\lambda\tau}\right] e^{\lambda} \\
&\leq \gamma_0\gamma_1\cdots\gamma_{l-1}\gamma_l\overrightarrow{u(m_0)} \left[-p + q\max_{k\in\mathbb{Z}_+}\left\{\frac{1}{\gamma_k},\ 1\right\} e^{\lambda\tau}\right] e^{\lambda} \\
&\leq \gamma_0\gamma_1\cdots\gamma_{l-1}\gamma_l\overrightarrow{u(m_0)},
\end{aligned}
$$

which is a contradiction. So (2.8) holds. Thus by mathematical induction, we obtain that (2.1) holds for all $m \in N[m_0, \infty)$. The proof is complete. □

Remark 2.2. From the proof of Lemma 2.1, we can find that condition (iii) $\tau \leq m_k - m_{k-1}$, $k \in \mathbb{Z}_+$ can be removed if $\gamma_k \geq 1$ for all $k \in \mathbb{Z}_+$.

We next present a theorem which provides sufficient conditions for oscillation of (1.1) by using Lemma 2.1.

Theorem 2.3. *Assume that* $\tau \leq m_k - m_{k-1}$, $k \in \mathbb{Z}_+$ *and the following conditions hold:*

(H_1) $|F_j(s)| \leq L_j^F|s|$ *and* $|G_j(s)| \leq L_j^G|s|$ *for all* $s \in \mathbb{R}$, $j = 1, 2, \cdots, N_0$, *where* $L_j^F, L_j^G > 0$ *are constants;*

(H_2) *we have*

$$\sum_{j=1}^{N_0}|b_j|L_j^F + \sum_{j=1}^{N_0}|c_j|L_j^G \max_{k\in\mathbb{Z}_+}\left\{\frac{1}{\beta_k}, 1\right\} < a + 1;$$

(H_3) *there exist constants* $N^\star \geq m_0$, $M^\star > 0$, $\delta \geq 0$ *such that* $\delta < \lambda$ *and the inequality*

$$\sum_{m_k\in N[n,m]} \ln\beta_k - \delta(m - n) < M^\star \quad \textit{for all} \quad m \geq n,\ n \in N[N^\star, \infty)$$

holds, where λ *satisfies the inequality*

$$\sum_{j=1}^{N_0}|b_j|L_j^F + \sum_{j=1}^{N_0}|c_j|L_j^G e^{\lambda\tau} \max_{k\in\mathbb{Z}_+}\left\{\frac{1}{\beta_k}, 1\right\} \leq a + e^{-\lambda}. \tag{2.9}$$

Then all solutions of system (1.1) *either oscillate or tend to zero.*

Proof. Suppose that there is a solution $x(m)$ of system (1.1) which is nonoscillatory. Without loss of generality, we may suppose that $x(m) > 0$ for $m \in N[M_0, \infty)$, $M_0 \geq N^\star$. Next we can prove that $x(m)$ tends to zero. First, from (H_1) and (H_2), we get, for $m \in N[m_k, m_{k+1})$, $m_k \geq M_0 + \tau + 1$, $k \in \mathbb{Z}_+$

$$\begin{aligned} x(m) &= -ax(m-1) + \sum_{j=1}^{N_0} b_j F_j(x(m-1)) + \sum_{j=1}^{N_0} c_j G_j(x(m-1-\tau_j(m))) \\ &\leq -ax(m-1) + \sum_{j=1}^{N_0} |b_j| L_j^F x(m-1) + \sum_{j=1}^{N_0} |c_j| L_j^G x(m-1-\tau_j(m)) \\ &\leq -ax(m-1) + \sum_{j=1}^{N_0} |b_j| L_j^F x(m-1) + \sum_{j=1}^{N_0} |c_j| L_j^G \overrightarrow{x(m-1)} \\ &\leq -\left[a - \sum_{j=1}^{N_0} |b_j| L_j^F\right] x(m-1) + \sum_{j=1}^{N_0} |c_j| L_j^G \overrightarrow{x(m-1)}, \end{aligned}$$

where $\overrightarrow{x(m-1)} = \sup_{n \in N[m-1-\tau, m-1]} x(n)$. By Lemma 2.1 and noting that $x(m_k) = \beta_k x(m_k^-)$, we have

$$x(m) \leq \overrightarrow{x(M_0+\tau+1)} \left(\prod_{m_k \in N[M_0+\tau+1, m]} \beta_k \right) e^{-\lambda(m-M_0-\tau-1)}, \qquad m \in N[M_0+\tau+1, \infty).$$

In view of condition (H_3), we furthermore obtain

$$x(m) \leq \overrightarrow{x(M_0+\tau+1)} e^{-(\lambda-\delta)(m-M_0-\tau-1)}, \; m \in N[M_0+\tau+1, \infty), \tag{2.10}$$

where $\lambda > 0$ satisfies the inequality (2.9).

On the other hand, suppose that (1.1) has an eventually negative solution $x(m)$ which is defined $[m_0, \infty)$ and $x(m) < 0$ for $m \in N[M_0, \infty)$, $M_0 \geq N^\star$. Let $x(m) = -y(m)$. Then $y(m) > 0$ for $m \in N[M_0, \infty)$. Similarly, we arrive for $m \in N[m_k, m_{k+1})$, $m_k \geq M_0 + \tau + 1$, $k \in \mathbb{Z}_+$ at

$$\begin{aligned} y(m) &\leq -ay(m-1) + \sum_{j=1}^{N_0} |b_j| L_j^F y(m-1) + \sum_{j=1}^{N_0} |c_j| L_j^G y(m-1-\tau_j(m)) \\ &\leq -\left[a - \sum_{j=1}^{N_0} |b_j| L_j^F\right] y(m-1) + \sum_{j=1}^{N_0} |c_j| L_j^G \overrightarrow{y(m-1)}. \end{aligned}$$

Since $y(m_k) = \beta_k y(m_k^-)$, by Lemma 2.1, we finally get

$$|x(m)| \leq \overrightarrow{|x(M_0+\tau+1)|}e^{-(\lambda-\delta)(m-M_0-\tau-1)},\ m \in N[M_0+\tau+1,\infty), \qquad (2.11)$$

where $\lambda > 0$ satisfies the inequality (2.9).

The inequalities (2.10) and (2.11) imply that $|x(m)| \to 0$ as $m \to \infty$. The proof is complete. □

Remark 2.4. In Theorem 2.3, if $\sup\limits_{n\in\mathbb{Z}_+}\left(\prod\limits_{k=1}^{n}\beta_k\right) < \infty$, then we can choose $\delta = 0$ in condition (H$_3$).

Corollary 2.5. *Assume that* $\beta_k \geq 1,\ k \in \mathbb{Z}_+$ *and condition* (H_1) *from Theorem 2.3 holds. Moreover, suppose that*

(H_2') *we have*

$$\sum_{j=1}^{N_0}|b_j|L_j^F + \sum_{j=1}^{N_0}|c_j|L_j^G < a+1;$$

(H_3') *there exist constants* $N^\star \geq m_0$, $M^\star > 0$, $\delta \geq 0$ *such that* $\delta < \lambda$ *and the inequality*

$$\sum_{m_k\in N[n,m]}\ln\beta_k - \delta(m-n) < M^\star \quad \textit{for all} \quad m \geq n,\ n \in N[N^\star,\infty)$$

holds, where λ *satisfies the inequality*

$$\sum_{j=1}^{N_0}|b_j|L_j^F + \sum_{j=1}^{N_0}|c_j|L_j^G e^{\lambda\tau} \leq a + e^{-\lambda}.$$

Then all solutions of system (1.1) *either oscillate or tend to zero.*

Corollary 2.6. *Assume that* $\beta_k \leq 1$, $\tau \leq m_k - m_{k-1}$, $k \in \mathbb{Z}_+$ *and conditions* (H_1) *from Theorem 2.3 holds. Moreover, suppose that*

$$\sum_{j=1}^{N_0}|b_j|L_j^F + \frac{\sum\limits_{j=1}^{N_0}|c_j|L_j^G}{\min\limits_{k\in\mathbb{Z}_+}\beta_k} < a+1.$$

Then all solutions of system (1.1) *either oscillate or tend to zero.*

3 An Example

Example 3.1. Consider the impulsive difference equation with time-varying delays

$$\begin{cases} x(m) = -0.8x(m-1) + 0.2\sin x(m-1) + 0.1\cos x(m-1) \\ \qquad - 0.6\left|x\left(m-1-\left[\sin\frac{m\pi}{2}\right]\right)\right| + 0.4\left|x\left(m-1-3\left[\cos\frac{m\pi}{2}\right]\right)\right|, \\ \qquad\qquad m \neq m_k,\ m \in N[m_0,\infty), \\ x(m_k) = \beta_k x(m_k^-),\ k \in \mathbb{Z}_+, \\ x(m_0+\theta) = \phi(\theta),\ \theta \in N[-\tau-1,0], \end{cases} \tag{3.1}$$

where $\tau = 3$, $m_0 = 1$, $m_k - m_{k-1} = 3$ and $\beta_k = e^{0.07}$. Choose $L_j^F = L_j^G = 1$, $j = 1, 2$. One can check that

$$\sum_{j=1}^{N_0} |b_j| L_j^F + \sum_{j=1}^{N_0} |c_j| L_j^G = 1.3 < 1.8 = a + 1.$$

Then we can choose $\lambda = 0.0953$ such that

$$\sum_{j=1}^{N_0} |b_j| L_j^F + \sum_{j=1}^{N_0} |c_j| L_j^G e^{\lambda\tau} \approx 1.631 \leq 1.709 \approx a + e^{-\lambda}.$$

Furthermore, we can choose $\delta = 0.0803 < \lambda$, $N^\star = m_0$, $M^\star = 1$ such that

$$\sum_{m_k \in N[n,m]} \ln\beta_k - \delta(m-n) = -0.0103(m-n) \leq 0 \text{ for all } m \geq n, n \in N[m_0,\infty).$$

By Corollary 2.5, all solutions of system (3.1) either oscillate or tend to zero.

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