International Journal of Difference Equations
ISSN 0973-6069, Volume 4, Number 1, pp. 115–136 (2009)
http://campus.mst.edu/ijde

Two Modifications of the Beverton–Holt Equation

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Abstract

In this paper we study the boundedness, the persistence of the positive solutions, the existence of a unique positive equilibrium, the convergence of the positive solutions to the positive equilibrium and the stability of two systems of rational difference equations which are modifications of the Beverton–Holt equation.

AMS Subject Classifications: 39A10.
Keywords: Beverton–Holt equation, difference equation, boundedness, persistence, convergence, stability.

1 Introduction

Difference equations have many applications in several applied sciences, such as biology, ecology, economics, population dynamics, genetics, etc. (see [4,8,11–13,15,21,28] and the references cited therein). For this reason, there exist an increasing interest in studying difference equations and systems of difference equations (see [1–11, 13–21, 23–25, 27–30] and the references cited therein).

In [13] the authors studied some discrete competition models. It is known that if $b > 1$, then all solutions of the Beverton–Holt equation

$$x_{n+1} = \frac{bx_n}{1 + c_{11}x_n}, \ \ n = 0, 1, \ldots,$$

where $x_0 > 0$ tend monotonically to the equilibrium $x = \dfrac{b-1}{c_{11}}$. This difference equation is the discrete analog of the logistic differential equation studied in [12]. The

Received September 30, 2008; Accepted October 15, 2008
Communicated by Gerasimos Ladas

Leslie/Gower (difference equation) competition model (see [21])

$$x_{n+1} = \frac{b_1 x_n}{1 + c_{11} x_n + c_{12} y_n},$$
$$y_{n+1} = \frac{b_2 y_n}{1 + c_{21} x_n + c_{22} y_n}$$

is a modification of the Beverton–Holt equation which has played a key role in theoretical ecology.

In [29] the authors studied the global stability of the rational difference equation

$$x_{n+1} = \frac{b x_n}{1 + b_0 x_n + b_1 x_{n-1} + \cdots + b_k x_{n-k}}, \quad n = 0, 1, \ldots,$$

where b, b_i, $i = 0, 1, \ldots, k$ are positive constants and the initial values x_i, $i = -k, -k + 1, \ldots, 0$ are positive numbers.

In this paper we consider two systems of rational difference equations which are modifications of the Beverton–Holt equation, of the form

$$\begin{aligned} x_{n+1} &= \frac{a x_n}{1 + \sum_{i=0}^{p} b_i x_{n-i} + \sum_{i=0}^{m} c_i y_{n-i}}, \\ y_{n+1} &= \frac{d y_n}{1 + \sum_{i=0}^{q} e_i y_{n-i} + \sum_{i=0}^{s} k_i x_{n-i}}, \quad n = 0, 1, \ldots, \end{aligned} \tag{1.1}$$

$$\begin{aligned} x_{n+1} &= \frac{a y_n}{1 + \sum_{i=0}^{m} c_i y_{n-2i} + \sum_{i=0}^{p} b_i x_{n-2i-1}}, \\ y_{n+1} &= \frac{d x_n}{1 + \sum_{i=0}^{s} k_i x_{n-2i} + \sum_{i=0}^{q} e_i y_{n-2i-1}}, \quad n = 0, 1, \ldots, \end{aligned} \tag{1.2}$$

where a, d, b_i, $i = 0, 1, \ldots, p$, c_i, $i = 0, 1, \ldots, m$, e_i, $i = 0, 1, \ldots, q$, $k_i = 0, 1, \ldots, s$ are nonnegative constants, the initial values of (1.1) x_i, $i = -\pi, -\pi + 1, \ldots, 0$, y_i, $i = -\tau, -\tau + 1, \ldots, 0$, $\pi = \max\{p, s\}$, $\tau = \max\{m, q\}$ are positive real numbers and the initial values of (1.2) x_i, $i = -\lambda, -\lambda + 1, \ldots, 0$, y_i, $i = -\mu, -\mu + 1, \ldots, 0$, $\lambda = \max\{2p+1, 2s\}$, $\mu = \max\{2q+1, 2m\}$ are also positive real numbers. More precisely, we study the boundedness, the persistence of the positive solutions of (1.1) and (1.2), the existence of the unique positive equilibrium of (1.1) and (1.2), the convergence of the positive solutions of (1.1) to the unique positive equilibrium of (1.1). In addition, we study the convergence of the positive solutions of the system

$$\begin{aligned} x_{n+1} &= \frac{ay_n}{1 + c_0 y_n + \sum_{i=0}^{p} b_i x_{n-2i-1}} \\ y_{n+1} &= \frac{dx_n}{1 + k_0 x_n + \sum_{i=0}^{q} e_i y_{n-2i-1}}, \quad n = 0, 1, \ldots, \end{aligned} \tag{1.3}$$

where the constants c_0, k_0, b_i, $i = 0, 1, \ldots, p$, e_i, $i = 0, 1, \ldots, q$ are nonnegative real numbers and the initial values are also positive real numbers to the unique positive equilibrium of (1.3). Finally, we study the global asymptotic stability of

$$\begin{aligned} x_{n+1} &= \frac{ax_n}{1 + bx_n + cy_{n-1}}, \\ y_{n+1} &= \frac{dy_n}{1 + ey_n + kx_{n-1}}, \quad n = 0, 1, \ldots, \end{aligned} \tag{1.4}$$

$$\begin{aligned} x_{n+1} &= \frac{ay_n}{1 + cy_n + bx_{n-1}}, \\ y_{n+1} &= \frac{dx_n}{1 + kx_n + ey_{n-1}}, \quad n = 0, 1, \ldots, \end{aligned} \tag{1.5}$$

where a, b, c, d, e, k are positive constants and the initial values are also positive real numbers.

2 Study of System (1.1)

In this section we study system (1.1). First we find conditions so that system (1.1) has a unique positive equilibrium.

Proposition 2.1. *Consider system* (1.1), *where*

$$a > 1, \quad d > 1. \tag{2.1}$$

Suppose that either

$$(a-1)E > (d-1)C, \quad (d-1)B > (a-1)K \tag{2.2}$$

or

$$(a-1)E < (d-1)C, \quad (d-1)B < (a-1)K \tag{2.3}$$

hold, where

$$B = \sum_{i=0}^{p} b_i, \quad C = \sum_{i=0}^{m} c_i, \quad E = \sum_{i=0}^{q} e_i, \quad K = \sum_{i=0}^{s} k_i. \tag{2.4}$$

Then system (1.1) *has a unique positive equilibrium* $(\bar{x}, \bar{y})$.

Proof. In order $(\bar{x}, \bar{y})$ to be a positive equilibrium for (1.1), we must have

$$\bar{x} = \frac{a\bar{x}}{1 + B\bar{x} + C\bar{y}}, \quad \bar{y} = \frac{d\bar{y}}{1 + E\bar{y} + K\bar{x}}$$

or equivalently

$$B\bar{x} + C\bar{y} = a - 1, \quad K\bar{x} + E\bar{y} = d - 1.$$

So using (2.1) and since either (2.2) or (2.3) hold, we get

$$\bar{x} = \frac{(a-1)E - (d-1)C}{BE - KC} > 0, \quad \bar{y} = \frac{(d-1)B - (a-1)K}{BE - KC} > 0.$$

This completes the proof. □

In the following proposition we study the boundedness and persistence of the positive solution of (1.1).

Proposition 2.2. *Consider system* (1.1)*, where relations* (2.1) *and*

$$(a-1)e_0 > (d-1)C, \quad (d-1)b_0 > (a-1)K \tag{2.5}$$

hold. Then every positive solution of (1.1) *is bounded and persists.*

Proof. Let (x_n, y_n) be an arbitrary solution of (1.1). Since from (2.5), $b_0 \neq 0$, $e_0 \neq 0$, then from (1.1), we have

$$x_n \leq \frac{a}{b_0}, \quad y_n \leq \frac{d}{e_0}, \quad n = 1, 2, \ldots, \tag{2.6}$$

and so (x_n, y_n) is a bounded solution. We prove now that (x_n, y_n) persists. Suppose that x_n does not persists. Then we may suppose that there exists a subsequence x_{n_r} of x_n such that

$$\lim_{r \to \infty} x_{n_r} = 0, \quad x_{n_r} = \min\{x_s,\ 0 \leq s \leq n_r\}. \tag{2.7}$$

Firstly, suppose that there exists a $v \in N$ such that

$$y_v \leq \frac{d-1}{e_0}. \tag{2.8}$$

Then from (1.1) and (2.8) we obtain

$$y_{v+1} = \frac{dy_v}{1 + \sum_{i=0}^{q} e_i y_{v-i} + \sum_{i=0}^{s} k_i x_{v-i}} \leq \frac{dy_v}{1 + e_0 y_v} \leq \frac{d\left(\frac{d-1}{e_0}\right)}{1 + e_0\left(\frac{d-1}{e_0}\right)} = \frac{d-1}{e_0}$$

and working inductively we have

$$y_n \leq \frac{d-1}{e_0}, \quad n \geq v. \tag{2.9}$$

Moreover, since from (1.1)

$$x_{n_r} = \frac{ax_{n_r-1}}{1 + \sum_{i=0}^{p} b_i x_{n_r-i-1} + \sum_{i=0}^{m} c_i y_{n_r-i-1}} \tag{2.10}$$

and

$$x_{n_r+1} = \frac{ax_{n_r}}{1 + \sum_{i=0}^{p} b_i x_{n_r-i} + \sum_{i=0}^{m} c_i y_{n_r-i}},$$

then, using (2.6) and (2.7), it follows that

$$\lim_{r\to\infty} x_{n_r-1} = 0, \quad \lim_{r\to\infty} x_{n_r+1} = 0.$$

Then working inductively we have

$$\lim_{r\to\infty} x_{n_r+\tau} = 0, \quad \tau = .., -1, 0, 1, \ldots. \tag{2.11}$$

Furthermore, from (2.5) there exists sufficiently small positive ϵ such that

$$(a-1)e_0 - C(d-1) - \epsilon B e_0 > 0. \tag{2.12}$$

Using (2.11) for sufficiently large n_r we have

$$x_{n_r-j} \leq \epsilon, \quad j = 1, 2, \ldots, p+1, \tag{2.13}$$

where ϵ satisfies (2.12). Therefore from (2.9), (2.10), (2.12) and (2.13) for sufficiently large n_r it follows that

$$x_{n_r} \geq \frac{ax_{n_r-1}}{1 + B\epsilon + C\left(\frac{d-1}{e_0}\right)} > x_{n_r-1}$$

which contradicts to (2.7). Therefore, x_n persists in the case where (2.8) holds.

Suppose now

$$y_n > \frac{d-1}{e_0}, \quad n = 1, 2, \ldots. \tag{2.14}$$

From (1.1) and (2.14) we have

$$\begin{aligned} y_{n+1} - y_n &= \frac{dy_n}{1 + \sum_{i=0}^{q} e_i y_{n-i} + \sum_{i=0}^{s} k_i x_{n-i}} - y_n \\ &= \frac{y_n\Big(d - 1 - e_0 y_n - \sum_{i=1}^{q} e_i y_{n-i} - \sum_{i=0}^{s} k_i x_{n-i}\Big)}{1 + \sum_{i=0}^{q} e_i y_{n-i} + \sum_{i=0}^{s} k_i x_{n-i}} < 0. \end{aligned} \tag{2.15}$$

So, from (2.14) and (2.15), there exists the $\lim_{n\to\infty} y_n$ and it is different from zero. Let

$$\lim_{n\to\infty} y_n = l \neq 0. \tag{2.16}$$

Then from (2.11), (2.16) and since from (1.1),

$$y_{n_r+1} = \frac{dy_{n_r}}{1 + \sum_{i=0}^{q} e_i y_{n_r-i} + \sum_{i=0}^{s} k_i x_{n_r-i}}$$

we can prove that

$$l = \frac{d-1}{E}. \tag{2.17}$$

Moreover, from (2.5) there exists a sufficiently small $\epsilon > 0$ such that

$$(a-1)E - (d-1)C - \epsilon E(B+C) > (a-1)e_0 - (d-1)C - \epsilon E(B+C) > 0. \tag{2.18}$$

From (2.16) and (2.17), there exists a sufficiently large n_r such that (2.13) and

$$y_{n_r-1-i} \leq \frac{d-1}{E} + \epsilon, \quad i \in \{0, 1, \ldots, m\} \tag{2.19}$$

hold. Then from (2.10), (2.13), (2.18) and (2.19), we have

$$x_{n_r} \geq \frac{ax_{n_r-1}}{1 + B\epsilon + C\Big(\frac{d-1}{E} + \epsilon\Big)} > x_{n_r-1}$$

which contradicts again to (2.7). Therefore, x_n persists. Working similarly, we can prove that y_n persists. This completes the proof. □

In the following proposition we study the convergence of the positive solution of (1.1) to the unique positive equilibrium of (1.1).

Proposition 2.3. *Consider system* (1.1)*, where relations* (2.1) *and* (2.5) *are satisfied. Suppose also that*

$$b_0 > B_1, \quad e_0 > E_1, \quad (b_0 - B_1)(e_0 - E_1) > CK, \quad B_1 = \sum_{i=1}^{p} b_i, E_1 = \sum_{i=1}^{q} e_i. \tag{2.20}$$

Then every positive solution of (1.1) *tends to the unique positive equilibrium of* (1.1)*.*

Proof. From Proposition 2.2 there exist

$$\begin{aligned} L_1 &= \limsup_{n\to\infty} x_n < \infty, \quad l_1 = \liminf_{n\to\infty} x_n > 0, \\ L_2 &= \limsup_{n\to\infty} y_n < \infty, \quad l_2 = \liminf_{n\to\infty} y_n > 0. \end{aligned} \tag{2.21}$$

Then from (1.1) we get

$$L_1 \le \frac{aL_1}{1 + b_0 L_1 + B_1 l_1 + C l_2}, \quad l_1 \ge \frac{a l_1}{1 + b_0 l_1 + B_1 L_1 + C L_2},$$
$$L_2 \le \frac{dL_2}{1 + e_0 L_2 + E_1 l_2 + K l_1}, \quad l_2 \ge \frac{d l_2}{1 + e_0 l_2 + E_1 L_2 + K L_1},$$

or equivalently

$$\begin{aligned} b_0 L_1 + B_1 l_1 + C l_2 \le a - 1 \le b_0 l_1 + B_1 L_1 + C L_2, \\ e_0 L_2 + E_1 l_2 + K l_1 \le d - 1 \le e_0 l_2 + E_1 L_2 + K L_1. \end{aligned} \tag{2.22}$$

Therefore, from (2.22) we get

$$\begin{aligned} (b_0 - B_1)(L_1 - l_1) \le C(L_2 - l_2), \\ (e_0 - E_1)(L_2 - l_2) \le K(L_1 - l_1). \end{aligned} \tag{2.23}$$

Hence, using (2.20) and multiplying both sides of (2.23) we obtain

$$(b_0 - B_1)(e_0 - E_1)(L_1 - l_1)(L_2 - l_2) \le CK(L_1 - l_1)(L_2 - l_2). \tag{2.24}$$

So relations (2.20) and (2.24) imply that $(L_1 - l_1)(L_2 - l_2) \le 0$. Therefore, either $L_1 = l_1$ or $L_2 = l_2$. If $L_1 = l_1$ (resp. $L_2 = l_2$), then from (2.23) $L_2 = l_2$ (resp. $L_1 = l_1$). This completes the proof. □

In the last proposition of this section we study the global asymptotic stability of the positive equilibrium $(\bar{x}, \bar{y})$ of (1.4). We need the following lemma which has been proved in [22]. For readers' convenience we state it here without its proof.

Lemma 2.4. *Consider the equation*

$$x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0, \tag{2.25}$$

where $a_i,\ i = 0,1,2,3$ *are positive constants. Then every solution of* (2.25) *is of modulus less than* 1 *if and only if the following conditions are satisfied:*

$$K_1 > 0,\ \ K_2 > 0,\ \ K_3 > 0,\ \ K_5 > 0,\ \ K_4K_2K_3 - K_4^2K_1 > K_2^2K_5, \tag{2.26}$$

where

$$K_1 = 1 + a_0 + a_1 + a_2 + a_3,\ \ K_2 = 4 + 2a_3 - 2a_1 - 4a_0,\ \ K_3 = 6 - 2a_2 + 6a_0,$$

$$K_4 = 4 - 2a_3 + 2a_1 - 4a_0,\ \ K_5 = 1 - a_3 + a_2 - a_1 + a_0.$$

After some calculations we can see that

$$\begin{aligned} &K_4K_2K_3 - K_4^2K_1 - K_2^2K_5 \\ &\qquad = a_0^3 + a_0a_1a_3 + a_1a_3 + 2a_0a_2 + 1 - a_0 - a_2 - a_0^2 - a_1^2 - a_0^2a_2 - a_0a_3^2. \end{aligned}$$

In addition if $a_1 = 0$, then

$$K_4K_2K_3 - K_4^2K_1 - K_2^2K_5 = (a_0 - 1)^2(1 + a_0 - a_2) - a_0a_3^2. \tag{2.27}$$

Proposition 2.5. *Consider system* (1.4)*, where* (2.1) *and the relations*

$$(a-1)e > (d-1)c, \quad (d-1)b > (a-1)k \tag{2.28}$$

hold. Then the unique positive equilibrium $(\bar{x}, \bar{y})$ *of* (1.4) *is globally asymptotically stable.*

Proof. We prove that $(\bar{x}, \bar{y})$ is locally asymptotically stable. The linearized system about the positive equilibrium of (1.4) is

$$\begin{aligned} x_{n+1} &= \frac{a(1 + c\bar{y})}{(1 + c\bar{y} + b\bar{x})^2}x_n - \frac{ac\bar{x}}{(1 + c\bar{y} + b\bar{x})^2}y_{n-1}, \\ y_{n+1} &= \frac{d(1 + k\bar{x})}{(1 + e\bar{y} + k\bar{x})^2}y_n - \frac{dk\bar{y}}{(1 + e\bar{y} + k\bar{x})^2}x_{n-1}. \end{aligned} \tag{2.29}$$

System (2.29) is equivalent to system

$$w_{n+1} = Aw_n,\ \ A = \begin{pmatrix} H & 0 & 0 & T \\ 0 & M & N & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ \ w_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix},$$

$$H = \frac{a(1 + c\bar{y})}{(1 + c\bar{y} + b\bar{x})^2}, \quad T = -\frac{ac\bar{x}}{(1 + c\bar{y} + b\bar{x})^2},$$

$$M = \frac{d(1 + k\bar{x})}{(1 + k\bar{x} + e\bar{y})^2}, \quad N = -\frac{dk\bar{y}}{(1 + k\bar{x} + e\bar{y})^2}.$$

The characteristic equation of A is

$$\lambda^4 - \lambda^3(H+M) + \lambda^2 HM - NT = 0. \tag{2.30}$$

Using Lemma 2.4, all the roots of (2.30) are of modulus less than 1 if and only if (2.26) are satisfied, where

$$K_1 = 1 - H - M + HM - NT, \quad K_2 = 4 - 2(H+M) + 4NT,$$

$$K_3 = 6 - 2HM - 6NT, \quad K_4 = 4 + 2(H+M) + 4NT,$$

$$K_5 = 1 + H + M + HM - NT.$$

Since $(\bar{x}, \bar{y})$ is a positive equilibrium of (1.4), we have

$$1 + c\bar{y} + b\bar{x} = a, \quad 1 + e\bar{y} + k\bar{x} = d, \tag{2.31}$$

and so

$$\bar{x} = \frac{e(a-1) - c(d-1)}{be - ck}, \quad \bar{y} = \frac{b(d-1) - k(a-1)}{be - ck}. \tag{2.32}$$

In addition, from (2.31) we get

$$H = \frac{1 + c\bar{y}}{a}, \quad T = \frac{-c\bar{x}}{a}, \quad M = \frac{1 + k\bar{x}}{d}, \quad N = \frac{-k\bar{y}}{d}. \tag{2.33}$$

Using (2.28), (2.31), (2.32), (2.33) and after some calculations, we have

$$\begin{aligned} K_1 &= \frac{1}{ad}\Big((d-1)(a-1) + c(1-d)\bar{y} + k(1-a)\bar{x}\Big) \\ &= \frac{1}{ad(be-ck)}\Big(e(a-1) - c(d-1)\Big)\Big(b(d-1) - k(a-1)\Big) > 0, \end{aligned}$$

$$\begin{aligned} K_2 &= \frac{2}{ad}\Big(d(a-1) + a(d-1) - cd\bar{y} - ak\bar{x} + 2ck\bar{x}\bar{y}\Big) \\ &= \frac{2}{ad}\Big(d(c\bar{y} + b\bar{x}) + a(e\bar{y} + k\bar{x}) - cd\bar{y} - ak\bar{x} + 2ck\bar{x}\bar{y}\Big) \\ &= \frac{2}{ad}\Big(db\bar{x} + ae\bar{y} + 2ck\bar{x}\bar{y}\Big) > 0, \end{aligned}$$

$$\begin{aligned} K_3 &= \frac{2}{ad}\Big(3ad - 1 - k\bar{x} - c\bar{y} - 4ck\bar{x}\bar{y}\Big) \\ &= \frac{2}{ad}\Big(3(1 + c\bar{y} + b\bar{x})(1 + e\bar{y} + k\bar{x}) - 1 - k\bar{x} - c\bar{y} - 4ck\bar{x}\bar{y}\Big) \\ &= \frac{2}{ad}\Big(2 + 3b\bar{x} + 3e\bar{y} + 2k\bar{x} + 2c\bar{y} + 3ce\bar{y}^2 + 3bk\bar{x}^2 + (3be - ck)\bar{x}\bar{y}\Big) > 0, \end{aligned}$$

$$K_5 \quad = 1 + \frac{1+c\bar{y}}{a} + \frac{1+k\bar{x}}{d} + \frac{1+k\bar{x}+c\bar{y}}{ad} > 0.$$

Moreover, from (2.27) and (2.31), we have

$$\begin{aligned}
&K_4K_2K_3 - K_4^2K_1 - K_2^2K_5 \\
&= \Big(\frac{ck\bar{x}\bar{y}}{ad}+1\Big)^2\Big(1-\frac{ck\bar{x}\bar{y}}{ad}-\frac{(1+c\bar{y})(1+k\bar{x})}{ad}\Big)+\frac{ck\bar{x}\bar{y}}{ad}\Big(\frac{1+c\bar{y}}{a}+\frac{1+k\bar{x}}{d}\Big)^2 \\
&= \frac{1}{ad}\Big(1+\frac{ck\bar{x}\bar{y}}{ad}\Big)^2\Bigg((1+c\bar{y}+b\bar{x})(1+e\bar{y}+k\bar{x})-ck\bar{x}\bar{y}-(1+c\bar{y})(1+k\bar{x})\Bigg) \\
&\quad +\frac{ck\bar{x}\bar{y}}{ad}\Big(\frac{1+c\bar{y}}{a}+\frac{1+k\bar{x}}{d}\Big)^2 \\
&= \frac{1}{ad}\Big(1+\frac{ck\bar{x}\bar{y}}{ad}\Big)^2\Big(e\bar{y}+b\bar{x}+ce\bar{y}^2+bk\bar{x}^2+(be-ck)\bar{x}\bar{y}\Big) \\
&\quad +\frac{ck\bar{x}\bar{y}}{ad}\Big(\frac{1+c\bar{y}}{a}+\frac{1+k\bar{x}}{d}\Big)^2 > 0.
\end{aligned}$$

Then from Lemma 2.4, all the roots of (2.30) are of modulus less than 1 and so the positive equilibrium $(\bar{x},\bar{y})$ of (1.4) is locally asymptotically stable. So from Proposition 2.3, $(\bar{x},\bar{y})$ is globally asymptotically stable. This completes the proof. □

3 Study of System (1.2)

In the first proposition we study the existence of a positive equilibrium of (1.2).

Proposition 3.1. *Consider system* (1.2) *such that*

$$ad > 1. \tag{3.1}$$

Then system (1.2) *has a unique positive equilibrium.*

Proof. We consider the system of algebraic equations

$$x = \frac{ay}{1+Cy+Bx}, \quad y = \frac{dx}{1+Kx+Ey}, \tag{3.2}$$

where the constants C, B, K, E are defined in (2.4). System (3.2) is equivalent to the system

$$\begin{aligned} f(x) \quad &= B(EB-KC)x^3 + (2BE+KBa-CK-BC-dC^2)x^2 \\ &+(aK+aB-C+2adC+E)x + a(1-da) = 0, \end{aligned} \tag{3.3}$$

$$y = \frac{Bx^2+x}{a-Cx}. \tag{3.4}$$

Suppose that

$$EB - KC < 0. \tag{3.5}$$

From (3.1) and (3.3), it follows that

$$f\left(\frac{a}{C}\right) = \frac{B^2Ea^3}{C^3} + \frac{2BEa^2}{C^2} + \frac{Ea}{C} > 0, \quad f(0) = a(1 - da) < 0. \tag{3.6}$$

Moreover, from (3.3) and (3.5), we have

$$f(-\infty) > 0, \quad f(\infty) < 0. \tag{3.7}$$

Therefore, from (3.6) and (3.7), it follows that equation (3.3) has one solution in the interval $(-\infty, 0)$, one solution in $\left(0, \frac{a}{C}\right)$ and one solution in the interval $\left(\frac{a}{C}, \infty\right)$. Therefore, equation (3.3) has a unique solution in the interval $\left(0, \frac{a}{C}\right)$ and so system (3.2) has a unique positive solution $(\bar{x}, \bar{y})$ such that $\bar{x} \in \left(0, \frac{a}{C}\right)$ and $\bar{y}$ satisfies (3.4).

Now, suppose that

$$EB - KC > 0. \tag{3.8}$$

Then either inequality

$$E > C \tag{3.9}$$

or

$$B > K \tag{3.10}$$

holds. Firstly, consider that (3.9) is satisfied. Using (3.6), we have that equation (3.3) has a solution in the interval $\left(0, \frac{a}{C}\right)$. We prove that (3.3) has a unique solution in $\left(0, \frac{a}{C}\right)$. We set

$$\theta = EB^2 - BKC, \quad \lambda = 2BE + KBa - CK - BC - dC^2,$$
$$\mu = aK + aB - C + 2adC + E, \quad \nu = a(1 - da).$$

Let

$$\lambda \geq 0. \tag{3.11}$$

Then if p_1, p_2, p_3 are the roots of (3.3), from (3.1), (3.8) and (3.11), we get

$$p_1 + p_2 + p_3 = -\frac{\lambda}{\theta} \leq 0, \quad p_1p_2p_3 = -\frac{\nu}{\theta} > 0,$$

and so equation (3.3) has a unique solution in the interval $\left(0, \frac{a}{C}\right)$.

Suppose now that

$$\lambda < 0. \tag{3.12}$$

Consider equation

$$f'(x) = 3\theta x^2 + 2\lambda x + \mu = 0. \tag{3.13}$$

If

$$\Delta = \lambda^2 - 3\theta\mu \leq 0,$$

then $f'(x) > 0$ for every $x \in (-\infty, \infty)$ and so equation (3.3) has a unique solution in the interval $\left(0, \dfrac{a}{C}\right)$. Suppose now that

$$\Delta > 0. \tag{3.14}$$

After some calculations and using (3.8) and (3.9), we take

$$f'\left(\frac{a}{C}\right) = \frac{(aB+C)\Big(a(BE-CK) + 2aBE + C(E-C)\Big)}{C^2} > 0. \tag{3.15}$$

Let q_1, q_2 be the roots of (3.13) such that $q_1 < q_2$. Then using (3.15) we have either relation

$$\frac{a}{C} < q_1 \tag{3.16}$$

or

$$q_2 < \frac{a}{C} \tag{3.17}$$

holds. If (3.16) is satisfied, then $f'(x) > 0$ for every $x \in \left(0, \dfrac{a}{C}\right)$ and so (3.3) has a unique solution in the interval $\left(0, \dfrac{a}{C}\right)$.

We prove that (3.17) does not hold. Suppose on the contrary that (3.17) is true. Then we must have

$$C\lambda + 3\theta a > 0. \tag{3.18}$$

Then from (3.12) and (3.18) we have

$$P < d < Q, \tag{3.19}$$

$$P = \frac{BE - CK + B(E-C) + KBa}{C^2},$$

$$Q = \frac{(3aB+C)(BE-CK) + BC(E-C) + KBCa}{C^3}.$$

After some calculations, we get

$$\begin{aligned} \Delta(d) \quad &= \lambda^2 - 3\theta\mu = -3B(BE-CK)(-C+E+aB+2Cad+aK) \\ &\quad +(C^2 d + CK + BC - 2EB - aBK)^2. \end{aligned} \tag{3.20}$$

Moreover, using (3.8), (3.9) and (3.20), we have

$$\begin{aligned} &\Delta(P) \\ &= -\frac{3B(BE-CK)\Big(a(BE-CK) + 2aBE + (aB+C)(E-C) + 2KBa^2\Big)}{C} \\ &\qquad < 0, \end{aligned} \tag{3.21}$$

$$\Delta(Q) = -\frac{3B(aB+C)(BE-CK)\Big(a(BE-CK)+2aBE+C(E-C)\Big)}{C^2} < 0. \tag{3.22}$$

Therefore from (3.21) and (3.22) it follows that

$$\Delta(d) < 0, \;\; P < d < Q$$

which contradicts to (3.14). So (3.17) is not true which means that if relations (3.8) and (3.9) are satisfied, then equation (3.3) has a unique solution in $\left(0, \frac{a}{C}\right)$. Therefore, system (3.2) has a unique solution $(\bar{x}, \bar{y})$ such that $\bar{x} \in \left(0, \frac{a}{C}\right)$ and $\bar{y}$ satisfies (3.4).

Suppose now that (3.10) holds. System (3.2) is equivalent to the system

$$\begin{array}{l} E(EB-KC)y^3 + (2BE+CEd-EK-KC-aK^2)y^2 \\ +(dE+dC-K+2adK+B)y + d(1-da) = 0, \end{array} \tag{3.23}$$

$$x = \frac{Ey^2+y}{d-Ky}. \tag{3.24}$$

Then arguing as above, we can prove that system (3.2) has a unique solution $(\bar{x}, \bar{y})$ such that relations (3.23) and (3.24) are satisfied.

Finally, suppose that

$$EB - KC = 0. \tag{3.25}$$

Then equation (3.3) becomes

$$h(x) = \zeta x^2 + \mu x + \nu = 0, \;\; \zeta = BE + KBa - BC - dC^2. \tag{3.26}$$

If $\zeta \neq 0$, then using

$$h\left(\frac{a}{C}\right) = \frac{a^2BE}{C^2} + \frac{aE}{C} + \frac{a^3BK}{C^2} + \frac{a^2K}{C} > 0, \;\; h(0) = \nu < 0,$$

we have that equation (3.26) has a unique solution in $\left(0, \frac{a}{C}\right)$.

Finally, suppose that $\zeta = 0$. Then since

$$h\left(\frac{a}{C}\right) = \frac{a^2K}{C} + \frac{a^2B}{C} + \frac{aE}{C} + a^2d > 0, \;\; h(0) = \nu < 0,$$

equation (3.26) has a unique solution in $\left(0, \frac{a}{C}\right)$. This completes the proof. □

In the following proposition we study the boundedness and persistence of the positive solutions of system (1.2).

Proposition 3.2. *Consider system* (1.2)*, where* (3.1) *holds and*

$$c_0 \neq 0, \;\; k_0 \neq 0. \tag{3.27}$$

Then every positive solution of (1.2) *is bounded and persists.*

Proof. Let (x_n, y_n) be an arbitrary solution of (1.2). Then using (1.2) and (3.27) it is obvious that

$$x_n \leq \frac{a}{c_0}, \;\; y_n \leq \frac{d}{k_0}, \;\; n = 1, 2, \ldots, \tag{3.28}$$

which implies that (x_n, y_n) is a bounded solution. We prove that (x_n, y_n) persists. Suppose that x_n does not persist. Without loss of generality we may assume that there exists a subsequence n_r such that relations (2.7) are satisfied. Hence, from (1.2)

$$x_{n_r} = \frac{a y_{n_r-1}}{1 + \sum_{i=0}^{m} c_i y_{n_r-1-2i} + \sum_{i=0}^{p} b_i x_{n_r-2i-2}}.$$

Then from (2.7) and (3.28), we get

$$\lim_{r\to\infty} y_{n_r-1} = 0. \tag{3.29}$$

In addition, since from (1.2)

$$y_{n_r-1} = \frac{d x_{n_r-2}}{1 + \sum_{i=0}^{s} k_i x_{n_r-2-2i} + \sum_{i=0}^{q} e_i y_{n_r-2i-3}},$$

relation (3.29) implies that

$$\lim_{r\to\infty} x_{n_r-2} = 0.$$

Working inductively, we can prove that

$$\lim_{r\to\infty} x_{n_r-2i} = 0, \;\; \lim_{r\to\infty} y_{n_r-2i-1} = 0, \;\; i = 0, 1, \ldots. \tag{3.30}$$

Therefore, from (3.1) and (3.30) for sufficiently large n_r, we get

$$x_{n_r-2j} \leq \epsilon, \; j \in \{1, 2, \ldots, \phi+1\}, \;\; y_{n_r-2w-1} \leq \epsilon, \; w \in \{0, 1, \ldots, \psi+1\}, \tag{3.31}$$

where $\phi = \max\{p, s\}$, $\psi = \max\{m, q\}$ and ϵ is a sufficiently small positive number such that

$$\frac{ad}{(1+\epsilon(C+B))(1+\epsilon(K+E))} > 1. \tag{3.32}$$

Moreover, from (1.4), we have

$$x_{n_r} = \frac{a d x_{n_r-2}}{\left(1 + \sum_{i=0}^{m} c_i y_{n_r-1-2i} + \sum_{i=0}^{p} b_i x_{n_r-2i-2}\right)\left(1 + \sum_{i=0}^{s} k_i x_{n_r-2-2i} + \sum_{i=0}^{q} e_i y_{n_r-2i-3}\right)}. \tag{3.33}$$

Therefore, from relations (3.31)–(3.33), it follows that

$$x_{n_r} > \frac{adx_{n_r-2}}{(1+(C+B)\epsilon)(1+(K+E)\epsilon)} > x_{n_r-2}$$

which contradicts to (2.7). Therefore x_n persists. Using the same argument we can easily prove that also y_n persists. This completes the proof. □

In the next proposition we study the convergence of the positive solutions of the system (1.3) to the unique positive equilibrium.

Proposition 3.3. *Suppose that relations* (3.1), (3.27) *and*

$$c_0 \geq E, \;\; k_0 \geq B \tag{3.34}$$

hold. Then every positive solution of (1.3) *tends to the unique positive equilibrium of* (1.3).

Proof. Suppose that either $c_0 \neq E$ or $k_0 \neq B$ holds. Using Proposition 3.2 we have that (2.21) are satisfied. Then from (1.3) we take

$$\begin{aligned} L_1 &\leq \frac{aL_2}{1+c_0L_2+Bl_1}, \;\; l_1 \geq \frac{al_2}{1+c_0l_2+BL_1}, \\ L_2 &\leq \frac{dL_1}{1+k_0L_1+El_2}, \;\; l_2 \geq \frac{dl_1}{1+k_0l_1+EL_2}. \end{aligned} \tag{3.35}$$

Then relations (3.35) imply that

$$\begin{aligned} L_1L_2 &\leq \frac{adL_1L_2}{(1+c_0L_2+Bl_1)(1+k_0L_1+El_2)}, \\ l_1l_2 &\geq \frac{adl_1l_2}{(1+c_0l_2+BL_1)(1+k_0l_1+EL_2)} \end{aligned}$$

which implies that

$$(1+c_0L_2+Bl_1)(1+k_0L_1+El_2) \leq (1+c_0l_2+BL_1)(1+k_0l_1+EL_2). \tag{3.36}$$

So from (3.34), (3.36), we have

$$(k_0-B)(L_1-l_1)+(c_0-E)(L_2-l_2)+(c_0k_0-BE)(L_1L_2-l_1l_2) \leq 0$$

which implies that $L_1 = l_1$, $L_2 = l_2$.

Now, suppose that

$$c_0 = E, \;\; k_0 = B. \tag{3.37}$$

Then from (3.35) we get

$$d\frac{l_1}{l_2} \leq 1+Bl_1+c_0L_2 \leq a\frac{L_2}{L_1}, \;\; a\frac{l_2}{l_1} \leq 1+c_0l_2+BL_1 \leq d\frac{L_1}{L_2},$$

from which we take

$$dl_1 L_1 = al_2 L_2. \tag{3.38}$$

From (3.35), (3.37) and (3.38), we get

$$\begin{aligned}
L_1 l_2 &\leq \frac{aL_2 l_2}{1 + c_0 L_2 + Bl_1} = \frac{dL_1 l_1}{1 + c_0 L_2 + Bl_1},\\
l_1 L_2 &\geq \frac{aL_2 l_2}{1 + c_0 l_2 + BL_1} = \frac{dL_1 l_1}{1 + c_0 l_2 + BL_1},\\
L_2 l_1 &\leq \frac{dL_1 l_1}{1 + BL_1 + c_0 l_2} = \frac{aL_2 l_2}{1 + BL_1 + c_0 l_2},\\
L_1 l_2 &\geq \frac{dL_1 l_1}{1 + Bl_1 + c_0 L_2} = \frac{aL_2 l_2}{1 + Bl_1 + c_0 L_2}.
\end{aligned} \tag{3.39}$$

Then relations (3.35), (3.37) and (3.39) imply that

$$\begin{aligned}
l_2 &= \frac{dl_1}{1 + c_0 L_2 + Bl_1}, \quad L_2 = \frac{dL_1}{1 + c_0 l_2 + BL_1},\\
l_1 &= \frac{al_2}{1 + BL_1 + c_0 l_2}, \quad L_1 = \frac{aL_2}{1 + Bl_1 + c_0 L_2}.
\end{aligned} \tag{3.40}$$

Without loss of generality we may assume that there exists a subsequence n_r such that

$$\begin{aligned}
&\lim_{r\to\infty} x_{n_r} = L_1, \quad \lim_{r\to\infty} x_{n_r - i} = M_i, \quad i = 2, 3, \ldots, 2p+2,\\
&\lim_{r\to\infty} y_{n_r - i} = R_i, \quad i = 1, 2, \ldots, 2q+3.
\end{aligned} \tag{3.41}$$

From (1.3) we have

$$x_{n_r} = \frac{ay_{n_r-1}}{1 + c_0 y_{n_r-1} + \sum_{i=0}^{p} b_i x_{n_r-2i-2}}$$

and so from (3.41) we get

$$L_1 = \frac{aR_1}{1 + c_0 R_1 + \sum_{i=0}^{p} b_i M_{2i+2}} \leq \frac{aL_2}{1 + Bl_1 + c_0 L_2}. \tag{3.42}$$

Thus, from (3.40) and (3.42) we have

$$L_2 = R_1, \quad M_{2i+2} = l_1, \quad i = 0, 1, \ldots, p. \tag{3.43}$$

In addition, from (1.3) we get

$$y_{n_r-1} = \frac{dx_{n_r-2}}{1 + k_0 x_{n_r-2} + \sum_{i=0}^{q} e_i y_{n-2i-3}},$$

and so from (3.41) we have

$$L_2 = \frac{dM_2}{1 + k_0 M_2 + \sum_{i=0}^{q} e_i R_{2i+3}} \leq \frac{dL_1}{1 + k_0 L_1 + E l_2}. \tag{3.44}$$

Then from (3.40) and (3.44) we get

$$M_2 = L_1, \quad R_{2i+3} = l_2, \quad i = 0, 1, \dots, q. \tag{3.45}$$

Therefore, from (3.43) and (3.45) we take

$$M_2 = L_1 = l_1. \tag{3.46}$$

Finally, using (3.40) and (3.46) it is obvious that

$$L_2 = l_2.$$

This completes the proof. □

In the last proposition we study the global asymptotic stability of the positive equilibrium of (1.5). We need the following lemma which has been proved in [26]. For readers' convenience we state it here without its proof.

Lemma 3.4. *Consider the algebraic equation*

$$x^2 + a_1 x + a_0 = 0. \tag{3.47}$$

Then all roots of (3.47) *are of modulus less than 1 if and only if*

$$|a_1| < a_0 + 1 < 2. \tag{3.48}$$

Proposition 3.5. *Consider system* (1.5)*, where* (3.1) *holds. Suppose also that*

$$c > e, \quad k > b. \tag{3.49}$$

Then the unique positive equilibrium $(\bar{x}, \bar{y})$ *of* (1.5) *is globally asymptotically stable.*

Proof. We prove that $(\bar{x}, \bar{y})$ is locally asymptotically stable. The linearized system of (1.5) about $(\bar{x}, \bar{y})$ is

$$\begin{aligned} x_{n+1} &= -\frac{ab\bar{y}}{(1 + c\bar{y} + b\bar{x})^2} x_{n-1} + \frac{a(1 + b\bar{x})}{(1 + c\bar{y} + b\bar{x})^2} y_n, \\ y_{n+1} &= \frac{d(1 + e\bar{y})}{(1 + k\bar{x} + e\bar{y})^2} x_n - \frac{ed\bar{x}}{(1 + k\bar{x} + e\bar{y})^2} y_{n-1}. \end{aligned} \tag{3.50}$$

It is obvious that system (3.50) is equivalent to the system

$$w_{n+1} = Aw_n, \quad A = \begin{pmatrix} 0 & H & T & 0 \\ M & 0 & 0 & N \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad w_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix},$$

$$H = \frac{a(1+b\bar{x})}{(1+c\bar{y}+b\bar{x})^2}, \quad T = -\frac{ab\bar{y}}{(1+c\bar{y}+b\bar{x})^2},$$

$$M = \frac{d(1+e\bar{y})}{(1+k\bar{x}+e\bar{y})^2}, \quad N = -\frac{ed\bar{x}}{(1+k\bar{x}+e\bar{y})^2}.$$

Then the characteristic equation of A is

$$\lambda^4 - \lambda^2(HM+N+T) + NT = 0. \tag{3.51}$$

Since $(\bar{x}, \bar{y})$ is the positive equilibrium of (1.5), we have

$$\frac{1}{1+c\bar{y}+b\bar{x}} = \frac{\bar{x}}{a\bar{y}}, \quad \frac{1}{1+k\bar{x}+e\bar{y}} = \frac{\bar{y}}{d\bar{x}}. \tag{3.52}$$

Hence,

$$H = \frac{(1+b\bar{x})\bar{x}^2}{a\bar{y}^2}, \quad T = -\frac{b\bar{x}^2}{a\bar{y}}, \quad M = \frac{(1+e\bar{y})\bar{y}^2}{d\bar{x}^2}, \quad N = -\frac{e\bar{y}^2}{d\bar{x}}. \tag{3.53}$$

Relations (3.49), (3.52) and (3.53) imply that

$$NT = \frac{be}{ad}\bar{x}\bar{y} < \frac{be}{ad}\frac{a}{c}\frac{d}{k} = \frac{be}{ck} < 1. \tag{3.54}$$

Moreover, from (3.1), (3.52) we have

$$\begin{aligned} HM+N+T-NT &= \frac{1}{ad}\Big(1+e\bar{y}+b\bar{x}-bd\frac{\bar{x}^2}{\bar{y}}-ea\frac{\bar{y}^2}{\bar{x}}\Big) \\ &= \frac{1}{ad}\Big(1+e\bar{y}+b\bar{x}-b\bar{x}-bk\bar{x}^2-be\bar{x}\bar{y}-e\bar{y}-ce\bar{y}^2-eb\bar{x}\bar{y}\Big) \\ &= \frac{1}{ad}\Big(1-bk\bar{x}^2-2be\bar{x}\bar{y}-ce\bar{y}^2\Big) < 1, \end{aligned} \tag{3.55}$$

$$\begin{aligned} HM+N+T+NT+1 &= \frac{1}{ad}\Big(1+e\bar{y}+b\bar{x}+2be\bar{x}\bar{y}-bd\frac{\bar{x}^2}{\bar{y}}-ea\frac{\bar{y}^2}{\bar{x}}\Big)+1 \\ &= \frac{1}{ad}\Big(1+e\bar{y}+b\bar{x}+2be\bar{x}\bar{y}-e\bar{y}-ce\bar{y}^2-eb\bar{x}\bar{y}-b\bar{x}-bk\bar{x}^2-be\bar{x}\bar{y}\Big)+1 \\ &= \frac{1}{ad}\Big(ad+1-bk\bar{x}^2-ce\bar{y}^2\Big) \\ &= \frac{1}{ad}\Big((1+c\bar{y}+b\bar{x})(1+k\bar{x}+e\bar{y})+1-bk\bar{x}^2-ce\bar{y}^2\Big) > 0. \end{aligned} \tag{3.56}$$

Therefore, relations (3.54), (3.55) and (3.56) imply that all conditions of Lemma 3.4 are satisfied. Therefore, all the roots of equation (3.51) are of modulus less than 1 which implies that $(\bar{x}, \bar{y})$ is locally asymptotically stable. Using Proposition 3.3, $(\bar{x}, \bar{y})$ is globally asymptotically stable. This completes the proof. □

4 Conclusion

In this paper, we considered two systems of rational difference equations of the form (1.1) and (1.2). These systems are modifications of the Beverton–Holt equation which is the discrete analog of the logistic differential equation studied in [12]. Systems of this form are worthwhile studying since many authors studied discrete competition models (see [13] and the references cited therein).

The main results of this paper were presented in two sections. In Section 2 we studied system (1.1) and in Section 3 system (1.2). Summarizing the results of Sections 2 and 3 we get the following statements, concerning both systems.

(i) We studied the existence and the uniqueness of the positive equilibrium of the systems.

(ii) We found conditions so that every positive solution of the systems is bounded and persists.

(iii) We investigated the convergence of the positive solutions of the system (1.1) and system (1.3) which is a special case of system (1.2).

(iv) We studied the global asymptotic stability to the unique positive equilibrium of systems (1.4) and (1.5), which are special cases of systems (1.1) and (1.2) respectively.

Finally, we state the following open problems.

Open Problem 4.1. Consider the systems of difference equations (1.1) and (1.2), where a, d, b_i, $i = 0, 1, \dots, p$, c_i, $i = 0, 1, \dots, m$, e_i, $i = 0, 1, \dots, q$, $k_i \in 0, 1, \dots, s$ are nonnegative constants, the initial values of (1.1) x_i, $i = -\pi, -\pi + 1, \dots, 0$, y_i, $i = -\tau, -\tau + 1, \dots, 0$, $\pi = \max\{p, s\}$, $\tau = \max\{m, q\}$ are positive real numbers and the initial values of (1.2) x_i, $i = -\lambda, -\lambda + 1, \dots, 0$, y_i, $i = -\mu, -\mu + 1, \dots, 0$, $\lambda = \max\{2p + 1, 2s\}$, $\mu = \max\{2q + 1, 2m\}$ are also positive real numbers. Prove that:

I. If relations (2.1) and (2.5) are satisfied, then every positive solution of (1.1) tends to the unique positive equilibrium of (1.1).

II. If relations (3.1) and (3.27) hold, then every positive solution of (1.2) tends to the unique positive equilibrium of (1.2).

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