

## Unboundedness for some Classes of Rational Difference Equations

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### Abstract

We study the rational difference equation

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}.$$

Particularly, we show that for nonnegative  $\alpha$  and  $C$ , whenever  $C\alpha = 0$  and  $C + \alpha > 0$ , unbounded solutions exist for some choice of nonnegative initial conditions. Moreover, we study the rational difference equation

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, \quad n \in \mathbb{N}.$$

Particularly, we show that whenever  $0 < \beta < \frac{1}{3}$  and  $\alpha \in [0, 1]$ , unbounded solutions exist for some choice of nonnegative initial conditions.

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## 1 Introduction

In [1] Camouzis and Ladas devote a chapter to the study of unbounded solutions for the  $k$ th order rational difference equation with nonnegative parameters and nonnegative initial conditions

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j}}, \quad n \in \mathbb{N}.$$

In the introduction of said chapter, the authors of [1] pose five conjectures regarding the boundedness character of five different special cases of the third order rational difference equation. Particularly we are referring to the special cases #28, #44, #56, #70, and #120. These are the only remaining cases of third order for which the boundedness character has not been established.

First we study special cases #56 and #120

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, \quad n \in \mathbb{N}.$$

Using a standard induction technique, we show that whenever  $0 < \beta < \frac{1}{3}$  and  $\alpha \in [0, 1]$ , unbounded solutions exist for some choice of nonnegative initial conditions.

We then study special cases #44 and #28

$$x_n = \frac{\alpha + x_{n-1}}{C x_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}.$$

We show that for nonnegative  $\alpha$  and  $C$ , whenever  $C\alpha = 0$  and  $C + \alpha > 0$ , unbounded solutions exist for some choice of nonnegative initial conditions. The proof is presented in two special cases. The case where  $\alpha > 0$  and the case where  $C > 0$ .

## 2 Todd's Equation

Consider the third order rational difference equation

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, \quad n \in \mathbb{N}. \quad (2.1)$$

There have been significant results concerning the case where  $\beta = 1$ . In this case the equation is generally referred to by the cognomen "Todd's equation" and possesses the invariant

$$(\alpha + x_n + x_{n-1} + x_{n-2}) \left(1 + \frac{1}{x_n}\right) \left(1 + \frac{1}{x_{n-1}}\right) \left(1 + \frac{1}{x_{n-2}}\right) = \text{constant}.$$

For more information regarding Todd's equation see [5–7]. In the following theorem we show that whenever  $0 < \beta < \frac{1}{3}$  and  $\alpha \in [0, 1]$ , (2.1) has unbounded solutions for some choice of nonnegative initial conditions.

**Theorem 2.1.** Suppose  $0 < \beta < \frac{1}{3}$  and  $\alpha \in [0, 1]$ . Then Equation (2.1) has unbounded solutions for some initial conditions.

*Proof.* Choose initial conditions so that

$$\min\{x_0, x_{-2}\} > \max\left\{\frac{1}{\beta}, \frac{x_{-1}}{\beta}\right\}.$$

We shall first prove by induction that for all  $j \in \mathbb{N}$ ,

$$\min\{x_{2j}, x_{2j-2}\} > \max\left\{\frac{1}{\beta}, \frac{x_{2j-1}}{\beta}\right\}. \quad (2.2)$$

The initial conditions provide the base case. Assume the following holds for some  $j \in \mathbb{N}$ :

$$\min\{x_{2j-2}, x_{2j-4}\} > \max\left\{\frac{1}{\beta}, \frac{x_{2j-3}}{\beta}\right\}.$$

Since  $\beta x_{2j-2} > x_{2j-3}, x_{2j-2} > \frac{1}{\beta}, \beta x_{2j-2} > 1 \geq \alpha$ , and  $x_{2j-4} > \frac{1}{\beta} > 3$ , we see that

$$x_{2j-1} = \frac{\alpha + \beta x_{2j-2} + x_{2j-3}}{x_{2j-4}} < \frac{3\beta x_{2j-2}}{x_{2j-4}} < \beta x_{2j-2}.$$

Thus we have shown

$$x_{2j-2} > \max\left\{\frac{1}{\beta}, \frac{x_{2j-1}}{\beta}\right\}.$$

Since  $\beta x_{2j-4} > x_{2j-3}$  and  $0 < \beta < \frac{1}{3}$ , we have

$$x_{2j} = \frac{\alpha + \beta x_{2j-1} + x_{2j-2}}{x_{2j-3}} > \frac{x_{2j-2}}{x_{2j-3}} > \frac{x_{2j-2}}{\beta x_{2j-4}} > \frac{3x_{2j-2}}{x_{2j-4}} = \frac{3\beta x_{2j-2}}{\beta x_{2j-4}} > \frac{x_{2j-1}}{\beta}.$$

Also

$$x_{2j} > \frac{x_{2j-2}}{x_{2j-3}} > \frac{x_{2j-2}}{\beta x_{2j-2}} = \frac{1}{\beta}.$$

Thus

$$\min\{x_{2j}, x_{2j-2}\} > \max\left\{\frac{1}{\beta}, \frac{x_{2j-1}}{\beta}\right\}.$$

This completes the induction proof for (2.2).

Using (2.2), we now prove that  $x_{8\eta} > \frac{x_{8\eta-8}}{9\beta^2}$  for all  $\eta \in \mathbb{N}$ :

$$\begin{aligned} x_{8\eta} &= \frac{\alpha + \beta x_{8\eta-1} + x_{8\eta-2}}{x_{8\eta-3}} > \frac{x_{8\eta-2}}{x_{8\eta-3}} \\ &= \left( \frac{x_{8\eta-6}}{\alpha + \beta x_{8\eta-4} + x_{8\eta-5}} \right) \left( \frac{\alpha + \beta x_{8\eta-3} + x_{8\eta-4}}{x_{8\eta-5}} \right) \\ &> \left( \frac{x_{8\eta-6}}{3\beta x_{8\eta-4}} \right) \left( \frac{x_{8\eta-4}}{x_{8\eta-5}} \right) = \frac{x_{8\eta-6}}{3\beta x_{8\eta-5}} \\ &= \frac{x_{8\eta-6} x_{8\eta-8}}{3\beta(\alpha + \beta x_{8\eta-6} + x_{8\eta-7})} > \frac{x_{8\eta-6} x_{8\eta-8}}{9\beta^2 x_{8\eta-6}} = \frac{x_{8\eta-8}}{9\beta^2}. \end{aligned}$$

Since  $0 < \beta < \frac{1}{3}$ ,  $9\beta^2 < 1$ . Thus we have a subsequence of our solution which diverges to  $\infty$ . Hence the solution is unbounded.  $\square$

### 3 Special Case #44

We now study special case #44

$$x_n = \frac{\alpha + x_{n-1}}{x_{n-3}}, \quad n \in \mathbb{N}. \quad (3.1)$$

Particularly, we show that, whenever  $\alpha > 0$ , Equation (3.1) has unbounded solutions for some initial conditions.

The following lemma provides a useful technique for constructing divergent subsequences of solutions for rational difference equations.

**Lemma 3.1.** *Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence in  $[0, \infty)$ . Suppose that there exists  $D > 1$  and hypotheses  $H_1, \dots, H_k$  so that for all  $n \in \mathbb{N}$  there exists  $p_n \in \mathbb{N}$  so that the following holds: Whenever  $x_{n-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$ , then  $x_{n+p_n-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$  and  $x_{n+p_n-1} \geq Dx_{n-1}$ . Further assume that for some  $N \in \mathbb{N}$ ,  $x_{N-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$  and  $x_{N-1} > 0$ . Then  $\{x_n\}_{n=1}^{\infty}$  is unbounded. Particularly  $\{x_{z_m-1}\}_{m=1}^{\infty}$  is a subsequence of  $\{x_n\}_{n=1}^{\infty}$  which diverges to  $\infty$ , where  $z_m = z_{m-1} + p_{z_{m-1}}$  and  $z_0 = N$ .*

*Proof.* Let  $z_m = z_{m-1} + p_{z_{m-1}}$  and  $z_0 = N$ . Using induction, we prove that given  $m \in \mathbb{N}$  the following holds:  $x_{z_m-1} \geq D^m x_{N-1}$  and  $x_{z_m-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$ . By assumption,  $x_{N-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$  and  $x_{N-1} \geq D^0 x_{N-1}$ . This provides the base case. Assume  $x_{z_{m-1}-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$  and  $x_{z_{m-1}-1} \geq D^{m-1} x_{N-1}$ . Using our earlier assumption, this implies that there exists  $p_{z_{m-1}}$  so that  $x_{z_{m-1}+p_{z_{m-1}}-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$  and  $x_{z_{m-1}+p_{z_{m-1}}-1} \geq D x_{z_{m-1}-1} \geq (D) D^{m-1} x_{N-1} = D^m x_{N-1}$ . So we have shown that  $x_{z_m-1} \geq D^m x_{N-1}$  for all  $m \in \mathbb{N}$ . Hence the subsequence  $\{x_{z_m-1}\}_{m=1}^{\infty}$  of  $\{x_n\}_{n=1}^{\infty}$  clearly diverges to  $\infty$  since  $D > 1$ .  $\square$

The above argument merely simplifies the following arguments by removing a somewhat onerous construction.

**Theorem 3.2.** *If  $\alpha > 0$ , then Equation (3.1) has unbounded solutions for some initial conditions.*

*Proof.* We choose initial conditions so that

$$\begin{aligned}x_0 &> \max \left\{ \frac{2^{15}}{\alpha^3}, \frac{(\alpha + 1)^4 2^{15}}{\alpha} \right\}, \\x_{-1} &> \max \left\{ \frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11} \right\}, \\x_{-2} &> \frac{\alpha}{2}.\end{aligned}$$

We show that there exists  $D = \frac{4}{3}$  so that for all  $n \in \mathbb{N}$  there exists  $p_n \in \{7, 8\}$  so that the following holds: Whenever

$$\begin{aligned}x_{n-1} &> \max \left\{ \frac{2^{15}}{\alpha^3}, \frac{(\alpha + 1)^4 2^{15}}{\alpha} \right\}, \\x_{n-2} &> \max \left\{ \frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11} \right\}, \\x_{n-3} &> \frac{\alpha}{2},\end{aligned}$$

then we have

$$\begin{aligned}x_{n+p_n-1} &> \max \left\{ \frac{2^{15}}{\alpha^3}, \frac{(\alpha + 1)^4 2^{15}}{\alpha} \right\}, \\x_{n+p_n-2} &> \max \left\{ \frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11} \right\}, \\x_{n+p_n-3} &> \frac{\alpha}{2}, \\x_{n+p_n-1} &\geq \left( \frac{4}{3} \right) x_{n-1}.\end{aligned}$$

First assume

$$\begin{aligned}x_{n-1} &> \max \left\{ \frac{2^{15}}{\alpha^3}, \frac{(\alpha + 1)^4 2^{15}}{\alpha} \right\}, \\x_{n-2} &> \max \left\{ \frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11} \right\}, \\x_{n-3} &> \frac{\alpha}{2}.\end{aligned}$$

Since  $x_{n-1}, x_{n-2}, x_{n-3} > 0$ , we may write  $\eta = \log_2(x_{n-1})$ ,  $\ell = \log_2(x_{n-2})$ , and  $\rho = \log_2(x_{n-3})$ . Hence  $2^\eta = x_{n-1}$ ,  $2^\ell = x_{n-2}$ , and  $2^\rho = x_{n-3}$ . We use such representations for ease of computations. First we see that

$$x_n = \frac{\alpha + x_{n-1}}{x_{n-3}} = \frac{\alpha}{x_{n-3}} + \frac{x_{n-1}}{x_{n-3}} = \frac{\alpha}{2^\rho} + 2^{\eta-\rho}, \quad (3.2)$$

$$x_{n+1} = \frac{\alpha}{x_{n-2}} + \frac{x_n}{x_{n-2}} = \frac{\alpha}{2^\ell} + \left(\frac{\alpha}{2^\rho} + 2^{\eta-\rho}\right) \frac{1}{2^\ell} = \frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}, \quad (3.3)$$

$$\begin{aligned} x_{n+2} &= \frac{\alpha}{x_{n-1}} + \frac{x_{n+1}}{x_{n-1}} = \frac{\alpha}{2^\eta} + \left(\frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}\right) \left(\frac{1}{2^\eta}\right) \\ &= \frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}}, \end{aligned} \quad (3.4)$$

$$\begin{aligned} x_{n+3} &= \frac{\alpha}{x_n} + \frac{x_{n+2}}{x_n} \\ &= \frac{\alpha 2^\rho}{\alpha + 2^\eta} + \left(\frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}}\right) \left(\frac{2^\rho}{\alpha + 2^\eta}\right). \end{aligned} \quad (3.5)$$

We will make use of these identities later. We prove the result in two cases. Let us first assume  $\ell + \rho \geq \eta$ . We show that if this inequality is satisfied for some  $n \in \mathbb{N}$ , then  $p_n = 7$ . First we prove that  $x_{n+p_n-3} = x_{n+4} > \frac{\alpha}{2}$ . Notice that

$$x_{n+4} = \frac{\alpha + x_{n+3}}{x_{n+1}} > \frac{\alpha}{x_{n+1}}.$$

From (3.3) we see that

$$\frac{\alpha}{x_{n+1}} = \frac{\alpha}{\frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}} = \frac{\alpha}{\alpha 2^{-\ell} + \alpha 2^{-\ell-\rho} + 2^{\eta-\ell-\rho}}.$$

We now use the assumption  $\ell + \rho \geq \eta$ . This assumption implies that  $2^{\eta-\ell-\rho} \leq 2^0 = 1$ . Earlier we assumed that  $2^{-\rho} < \frac{2}{\alpha}$ . Moreover, from our assumptions,

$$2^\ell > (\alpha + 1)^2 2^{11} = (\alpha^2 + 2\alpha + 1) 2^{11}$$

so  $2^{-\ell} < \frac{1}{\alpha 2^{12}}$  and  $2^{-\ell} < 2^{-11}$ . Using these inequalities we obtain

$$\frac{\alpha}{\alpha 2^{-\ell} + \alpha 2^{-\ell-\rho} + 2^{\eta-\ell-\rho}} > \frac{\alpha}{\alpha \frac{1}{\alpha 2^{12}} + \alpha 2^{-11} \frac{2}{\alpha} + 1} = \frac{\alpha}{2^{-12} + 2^{-10} + 1} > \frac{\alpha}{2}.$$

So we have shown  $x_{n+p_n-3} = x_{n+4} > \frac{\alpha}{2}$ .

We now prove that  $x_{n+p_n-2} = x_{n+5} > \max\left\{\frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11}\right\}$ . Notice that

$$x_{n+5} = \frac{\alpha + x_{n+4}}{x_{n+2}} > \frac{x_{n+4}}{x_{n+2}} > \frac{\alpha}{2x_{n+2}}.$$

Since  $\ell + \rho \geq \eta$ ,  $2^{\ell+\rho} \geq 2^\eta$ . Moreover as we have recently shown  $2^\ell > 2^{11}$ , similarly  $2^\eta > 2^{15}$ . So  $\ell > 11 > 0$  and  $\eta > 15 > 0$ . So from (3.4),

$$x_{n+2} = \frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}} < \frac{\alpha}{2^\eta} + \frac{\alpha}{2^\eta} + \frac{\alpha}{2^\eta} + \frac{1}{2^\eta} = \frac{3\alpha + 1}{2^\eta}. \quad (3.6)$$

Hence,

$$x_{n+5} > \frac{\alpha}{2x_{n+2}} > \frac{\alpha}{2\frac{3\alpha+1}{2^\eta}} = \frac{\alpha}{3\alpha + 1} 2^{\eta-1} > \left(\frac{1}{3\alpha + 1}\right) \max\left\{\frac{2^{14}}{\alpha^2}, (\alpha + 1)^4 2^{14}\right\}.$$

So,

$$\begin{aligned} x_{n+5} &> \left(\frac{1}{3\alpha + 1}\right) \max\left\{\frac{2^{14}}{\alpha^2}, (\alpha + 1)^4 2^{14}\right\} \geq \frac{(\alpha + 1)^4 2^{14}}{3\alpha + 1} > \frac{(\alpha + 1)^4 2^{14}}{3\alpha + 3} \\ &= \frac{(\alpha + 1)^3 2^{14}}{3} > (\alpha + 1)^2 2^{11}. \end{aligned}$$

When  $\alpha \geq 1$ ,  $\frac{1}{\alpha^2} \leq 1 < (\alpha + 1)^2$  so

$$(\alpha + 1)^2 2^{11} = \max\left\{\frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11}\right\}.$$

Thus the only remaining case is when  $\alpha < 1$ . In this case we have the following:

$$x_{n+5} > \left(\frac{1}{3\alpha + 1}\right) \max\left\{\frac{2^{14}}{\alpha^2}, (\alpha + 1)^4 2^{14}\right\} \geq \frac{2^{14}}{(3\alpha + 1)\alpha^2} > \frac{2^{14}}{4\alpha^2} > \frac{2^{11}}{\alpha^2}.$$

So we have shown  $x_{n+p_n-2} = x_{n+5} > \max\left\{\frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11}\right\}$ .

We now prove that  $x_{n+p_n-1} = x_{n+6} \geq \left(\frac{4}{3}\right)x_{n-1} > \max\left\{\frac{2^{15}}{\alpha^3}, \frac{(\alpha + 1)^4 2^{15}}{\alpha}\right\}$ . First assume

$$\max\left\{(\alpha + 1)2^5, \frac{(\alpha + 1)2^5}{\alpha}\right\} \geq 2^{\eta-\rho}. \quad (3.7)$$

Notice that

$$x_{n+6} = \frac{\alpha + x_{n+5}}{x_{n+3}} > \frac{x_{n+5}}{x_{n+3}} = \frac{\alpha + x_{n+4}}{x_{n+2}x_{n+3}} > \frac{x_{n+4}}{x_{n+2}x_{n+3}} = \frac{\alpha + x_{n+3}}{x_{n+2}x_{n+3}x_{n+1}} > \frac{1}{x_{n+2}x_{n+1}}.$$

We use (3.3), our induction assumption, our assumption (3.7), and the fact that  $2^{-\rho} < \frac{2}{\alpha}$  to obtain

$$x_{n+1} = \frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{2^{\eta-\rho}}{2^\ell} < \frac{\alpha}{2^\ell} + \frac{1}{2^{\ell-1}} + \frac{\max\left\{(\alpha + 1)2^5, \frac{(\alpha+1)2^5}{\alpha}\right\}}{\max\left\{\frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11}\right\}}.$$

Notice that if  $\alpha \geq 1$ ,

$$\frac{\max \left\{ (\alpha + 1)2^5, \frac{(\alpha+1)2^5}{\alpha} \right\}}{\max \left\{ \frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11} \right\}} \leq \frac{(\alpha + 1)2^5}{(\alpha + 1)^2 2^{11}} = \frac{1}{(\alpha + 1)2^6} < \frac{1}{(\alpha + 1)2^3}.$$

Also if  $\alpha < 1$ ,

$$\frac{\max \left\{ (\alpha + 1)2^5, \frac{(\alpha+1)2^5}{\alpha} \right\}}{\max \left\{ \frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11} \right\}} \leq \frac{(\alpha + 1)2^5}{\alpha \left( \frac{2^{11}}{\alpha^2} \right)} = \frac{(\alpha + 1)\alpha}{2^6} < \frac{(\alpha + 1)^2}{2^6} < \frac{1}{(\alpha + 1)2^3}.$$

So

$$\begin{aligned} x_{n+1} &< \frac{\alpha}{2^\ell} + \frac{1}{2^{\ell-1}} + \frac{1}{(\alpha + 1)2^3} < \frac{\alpha}{(\alpha + 1)^2 2^{11}} + \frac{1}{(\alpha + 1)^2 2^{10}} + \frac{1}{(\alpha + 1)2^3} \\ &< \frac{1}{(\alpha + 1)2^{11}} + \frac{1}{(\alpha + 1)2^{10}} + \frac{1}{(\alpha + 1)2^3} = \frac{1}{\alpha + 1} (2^{-11} + 2^{-10} + 2^{-3}) \\ &< \frac{1}{4\alpha + 4}. \end{aligned}$$

Now using the inequality we have just shown and (3.6), we have

$$x_{n+6} > \frac{1}{x_{n+1}x_{n+2}} > \frac{4\alpha + 4}{3\alpha + 1} (2^\eta) > \frac{4\alpha + 4}{3\alpha + 3} (2^\eta) = \left( \frac{4}{3} \right) x_{n-1}.$$

Thus we have shown  $x_{n+p_n-1} = x_{n+6} \geq \left( \frac{4}{3} \right) x_{n-1} > \max \left\{ \frac{2^{15}}{\alpha^3}, \frac{(\alpha + 1)^4 2^{15}}{\alpha} \right\}$  when (3.7) holds. Now assume the opposite inequality in (3.7). Using (3.5) and (3.6), we have the following:

$$\begin{aligned} x_{n+3} &= \frac{\alpha 2^\rho}{\alpha + 2^\eta} + x_{n+2} \left( \frac{2^\rho}{\alpha + 2^\eta} \right) < \frac{\alpha 2^\rho}{\alpha + 2^\eta} + \left( \frac{3\alpha + 1}{2^\eta} \right) \left( \frac{2^\rho}{\alpha + 2^\eta} \right) \\ &< 2^{\rho-\eta} \left( \alpha + \frac{3\alpha + 1}{2^\eta} \right). \end{aligned}$$

So

$$\begin{aligned} x_{n+6} &> \frac{x_{n+5}}{x_{n+3}} > \frac{\alpha + x_{n+4}}{x_{n+3}x_{n+2}} > \frac{x_{n+4}}{x_{n+3}x_{n+2}} \\ &> \frac{\alpha}{x_{n+3}x_{n+2}x_{n+1}} > \frac{\alpha}{2^{\rho-\eta} \left( \alpha + \frac{3\alpha+1}{2^\eta} \right) x_{n+2}x_{n+1}}. \end{aligned}$$

Since  $2^\eta > \frac{(\alpha + 1)^4 2^{15}}{\alpha} > (\alpha + 1)^3 2^{15} > 3\alpha + 1$ , we see that  $\frac{3\alpha + 1}{2^\eta} < 1$  holds, and using (3.3), we get

$$x_{n+6} > \frac{\alpha}{2^{\rho-\eta}(\alpha + 1)x_{n+2}x_{n+1}} = \frac{\alpha}{2^{\rho-\eta}(\alpha + 1)x_{n+2} \left( \frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}} \right)}.$$



Distributing the  $2^{\rho-\eta}$ , we have

$$x_{n+6} > \frac{\alpha}{(\alpha + 1)x_{n+2} \left( \frac{\alpha}{2^{\ell+\eta-\rho}} + \frac{\alpha}{2^{\ell+\eta}} + \frac{1}{2^\ell} \right)}. \tag{3.8}$$

Now let us assume  $\alpha \geq 1$ . Then we have

$$x_{n+6} > \frac{\alpha}{(\alpha + 1)x_{n+2} \left( \frac{\alpha}{2^{\ell+\eta-\rho}} + \frac{\alpha}{2^{\ell+\eta}} + \frac{\alpha}{2^\ell} \right)} = \frac{2^\ell}{(\alpha + 1)x_{n+2} \left( \frac{1}{2^{\eta-\rho}} + \frac{1}{2^\eta} + 1 \right)}.$$

In this case  $2^\ell > \max \left\{ \frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11} \right\} \geq (\alpha + 1)^2 2^{11}$ , so

$$x_{n+6} > \frac{2^{11}(\alpha + 1)}{x_{n+2} \left( 1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^\eta} \right)}.$$

Since we assumed the reversed inequality in (3.7), we have that  $2^{\eta-\rho} > 2^5 > 1$ . Furthermore we know from earlier that  $2^\eta > 2^{15} > 1$ . Using this information, we obtain

$$x_{n+6} > \frac{2^{11}(\alpha + 1)}{x_{n+2} \left( 1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^\eta} \right)} > \frac{2^{11}(\alpha + 1)}{3x_{n+2}}.$$

Now we use (3.6), and we obtain

$$x_{n+6} > \frac{2^{11}(\alpha + 1)}{3(3\alpha + 1)}x_{n-1} > \frac{2^{11}(\alpha + 1)}{3(3\alpha + 3)}x_{n-1} = \left( \frac{2^{11}}{9} \right) x_{n-1} > \left( \frac{4}{3} \right) x_{n-1}.$$

We now prove the case when  $\alpha < 1$ . Here we continue from (3.8) with the following:

$$x_{n+6} > \frac{\alpha}{(\alpha + 1)x_{n+2} \left( \frac{1}{2^{\ell+\eta-\rho}} + \frac{1}{2^{\ell+\eta}} + \frac{1}{2^\ell} \right)} = \frac{\alpha 2^\ell}{(\alpha + 1)x_{n+2} \left( \frac{1}{2^{\eta-\rho}} + \frac{1}{2^\eta} + 1 \right)}.$$

In this case  $2^\ell > \max \left\{ \frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11} \right\} \geq \frac{2^{11}}{\alpha}$ , so we have

$$x_{n+6} > \frac{2^{11}}{(\alpha + 1)x_{n+2} \left( 1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^\eta} \right)}.$$

Since we assumed the reverse inequality in (3.7), we have that  $2^{\eta-\rho} > 2^5 > 1$ . Furthermore we know from earlier that  $2^\eta > 2^{15} > 1$ . Using this information, we obtain

$$x_{n+6} > \frac{2^{11}}{(\alpha + 1)x_{n+2} \left( 1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^\eta} \right)} > \frac{2^{11}}{3(\alpha + 1)x_{n+2}}.$$

Now we use (3.6) and the assumption  $\alpha < 1$  to obtain

$$x_{n+6} > \frac{2^{11}}{3(3\alpha + 1)(\alpha + 1)}(x_{n-1}) > \frac{2^{11}}{24}x_{n-1} > \left( \frac{4}{3} \right) x_{n-1}.$$

Thus we have shown  $x_{n+p_n-1} = x_{n+6} \geq \left(\frac{4}{3}\right)x_{n-1} > \max\left\{\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right\}$  when the opposite inequality of (3.7) holds. Therefore we have finished the case where  $\ell + \rho \geq \eta$ .

We now consider the case  $\ell + \rho < \eta$ . We show that if this inequality is satisfied for some  $n \in \mathbb{N}$ , then  $p_n = 8$ . First we prove that  $x_{n+p_n-3} = x_{n+5} > \frac{\alpha}{2}$ . Notice that since our assumptions have changed, (3.6) and (3.8) no longer hold. We will now make a new analogue for (3.6), namely the forthcoming (3.9). Since  $\ell + \rho < \eta$ , we have  $2^{\ell+\rho} < 2^\eta$ . Moreover, since  $2^\ell > 2^{11}$  and  $2^\eta > 2^{15}$ , we have  $\ell > 0$  and  $\eta > 0$ . So from (3.4),

$$x_{n+2} = \frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}} < \frac{\alpha}{2^{\ell+\rho}} + \frac{\alpha}{2^{\ell+\rho}} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho}} = \frac{3\alpha+1}{2^{\ell+\rho}}. \quad (3.9)$$

So we have

$$x_{n+5} = \frac{\alpha + x_{n+4}}{x_{n+2}} > \frac{\alpha}{x_{n+2}} > \frac{\alpha 2^{\ell+\rho}}{3\alpha+1}.$$

Notice that

$$\begin{aligned} \frac{\alpha 2^\ell}{3\alpha+1} &> \max\left\{\frac{\alpha 2^{11}}{(3\alpha+1)\alpha^2}, \frac{\alpha(\alpha+1)^2 2^{11}}{3\alpha+1}\right\} \\ &\geq \max\left\{\frac{2^{11}}{(3\alpha+1)\alpha}, \frac{\alpha(\alpha+1)^2 2^{11}}{3\alpha+3}\right\} \\ &= \max\left\{\frac{2^{11}}{(3\alpha+1)\alpha}, \frac{\alpha(\alpha+1)2^{11}}{3}\right\} > 1. \end{aligned}$$

So  $x_{n+5} > 2^\rho > \frac{\alpha}{2}$ .

We now prove that  $x_{n+p_n-2} = x_{n+6} > \max\left\{\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\}$ . Notice that from (3.5), we have

$$x_{n+3} = \frac{\alpha 2^\rho}{\alpha + 2^\eta} + x_{n+2} \left(\frac{2^\rho}{\alpha + 2^\eta}\right) < 2^{\rho-\eta}(\alpha + x_{n+2}). \quad (3.10)$$

Since  $x_{n+5} > 2^\rho$ , we get

$$x_{n+6} > \frac{x_{n+5}}{x_{n+3}} > 2^{\eta-\rho} \frac{x_{n+5}}{\alpha + x_{n+2}} > \frac{2^\eta}{\alpha + x_{n+2}}. \quad (3.11)$$

We assume  $\frac{1}{\alpha} \leq \alpha + 1$  and we use (3.9). We know that  $2^\rho > \frac{\alpha}{2}$  and  $2^\ell > (\alpha+1)^2 2^{11} \geq \frac{(\alpha+1)2^{11}}{\alpha}$ . So,

$$x_{n+2} < \frac{3\alpha+1}{2^{\ell+\rho}} < \frac{3\alpha+1}{\frac{(\alpha+1)2^{11}}{\alpha} \left(\frac{\alpha}{2}\right)} = \frac{3\alpha+1}{(\alpha+1)2^{10}} < \frac{3\alpha+3}{(\alpha+1)2^{10}} < 2^{-8}.$$

So, since  $\frac{1}{\alpha} \leq \alpha + 1$ ,

$$\begin{aligned} x_{n+6} &> \frac{2^\eta}{\alpha + x_{n+2}} > \frac{2^\eta}{\alpha + 2^{-8}} > \max \left\{ \frac{2^{15}}{(\alpha + 2^{-8})\alpha^3}, \frac{(\alpha + 1)^4 2^{15}}{(\alpha + 2^{-8})\alpha} \right\} \\ &\geq \frac{(\alpha + 1)^4 2^{15}}{(\alpha + 2^{-8})\alpha} > \frac{(\alpha + 1)^4 2^{15}}{(\alpha + 1)^2} = (\alpha + 1)^2 2^{15} > (\alpha + 1)^2 2^{11} \geq \frac{2^{11}}{\alpha^2}. \end{aligned}$$

We now assume  $\frac{1}{\alpha} > \alpha + 1$  and we use (3.9). We know that  $2^\rho > \frac{\alpha}{2}$  and  $2^\ell > \frac{2^{11}}{\alpha^2} \geq \frac{(\alpha + 1)2^{11}}{\alpha}$ . So,

$$x_{n+2} < \frac{3\alpha + 1}{2^{\ell+\rho}} < \frac{3\alpha + 1}{\frac{(\alpha+1)2^{11}}{\alpha} \left(\frac{\alpha}{2}\right)} = \frac{3\alpha + 1}{(\alpha + 1)2^{10}} < \frac{3\alpha + 3}{(\alpha + 1)2^{10}} < 2^{-8}.$$

We now use (3.11) and our assumption  $\frac{1}{\alpha} > \alpha + 1$  to obtain the following:

$$\begin{aligned} x_{n+6} &> \frac{2^\eta}{\alpha + x_{n+2}} > \frac{2^\eta}{\alpha + 2^{-8}} > \max \left\{ \frac{2^{15}}{(\alpha + 2^{-8})\alpha^3}, \frac{(\alpha + 1)^4 2^{15}}{(\alpha + 2^{-8})\alpha} \right\} \\ &\geq \frac{2^{15}}{(\alpha + 2^{-8})\alpha^3} > \frac{(\alpha + 1)2^{15}}{(\alpha + 1)\alpha^2} = \frac{2^{15}}{\alpha^2} > \frac{2^{11}}{\alpha^2} > (\alpha + 1)^2 2^{11}. \end{aligned}$$

Thus we have shown that  $x_{n+p_n-2} = x_{n+6} > \max \left\{ \frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11} \right\}$ .

Now we prove  $x_{n+p_n-1} = x_{n+7} \geq \left(\frac{4}{3}\right) x_{n-1} > \max \left\{ \frac{2^{15}}{\alpha^3}, \frac{(\alpha + 1)^4 2^{15}}{\alpha} \right\}$ . Notice that

$$x_{n+7} = \frac{\alpha + x_{n+6}}{x_{n+4}} > \frac{x_{n+6}}{x_{n+4}} = \frac{\alpha + x_{n+5}}{x_{n+4}x_{n+3}} > \frac{x_{n+5}}{x_{n+4}x_{n+3}} = \frac{\alpha + x_{n+4}}{x_{n+2}x_{n+3}x_{n+4}} > \frac{1}{x_{n+2}x_{n+3}}.$$

Using (3.9) and (3.10), we have

$$x_{n+7} > \frac{1}{x_{n+2}x_{n+3}} > \frac{2^{\ell+\rho}}{(3\alpha + 1)x_{n+3}} > \frac{2^{\ell+\eta}}{(3\alpha + 1)(\alpha + x_{n+2})}.$$

Earlier we demonstrated that  $x_{n+2} < 2^{-8}$ . Furthermore we have assumed that  $2^\ell > (\alpha + 1)^2 2^{11}$ . Thus,

$$\begin{aligned} x_{n+7} &> \frac{2^{\ell+\eta}}{(3\alpha + 1)(\alpha + x_{n+2})} > \frac{2^{\ell+\eta}}{(3\alpha + 3)(\alpha + 1)} > \frac{(\alpha + 1)^2 2^{11} 2^\eta}{(3\alpha + 3)(\alpha + 1)} \\ &= \left(\frac{2^{11}}{3}\right) x_{n-1} > \left(\frac{4}{3}\right) x_{n-1}. \end{aligned}$$

Hence  $x_{n+p_n-1} = x_{n+7} \geq \left(\frac{4}{3}\right) x_{n-1} > \max \left\{ \frac{2^{15}}{\alpha^3}, \frac{(\alpha + 1)^4 2^{15}}{\alpha} \right\}$ . Now we apply Lemma 3.1, and then the proof is done.  $\square$

## 4 Special Case #28

We now study special case #28

$$x_n = \frac{x_{n-1}}{Cx_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}. \quad (4.1)$$

Particularly, we show that whenever  $C > 0$ , Equation (4.1) has unbounded solutions for some initial conditions.

**Theorem 4.1.** *If  $C > 0$ , then Equation (4.1) has unbounded solutions for some initial conditions.*

*Proof.* We choose initial conditions so that

$$x_0 > \max \left\{ 1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C) \right\},$$

$$x_{-1} > \max \left\{ 10, \frac{10}{C}, \frac{1}{C^3} \right\}.$$

We show that there exists  $D = 2$  so that for all  $n \in \mathbb{N}$  there exists  $p_n \in \{7, 8\}$  so that the following holds: Whenever

$$x_{n-1} > \max \left\{ 1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C) \right\},$$

$$x_{n-2} > \max \left\{ 10, \frac{10}{C}, \frac{1}{C^3} \right\},$$

then we have

$$x_{n+p_n-1} > \max \left\{ 1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C) \right\},$$

$$x_{n+p_n-2} > \max \left\{ 10, \frac{10}{C}, \frac{1}{C^3} \right\},$$

$$x_{n+p_n-1} \geq 2x_{n-1}.$$

First assume

$$x_{n-1} > \max \left\{ 1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C) \right\},$$

$$x_{n-2} > \max \left\{ 10, \frac{10}{C}, \frac{1}{C^3} \right\}.$$

Using algebra, we immediately obtain the following:

$$\begin{aligned}x_n &= \frac{x_{n-1}}{Cx_{n-2} + x_{n-3}} < \frac{x_{n-1}}{Cx_{n-2}} < \frac{x_{n-1}}{10}, \\x_{n+1} &= \frac{x_n}{Cx_{n-1} + x_{n-2}} < \frac{x_n}{x_{n-2}} < \frac{x_n}{10}, \\x_{n+1} &= \frac{x_n}{Cx_{n-1} + x_{n-2}} < \frac{x_n}{Cx_{n-1}} < \frac{x_n}{1000C}, \\x_n &= \frac{x_{n-1}}{Cx_{n-2} + x_{n-3}} < \frac{x_{n-1}}{Cx_{n-2}} < \frac{x_{n-1}}{10C}, \\x_{n+2} &= \frac{x_{n+1}}{Cx_n + x_{n-1}} < \frac{x_{n+1}}{x_{n-1}} < \frac{x_{n+1}}{100C}, \\x_{n+2} &= \frac{x_{n+1}}{Cx_n + x_{n-1}} < \frac{x_{n+1}}{x_{n-1}} < \frac{x_{n+1}}{1000}.\end{aligned}$$

So we get the following inequalities:

$$100000x_{n+2} < 100x_{n+1} < 10x_n < x_{n-1}, \quad (4.2)$$

$$1000000C^3x_{n+2} < 10000C^2x_{n+1} < 10Cx_n < x_{n-1}. \quad (4.3)$$

Using (4.3) we get

$$\frac{x_{n+2}}{x_{n+3}} = Cx_{n+1} + x_n < 2x_n. \quad (4.4)$$

We use (4.2) and (4.3) to get

$$\begin{aligned}Cx_{n+3} + x_{n+2} &= x_{n+2} \left( 1 + \frac{C}{Cx_{n+1} + x_n} \right) < x_{n+2} \left( 1 + \frac{1}{x_{n+1}} \right) \\&< x_{n+2} \left( 1 + \frac{1}{1000x_{n+2}} \right) \\&= x_{n+2} + \frac{1}{1000} = \frac{x_n}{(Cx_n + x_{n-1})(Cx_{n-1} + x_{n-2})} + \frac{1}{1000} \\&< \frac{1}{Cx_{n-2}} + \frac{1}{1000} < \frac{1}{10} + \frac{1}{1000} < 1.\end{aligned}$$

In short,

$$Cx_{n+3} + x_{n+2} < 1.$$

Using (4.3) and (4.4), we have

$$\begin{aligned}x_{n+5} &= \frac{x_{n+4}}{Cx_{n+3} + x_{n+2}} = \frac{1}{Cx_{n+2} + x_{n+1}} \left( \frac{x_{n+3}}{Cx_{n+3} + x_{n+2}} \right), \\x_{n+5} &= \frac{1}{Cx_{n+2} + x_{n+1}} \left( \frac{1}{C + \frac{x_{n+2}}{x_{n+3}}} \right) > \frac{1}{2x_{n+1}} \left( \frac{1}{C + 2x_n} \right).\end{aligned} \quad (4.5)$$

Furthermore we have

$$x_{n+4} = \frac{x_{n+3}}{Cx_{n+2} + x_{n+1}} < \frac{x_{n+3}}{x_{n+1}} = \frac{1}{(Cx_{n+1} + x_n)(Cx_n + x_{n-1})} < \frac{1}{x_{n-1}x_n}. \quad (4.6)$$

Using (4.5) and (4.6), we obtain

$$\begin{aligned} x_{n+6} &= \frac{x_{n+5}}{Cx_{n+4} + x_{n+3}} > \left(\frac{1}{2x_{n+1}}\right) \left(\frac{1}{C + 2x_n}\right) \left(\frac{1}{\frac{C}{x_{n-1}x_n} + x_{n+3}}\right) \\ &= \left(\frac{x_{n-1}}{2x_{n+1}}\right) \left(\frac{1}{C + 2x_n}\right) \left(\frac{1}{\frac{C}{x_n} + x_{n-1}x_{n+3}}\right) \\ &= \left(\frac{Cx_{n-1} + x_{n-2}}{2C + 4x_n}\right) \left(\frac{Cx_{n-2} + x_{n-3}}{\frac{C}{x_n} + x_{n-1}x_{n+3}}\right) \\ &= \left(\frac{(Cx_{n-1} + x_{n-2})^2}{2C + 4x_n}\right) \left(\frac{Cx_{n-2} + x_{n-3}}{\frac{C}{x_{n+1}} + x_{n-1}x_{n+3}(Cx_{n-1} + x_{n-2})}\right). \end{aligned}$$

Using (4.2) and (4.3), we see

$$\begin{aligned} x_{n-1}x_{n+3}(Cx_{n-1} + x_{n-2}) &= \frac{x_{n-1}x_{n+2}(Cx_{n-1} + x_{n-2})}{Cx_{n+1} + x_n} \\ &= \frac{x_{n-1}x_{n+1}(Cx_{n-1} + x_{n-2})}{(Cx_{n+1} + x_n)(Cx_n + x_{n-1})} \\ &= \frac{x_{n-1}x_n}{(Cx_{n+1} + x_n)(Cx_n + x_{n-1})} \\ &< 1 < C^3x_{n-2} < \frac{C(Cx_{n-1} + x_{n-2})(Cx_{n-2} + x_{n-3})}{x_{n-1}} \\ &= \frac{C}{x_{n+1}}. \end{aligned}$$

Using this in the prior inequality, we get

$$x_{n+6} > \frac{x_{n-1}(Cx_{n-1} + x_{n-2})}{4C^2 + 8Cx_n}. \quad (4.7)$$

Using this fact, we have

$$\begin{aligned} x_{n+7} &= \frac{x_{n+6}}{Cx_{n+5} + x_{n+4}} = \frac{x_{n+6}}{\left(\frac{Cx_{n+4}}{Cx_{n+3} + x_{n+2}}\right) + x_{n+4}} = \left(\frac{x_{n+6}}{x_{n+4}}\right) \left(\frac{1}{1 + \frac{C}{Cx_{n+3} + x_{n+2}}}\right) \\ &= \left(\frac{x_{n+4}}{Cx_{n+3} + x_{n+2}}\right) \left(\frac{1}{Cx_{n+4} + x_{n+3}}\right) \left(\frac{1}{x_{n+4}}\right) \left(\frac{1}{1 + \frac{C}{Cx_{n+3} + x_{n+2}}}\right) \end{aligned}$$

$$\begin{aligned}
&= \left( \frac{1}{C + Cx_{n+3} + x_{n+2}} \right) \left( \frac{1}{Cx_{n+4} + x_{n+3}} \right) \\
&> \left( \frac{1}{Cx_{n+4} + x_{n+3}} \right) \left( \frac{1}{C + 1} \right).
\end{aligned}$$

We now use (4.2) and (4.3) to show

$$\begin{aligned}
x_{n+3} &= \frac{x_{n+2}}{Cx_{n+1} + x_n} = \frac{x_{n+1}}{(Cx_{n+1} + x_n)(Cx_n + x_{n-1})} \\
&< \frac{x_{n+1}}{(Cx_{n+1} + 500Cx_{n+1} + 5x_{n+1})(Cx_n + x_{n-1})} \\
&< \frac{1}{(501C + 5)x_{n-1}} < \frac{1}{(4C + 4)x_{n-1}}.
\end{aligned}$$

We now use this fact and (4.6) to obtain

$$x_{n+7} > \left( \frac{1}{\frac{C}{x_n x_{n-1}} + \frac{1}{x_{n-1}(4C+4)}} \right) \left( \frac{1}{C+1} \right) = \left( \frac{x_{n-1}}{\frac{C}{x_n} + \frac{1}{4(C+1)}} \right) \left( \frac{1}{C+1} \right). \quad (4.8)$$

Suppose  $x_n \leq 4C(C+1)$ . We will show that in this case  $p_n = 7$ . Using (4.5), we have

$$\begin{aligned}
x_{n+p_n-2} &= x_{n+5} > \frac{1}{2x_{n+1}} \left( \frac{1}{C + 2x_n} \right) \\
&= \frac{Cx_{n-1} + x_{n-2}}{2x_n} \left( \frac{1}{C + 2x_n} \right) > \frac{Cx_{n-1}}{2x_n(C + 2x_n)} \\
&\geq \frac{Cx_{n-1}}{8C(C+1)(C + 8C(C+1))} \\
&= \frac{x_{n-1}}{8(C+1)(C + 8C(C+1))} > \frac{x_{n-1}}{100(C+1)^3} > \max \left\{ 10, \frac{10}{C}, \frac{1}{C^3} \right\}.
\end{aligned}$$

Also using (4.7), we have

$$\begin{aligned}
x_{n+p_n-1} &= x_{n+6} > \frac{x_{n-1}(Cx_{n-1} + x_{n-2})}{4C^2 + 8Cx_n} > \frac{Cx_{n-1}^2}{4C^2 + 8C(4C(C+1))} \\
&= \frac{x_{n-1}^2}{4C + 8(4C(C+1))} > \frac{x_{n-1}^2}{50(C^2 + C)} > 2x_{n-1}.
\end{aligned}$$

Now suppose  $x_n > 4C(C+1)$ . We will show that in this case  $p_n = 8$ . Using (4.7), we have

$$\begin{aligned}
x_{n+p_n-2} &= x_{n+6} > \frac{x_{n-1}(Cx_{n-1} + x_{n-2})}{4C^2 + 8Cx_n} > \frac{x_{n-1}(Cx_{n-1} + x_{n-2})}{9Cx_n} \\
&= \frac{(Cx_{n-1} + x_{n-2})(Cx_{n-2} + x_{n-3})}{9C} > x_{n-2} > \max \left\{ 10, \frac{10}{C}, \frac{1}{C^3} \right\}.
\end{aligned}$$

Also using (4.8), we have

$$\begin{aligned} x_{n+p_n-1} &= x_{n+7} > \left( \frac{x_{n-1}}{\frac{C}{x_n} + \frac{1}{4(C+1)}} \right) \left( \frac{1}{C+1} \right) \\ &> \left( \frac{x_{n-1}}{\frac{C}{4C(C+1)} + \frac{1}{4(C+1)}} \right) \left( \frac{1}{C+1} \right) = 2x_{n-1}. \end{aligned}$$

Hence after application of Lemma 3.1, the proof is complete.  $\square$

## 5 Conclusion

Theorem 2.1 establishes the boundedness character of the special cases #56 and #120 in a range of their parameters. Theorems 3.2 and 4.1 establish the boundedness character of the special cases #44 and #28 respectively. Further work should focus on expanding the range for which the boundedness character of special cases #56 and #120 is known and resolving [1, Conjecture 3.0.1]. There remains only one special case for which Conjecture 3.0.1 has not yet been established. This is special case #70. It is worthwhile to note that special case #70 is part of the period-six trichotomy conjecture. The resolution of the period-six trichotomy conjecture will immediately resolve [1, Conjecture 3.0.1]. See [2] for more details regarding the period-six trichotomy conjecture. We restate, for the convenience of the reader, the period-six trichotomy conjecture.

*Conjecture 5.1.* Assume that  $\alpha, C \in [0, \infty)$ . Then the following period-six trichotomy result is true for the rational equation

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}. \quad (5.1)$$

- (i) Every solution of Equation (5.1) converges to its positive equilibrium if and only if  $\alpha C^2 > 1$ .
- (ii) Every solution of Equation (5.1) converges to a not necessarily prime period-six solution of Equation (5.1) if and only if  $\alpha C^2 = 1$ .
- (iii) Equation (5.1) has unbounded solutions if and only if  $\alpha C^2 < 1$ .

For more on boundedness character see [1–4].

## References

- [1] E. Camouzis and G. Ladas, *Dynamics of Third-Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC Press, Boca Raton, 2007.



- [2] E. Camouzis and G. Ladas, Three trichotomy conjectures, *J. Difference Equa. Appl.* 8 (2002), 495–500.
- [3] E. Camouzis, G. Ladas, F. Palladino, and E.P. Quinn, On the Boundedness Character of Rational Equations, Part 1, *J. Difference Equa. Appl.* 12 (2006), 503–523.
- [4] E.A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, Chapman & Hall/CRC Press, Boca Raton, 2005.
- [5] E.A. Grove and G. Ladas, Periodicity in nonlinear difference equations, *Revista Cubo* May 2002, 195–230.
- [6] V.L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, 1993.
- [7] M.R.S. Kulenovic and G. Ladas, *Dynamics of Second Order Rational Difference Equations With Open Problems and Conjectures*, Chapman & Hall/CRC Press, 2001.