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On Oscillation and Asymptotic Behaviour of a Higher Order Functional Difference Equation of Neutral Type

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Abstract

Necessary and sufficient conditions are obtained for the oscillation of all the solutions of the neutral functional difference equation

$$\Delta^m\big(y_n - p_n y_{\tau(n)}\big) + q_n G(y_{\sigma(n)}) = f_n \quad \text{for } n \geq n_0.$$

Different ranges of the sequence $\{p_n\}$ are considered. The positive integer m can take both odd and even values. The results hold when $\{f_n\} = \{0\}$ and $G(u) = u$ for $u \in \mathbb{R}$. This paper improves, generalizes and corrects some recent results.

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1 Introduction

In this paper, necessary and sufficient conditions are obtained so that every solution of

$$\Delta^m\big(y_n - p_n y_{\tau(n)}\big) + q_n G(y_{\sigma(n)}) = f_n \quad \text{for } n \geq n_0 \tag{1.1}$$

oscillates or asymptotically tends to zero or $\pm\infty$, where Δ is the forward difference operator given by $\Delta y_n = y_{n+1} - y_n$, $\{p_n\}, \{q_n\}$ and $\{f_n\}$ are infinite sequences of real numbers with $q_n \geq 0$ for every integer $n \geq n_0$, and $G \in C(\mathbb{R}, \mathbb{R})$. Further, we assume $\{\tau(n)\}, \{\sigma(n)\}$ are monotonic increasing sequences of integers such that $\tau(n), \sigma(n) \leq n$ for every integer $n \geq n_0$. Different ranges of $\{p_n\}$ are considered. The positive integer m, which is the order of the equation, can take both odd and even values.

Let $n_{-1} = \min\{\tau(n_0), \sigma(n_0)\}$. By a solution of (1.1), we mean a real sequence $\{y_n\}$ which is defined for every integer $n \geq n_{-1}$ and satisfies (1.1) for every integer $n \geq n_0$. Clearly, if the initial condition

$$y_n = \phi_n \quad \text{for } n_{-1} \leq n \leq n_0 + m - 1 \tag{1.2}$$

is given, then the equation (1.1) has a unique solution satisfying the given initial condition (1.2). A solution $\{y_n\}$ of (1.1) is said to be oscillatory if for every integer $n_1 \geq n_0$, there exists $n_2 \geq n_1$ such that $y_{n_2} y_{n_2+1} \leq 0$, otherwise $\{y_n\}$ is said to be nonoscillatory.

In the sequel, we shall need the following conditions.

(H_0) G is nondecreasing and $uG(u) > 0$ for all real $u \neq 0$.

(H_1) $\tau(n)/n \geq \kappa$ for all $n \geq n_0$ and some real $\kappa > 0$.

(H_2) $\sigma(n)/n \geq \mu$ for all $n \geq n_0$ and some real $\mu > 0$.

(H_3) $\liminf\limits_{|u|\to\infty} \big(G(u)/u\big) \geq \delta$ for some real $\delta > 0$

(H_4) $\displaystyle\sum_{i=n_0}^{\infty} i^{m-2} q_i = \infty$ when $m \geq 2$.

(H_5) $\displaystyle\sum_{i=n_0}^{\infty} q_i = \infty$.

(H_6) $\displaystyle\sum_{i=n_0}^{\infty} i^{m-1} q_i = \infty$.

(H_7) There exists a sequence $\{F_n\}$ such that $\{\Delta^m F_n\} = \{f_n\}$, and $\lim\limits_{n\to\infty} F_n = 0$.

(H_8) There exists a bounded sequence $\{F_n\}$ such that $\{\Delta^m F_n\} = \{f_n\}$.

In this paper, we assume that $\{p_n\}$ satisfies one of the following conditions.

(A_1) $0 \le p_n \le b < 1$,
(A_2) $-1 < -b \le p_n \le 0$,
(A_3) $-b_2 \le p_n \le -b_1 < -1$,
(A_4) $1 < b_1 \le p_n \le b_2$,
(A_5) $0 \le p_n \le b_2$,
(A_6) $-b_2 \le p_n \le 0$,
(A_7) $1 \le p_n \le b_2$,

where b, b_1, b_2 are positive real numbers.

In recent years, several papers on oscillation of solutions of neutral delay difference equations have appeared, see [1, 2, 10–17] and the references cited there in. Sufficient conditions for oscillation of

$$\Delta^m\big(y_n - p_n y_{n-k}\big) + q_n G(y_{n-l}) = f_n \quad \text{for } n \ge n_0 \tag{1.3}$$

are studied in [14]. In that paper, $\{p_n\}$ is confined to (A_2) only and G is restricted with a sublinear condition

$$\left| \int_0^{\pm\varepsilon} \frac{\mathrm{d}s}{G(s)} \right| < \infty \quad \text{for any } \varepsilon > 0. \tag{1.4}$$

In [16] the authors study

$$\Delta^m\big(y_n - p_n y_{n-k}\big) + q_n y_{n-l}^{\alpha} = 0 \quad \text{for } n \ge n_0, \tag{1.5}$$

where $1 > \alpha > 0$, is a quotient of odd integers and $\{p_n\}$ satisfies (A_1) or (A_2). They obtain the sufficient conditions of oscillation of (1.5) under the conditions

$$\sum_{i=n_0}^{\infty} q_i (i-l)^{\alpha(m-1)} = \infty \tag{1.6}$$

and

$$\sum_{i=n_0}^{\infty} q_i (1 + p_{i-l})^{\alpha(m-1)} = \infty \tag{1.7}$$

and presented the following results.

Theorem 1.1 (see [16, Theorem 2.1]). (*i*) *Let* m *be even. If* $0 \ge p_n > -1$ *for all large* n *and* (1.7) *hold, then all solutions of* (1.5) *are oscillatory.*

(*ii*) *Let* m *be odd. If* (A_2) *and* (1.6) *hold, then every solution of* (1.5) *oscillates or tends to zero at infinity.*

Theorem 1.2 (see [16, Theorem 2.2]). *If* (A_1) *and* (1.6) *hold, then every solution of* (1.5) *oscillates or asymptotically tends to zero.*

We may note that for $m \geq 2$, (H_4) implies (H_6) and further, if $1 > \alpha > 0$, then (1.6) implies (H_4) for $m \geq (2-\alpha)/(1-\alpha)$. Moreover, the equations (1.3) and (1.5) are particular cases of (1.1). The results in [11, 14, 16] do not hold for a class of equations, where G is either linear or superlinear, i.e., for example when $G(u) = u$ or $G(u) = u^3$ for $u \in \mathbb{R}$. Here, in this paper, an attempt is made to fill this existing gap in literature and obtain sufficient conditions for oscillatory behaviour of solutions of a more general equation (1.1) under the weaker conditions (H_4) or (H_6). Moreover, we observe that the existing papers in the literature do not have much to offer when $\{p_n\}$ satisfies (A_4), (A_6) or (A_7). In this direction, we find that the authors in [11] have obtained sufficient conditions for the oscillation of solutions of the equation

$$\Delta^m\big(y_n - p_n y_{n-k}\big) + q_n G(y_{n-l}) = 0 \quad \text{for } n \geq n_0, \tag{1.8}$$

with (A_4) or (A_7) and presented the following results.

Theorem 1.3 (see [11, Theorem 2.6]). *Let $\{p_n\}$ satisfy (A_7). If the condition (H_5) holds, then the following are valid statements.*

(*i*) *Every solution of* (1.8) *oscillates, if m is even.*

(*ii*) *Every solution of* (1.8) *oscillates or tends to zero asymptotically if m is odd.*

Theorem 1.4 (see [11, Theorem 2.7]). *Let $\{p_n\}$ satisfy (A_4). If (H_5) holds, then the following statements are true.*

(*i*) *Every solution of* (1.8) *oscillates for m even.*

(*ii*) *Every solution of* (1.8) *oscillates or tends to zero at infinity if m is odd.*

Theorem 1.5 (see [11, Theorem 2.10]). *Let $\{p_n\}$ be in (A_5) and $l \geq k$. Suppose that* (1.4) *and*

$$\sum_{i=n_1}^{\infty} q_i^* = \infty \tag{1.9}$$

hold, where $q_n^* := \min\{q_n, q_{\tau(n)}\}$ for $n \geq n_1 \geq n_0$. Then every solution of (1.8) *oscillates.*

Unfortunately, the following example contradicts all the above three theorems in [11].

Example 1.6. Consider the neutral equation

$$\Delta^m\big(y_n - 4y_{n-1}\big) + 4^{(n+1)/3}\sqrt[3]{y_{n-2}} = 0 \quad \text{for } n \geq 0, \tag{1.10}$$

where m may be any odd or even positive integer. Here, $\{p_n\}$ satisfies (A_4), (A_5) and (A_7). Also, note that $\{q_n\} = \{4^{(n+1)/3}\}$ satisfies (H_5) and it is monotonic increasing.

Hence (1.9) is satisfied (refer Remark 4.3 of this paper). Clearly, (1.10) satisfies all the conditions of Theorems 1.3–1.5. But, (1.10) has an unbounded positive solution $\{y_n\} = \{2^n\}$ which asymptotically tends to ∞. Thus, this example contradicts Theorems 1.3–1.5. Since $G(u) = \sqrt[3]{u}$ for $u \in \mathbb{R}$ is sublinear, this example further establishes that the results of [14, 16] do not hold when $\{p_n\}$ is in (A_4) or (A_7).

As the papers [11, 14, 16] deal with sublinear equations, and their results do not hold for linear or superlinear equations (i.e., (1.3) satisfying (H_3) or (1.5) with $\alpha \geq 1$), our objective is to complement their work, i.e., to extend these results to linear and superlinear equations. In this paper we study (1.1), with $\{p_n\}$ in almost all possible ranges. The results hold good for $\{f_n\} = \{0\}$ and $G(u) = u$ for $u \in \mathbb{R}$. In the last section existence of a bounded positive solution of (1.1) is shown. The last but not the least, this paper corrects, generalizes and improves some recent results in [10,11,14,16].

2 Some Lemmas

First, we state some lemmas that will be useful for our work. The following lemma which can be easily proved, generalizes [10, Lemma 2.1].

Lemma 2.1. *Let $\{f_n\}, \{q_n\}$ and $\{p_n\}$ be sequences of real numbers defined for all integers $n \geq n_0$ for some fixed integer n_0 such that*

$$f_n = q_n - p_n q_{\tau(n)} \quad \text{for all } n \geq n_1 \geq n_0,$$

where $\{\tau(n)\}$ is member of a monotonic increasing unbounded sequence satisfying $\tau(n) \leq n$ for all large n. Suppose that $\{p_n\}$ satisfies one of conditions (A_2), (A_3) or (A_5). If $q_n > 0$ for all $n \geq n_0$, $\liminf_{n\to\infty} q_n = 0$ and $\lim_{n\to\infty} f_n = L$ exists, then $L = 0$.

Lemma 2.2 (see [3, 11]). *Let $\{z_n\}$ be a real valued function defined for all integers $n \geq n_0$, where n_0 is a fixed integer, and $z_n > 0$ with $\Delta^m z_n$ of constant sign for all $n \geq n_0$ and not identically zero. Then there exists an integer p, $m-1 \geq p \geq 0$, with $(m+p)$ odd for $\Delta^m z_n \leq 0$ and $(m+p)$ even for $\Delta^m z_n \geq 0$, such that for all $n \geq n_0$,*

$$\Delta^j z_n > 0 \quad \text{for each } j \text{ with } p \geq j \geq 0$$

and

$$(-1)^{p+j} \Delta^j z_n > 0 \quad \text{for each } j \text{ with } m-1 \geq j \geq p+1.$$

Definition 2.3. Define the factorial function (see [8, pp. 20]) by

$$n^{(k)} := n(n-1)\cdots(n-k+1),$$

where $k \leq n$ and $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Note that $n^{(k)} = 0$, if $k > n$.

Then we have

$$\Delta n^{(k)} = kn^{(k-1)}, \tag{2.1}$$

where $n \in \mathbb{Z}$, $k \in \mathbb{N}$ and Δ is the forward difference operator. One can show by summing up (2.1) that

$$\sum_{i=m}^{n-1} i^{(k)} = \frac{1}{k+1}\big(n^{(k+1)} - m^{(k+1)}\big) \tag{2.2}$$

holds. Now, set

$$b_k(n,m) := \begin{cases} 1, & k=0 \\ \sum_{i=m}^{n} b_{k-1}(n,i), & k \in \mathbb{N}. \end{cases} \tag{2.3}$$

Here, we evaluate b_k by recursion. Clearly, for $k=1$ in (2.3), we have

$$b_1(n,m) = \sum_{i=m}^{n} b_0(n,i) = \sum_{i=m}^{n} 1 = (n+1-m) = (n+1-m)^{(1)}.$$

By (2.2) and for $k=2$ in (2.3), we get

$$\begin{aligned} b_2(n,m) =& \sum_{i=m}^{n} b_1(n,i) = \sum_{i=m}^{n} (n+1-i)^{(1)} = \sum_{i=1}^{n+1-m} i^{(1)} \\ =& = \frac{1}{2}(n+2-m)^{(2)} - \frac{1}{2}1^{(2)} = \frac{1}{2}(n+2-m)^{(2)}. \end{aligned}$$

Note that $1^{(2)} = 0$. By (2.2) and for $k=3$ in (2.3), we get

$$\begin{aligned} b_3(n,m) =& \sum_{i=m}^{n} b_2(n,i) = \frac{1}{2}\sum_{i=m}^{n} (n+2-i)^{(2)} = \frac{1}{2}\sum_{i=2}^{n+2-m} i^{(2)} \\ =& \frac{1}{6}\big[(n+3-m)^{(3)} - 2^{(3)}\big] = \frac{1}{3!}(n+3-m)^{(3)}. \end{aligned}$$

Using a simple induction, we obtain

$$b_k(n,m) = \frac{1}{k!}(n+k-m)^{(k)}. \tag{2.4}$$

Lemma 2.4. *Let p be a positive integer and $\{y_n\}$ be a nonoscillatory sequence which is eventually positive. If there exist a sequence $\{w_n\}$ and an integer p_0 with $p-1 \geq p \geq 0$ such that $\lim_{n\to\infty} \Delta^{p_0} w_n$ exists (finite) and $\lim_{n\to\infty} \Delta^j w_n = 0$ for all j with $p-1 \geq j \geq p_0+1$, then*

$$\Delta^p w_n = -y_n \tag{2.5}$$

implies

$$\Delta^{p_0} w_n = \lim_{n\to\infty} \Delta^{p_0} w_n + \frac{(-1)^{p-p_0-1}}{(p-p_0-1)!} \sum_{i=n}^{\infty} (i+p-p_0-1-n)^{(p-p_0-1)} y_i. \tag{2.6}$$

for all sufficiently large n.

Proof. Summing up (2.5) from n to ∞, we get

$$\lim_{n\to\infty} \Delta^{p-1} w_n - \Delta^{p-1} w_n = -\sum_{i=n}^{\infty} y_i$$

or simply

$$\Delta^{p-1} w_n = \sum_{i=n}^{\infty} y_i = \sum_{i=n}^{\infty} b_0(i,n) y_i \tag{2.7}$$

for all $n \geq n_1 \geq n_0$. Summing up (2.7) from n to ∞, we get

$$\begin{aligned} \Delta^{p-2} w_n &= \lim_{n\to\infty} \Delta^{p-2} w_n - \sum_{i=n}^{\infty} \sum_{j=i}^{\infty} b_0(j,i) y_i = -\sum_{j=n}^{\infty} \sum_{i=n}^{j} b_0(j,i) y_i \\ &= -\sum_{j=n}^{\infty} b_1(j,n) y_j = -\sum_{i=n}^{\infty} b_1(i,n) y_i \end{aligned} \tag{2.8}$$

for all $n \geq n_1$. Again summing up (2.8) from n to ∞, we obtain

$$\begin{aligned} \Delta^{p-3} w_n &= \lim_{n\to\infty} \Delta^{p-3} w_n + \sum_{j=n}^{\infty} \sum_{i=j}^{\infty} b_1(i,j) y_i = \sum_{i=n}^{\infty} \sum_{j=n}^{i} b_1(i,j) y_i \\ &= \sum_{i=n}^{\infty} b_2(i,n) y_i \end{aligned}$$

for all $n \geq n_1$. By the emerging pattern, we have

$$\Delta^{j} w_n = (-1)^{p-j-1} \sum_{i=n}^{\infty} b_{p-j-1}(i,n) y_i$$

for all $n \geq n_1$ and $j = p_0+1, \ldots, p-1$. Then by letting $j = p_0 + 1$, we get

$$\Delta^{p_0+1} w_n = (-1)^{p-p_0-2} \sum_{i=n}^{\infty} b_{p-p_0-2}(i,n) y_i \quad \text{for all } n \geq n_1. \tag{2.9}$$

Summing up (2.9) from n to ∞ and arranging we get

$$\Delta^{p_0} w_n = \lim_{n\to\infty} \Delta^{p_0} w_n + (-1)^{p-p_0-1} \sum_{i=n}^{\infty} b_{p-p_0-1}(i,n) y_i \quad \text{for all } n \geq n_1. \tag{2.10}$$

From (2.4) and (2.10) it follows that

$$\Delta^{p_0} w_n = \lim_{n\to\infty} \Delta^{p_0} w_n + \frac{(-1)^{p-p_0-1}}{(p-p_0-1)!} \sum_{i=n}^{\infty} (i+p-p_0-1-n)^{(p-p_0-1)} y_i$$

for all $n \geq n_1$. Hence, the proof is complete. □

Lemma 2.5. *If $\{w_n\}$ is a sequence of real numbers such that $\Delta^j w_n > 0$ for all sufficiently large n and $j = 0, 1, 2, \ldots, p$ for some integer $p \geq 1$, and $\Delta^{p+1} w_n < 0$ for all $n \geq n_0$ for some integer n_0, then there exists a real number $L > 0$ such that $w_n > Ln^{p-1}$ for all sufficiently large n.*

Proof. From the given conditions, it is clear that $\{\Delta^{p-1} w_n\}$ is increasing. Hence, we can find $n_1 \geq n_0$ and a real number $A > 0$ such that $n \geq n_1$ implies

$$\Delta^{p-1} w_n \geq A. \tag{2.11}$$

Choose $n \geq n_1 + 1$, then summing (2.11) from $i = n_1$ to $n - 1$, we obtain

$$\Delta^{p-2} w_n > A(n - n_1).$$

First taking $n \geq n_1 + 2$ and then summing up the above inequality from $i = n_1 + 1$ to $n - 1$, we obtain

$$\Delta^{p-3} w_n > \frac{1}{2} A(n - n_1)^{(2)}.$$

Continuing the above iteration for a total of $(p - 3)$ times more and using (2.2), we easily find

$$w_n > \frac{A(n - n_1)^{(p-1)}}{(p-1)!} \quad \text{for all } n \geq n_1 + p - 1.$$

Since $n^{(k)} \geq (n - k + 1)^k$ for all $n \geq k \geq 0$, it follows from the above inequality that

$$w_n > \frac{A(n - n_1 - p + 2)^{p-1}}{(p-1)!} \quad \text{for all } n \geq n_1 + p - 1.$$

Clearly, $\lim_{n\to\infty} \left(1 - \frac{n_1 + p - 2}{n}\right)^{p-1} = 1$. Hence, for any $1 > \varepsilon > 0$, we can find $n_2 \geq n_1 + p - 1$ such that $n \geq n_2$ implies

$$1 - \varepsilon < \left(1 - \frac{n_1 + p - 2}{n}\right)^{p-1} < 1 + \varepsilon.$$

Choose $0 < L < A/(p-1)! = B$ such that $L/B < 1 - \varepsilon$. Hence, for all $n \geq n_2$ we obtain $w_n > Ln^{p-1}$. □

Remark 2.6. Suppose that $\{w_n\}$ is a real sequence and L is a positive scalar as defined in Lemma 2.5. If $\{z_n\}$ is a sequence, which satisfies the condition that $z_n \geq w_n - \varepsilon$ for all $n \geq n_1 \geq n_0$, where $\varepsilon > 0$ is any preassigned arbitrary positive number, then there exist a positive scalar $L > C$ and a positive integer $n_2 \geq \max\left\{\left(\frac{\varepsilon}{L-C}\right)^{\frac{1}{p-1}}, n_1\right\}$ such that $n \geq n_2$ implies $z_n \geq Cn^{p-1}$.

We state another lemma, which is useful for our main results.

Lemma 2.7 (see [9]). *If* $\sum_{i=1}^{\infty} u_i$ *and* $\sum_{i=1}^{\infty} v_i$ *are positive term series with*

$$\lim_{n\to\infty} \frac{u_n}{v_n} = l,$$

where l *is a positive finite constant, then the two series converge or diverge together. If* $l = 0$*, then* $\sum_{i=1}^{\infty} v_i$ *is convergent implies* $\sum_{i=1}^{\infty} u_i$ *is convergent. If* $l = \infty$*, then* $\sum_{i=1}^{\infty} v_i$ *is divergent implies* $\sum_{i=1}^{\infty} u_i$ *is divergent.*

Remark 2.8. Since, $(n-k+1)^k < n^{(k)} < n^k$ is true for all $n \geq k$, where k is a fixed nonnegative integer, then by Lemma 2.7 the following statements are true.

(H_4) implies and is implied by the condition $\sum_{i=n_0}^{\infty}(i-n_0+m-2)^{(m-2)}q_i = \infty$.

(H_6) implies and is implied by the condition $\sum_{i=n_0}^{\infty}(i-n_0+m-1)^{(m-1)}q_i = \infty$.

Remark 2.9. Note that (H_7) implies (H_8). If the condition $\left|\sum_{i=n_0}^{\infty} i^{m-1} f_i\right| < \infty$ is satisfied, then (H_7) holds. Indeed, using Lemma 2.7 and Remark 2.8, we define

$$F_n = \frac{(-1)^m}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)} f_i \quad \text{for } n \geq n_0.$$

Then $\Delta^m F_n = f_n$ for all $n \geq n_0$ and $\lim_{n\to\infty} F_n = 0$. Thus, ($H_7$) holds. We further note that (H_7) implies and is implied by the condition "There exist a sequence $\{F_n\}$ and a constant η such that $\{\Delta^m F_n\} = \{f_n\}$, and $\lim_{n\to\infty} F_n = \eta$." The implies part is obvious. If $\eta \neq 0$, then we can replace $\{F_n\}$ by $\{L_n\} = \{F_n - \eta\}$. Then $\lim_{n\to\infty} L_n = 0$ and $\{\Delta^m L_n\} = \{f_n\}$. Thus, ($H_7$) holds and the equivalence is established.

3 Sufficient Conditions – I

In this section, we present the results to find sufficient conditions so that every solution of (1.1) oscillates or asymptotically tends to zero when $\{p_n\}$ satisfies one of the conditions (A_1)–(A_4).

Theorem 3.1. *Let $m \geq 2$. Suppose that $\{p_n\}$ satisfies one of the conditions (A_1) or (A_2). If (H_0), (H_2)–(H_4) and (H_8) hold, then every unbounded solution of* (1.1) *oscillates.*

Proof. Let $\{y_n\}$ be an unbounded nonoscillatory solution of (1.1). Then either $y_n > 0$ or $y_n < 0$ for all $n \geq n_1$ for some $n_1 \geq n_0$. Suppose $y_n > 0$ for all $n \geq n_1$. There exists an integer $n_2 \geq n_1$ such that $y_n, y_{\tau(n)} > 0$ and $y_{\sigma(n)} > 0$ for all $n \geq n_2$. For simplicity of notation, define

$$z_n := y_n - p_n y_{\tau(n)} \quad \text{for } n \geq n_2. \tag{3.1}$$

Set

$$w_n := z_n - F_n \quad \text{for } n \geq n_2. \tag{3.2}$$

Then using (3.1)–(3.2) in (1.1), we obtain

$$\Delta^m w_n = -q_n G(y_{\sigma(n)}) \leq 0 \quad \text{for all } n \geq n_2. \tag{3.3}$$

Hence, $w_n, \Delta w_n, \dots, \Delta^{m-1} w_n$ are monotonic and of single sign for all $n \geq n_3 \geq n_2$. Then $\lim_{n\to\infty} w_n = \lambda$, where $-\infty \leq \lambda \leq \infty$. Since $\{y_n\}$ is unbounded, there exists a subsequence $\{y_{n_k}\}$ such that

$$n_k \to \infty \text{ and } y_{n_k} \to \infty \quad \text{as } k \to \infty,$$

and

$$y(n_k) = \max\{y_n : n_3 \leq n \leq n_k\} \quad \text{for all } k \in \mathbb{N}. \tag{3.4}$$

Then from (H_8) it follows that we can find an integer $n_4 \geq n_3$ such that $k \geq n_4$ implies $\tau(n_k), \sigma(n_k) \geq n_3$ and $|F_{n_k}| < \varepsilon$ for some constant $\varepsilon > 0$. Hence, for all $k \geq n_4$, if (A_1) holds, then we have

$$w_{n_k} \geq y_{n_k}(1-b) - \varepsilon.$$

Similarly, if (A_2) holds, then for all $k \geq n_4$, we have

$$w_{n_k} \geq y_{n_k} - \varepsilon.$$

Taking $k \to \infty$, we find $\lim_{n\to\infty} w_n = \infty$ because of the monotonic nature of $\{w_n\}$. Hence, $w_n > 0, \Delta w_n > 0$ for all $n \geq n_4$. Since $\Delta^m w_n \not\equiv 0$ for all large n and is nonpositive, it follows from Lemma 2.2 that there exists a positive integer p such that $(m-p)$ is odd and for all $n \geq n_5 \geq n_4$, we have $\Delta^j w_n > 0$ for $j = 0, 1, \dots, p$ and $\Delta^j w_n \Delta^{j+1} w_n < 0$

for $j = p, p+1, \ldots, m-2$. Then $\lim_{n\to\infty} \Delta^p w_n = l$ (finite) exists. Hence, $p \geq 1$. Applying Lemma 2.4 to (3.3), we obtain

$$\Delta^p w_n = l + \frac{(-1)^{m-p-1}}{(m-p-1)!} \sum_{i=n}^{\infty} (i-n+m-p-1)^{(m-p-1)} q_i G(y_{\sigma(i)}) \tag{3.5}$$

for all $n \geq n_5$. This implies

$$\sum_{i=n}^{\infty} (i-n+m-p-1)^{(m-p-1)} q_i G(y_{\sigma(i)}) < \infty \quad \text{for all } n \geq n_5. \tag{3.6}$$

In view of Lemma 2.7 and Remark 2.8, we have

$$\sum_{i=n_5}^{\infty} i^{m-p-1} q_i G(y_{\sigma(i)}) < \infty. \tag{3.7}$$

Because of (H_4), the above inequality yields

$$\liminf_{n\to\infty} \frac{G(y_{\sigma(n)})}{n^{p-1}} = 0.$$

Then we claim $\liminf_{n\to\infty} \frac{y_{\sigma(n)}}{n^{p-1}} = 0$. Otherwise, there exists $n_6 \geq n_5$ and $\gamma > 0$ such that $n \geq n_6$ implies $y_{\sigma(n)} > \gamma n^{p-1}$. By ($H_0$) and ($H_3$), we obtain $\frac{G(y_{\sigma(n)})}{n^{p-1}} > \gamma\delta > 0$ for all $n \geq n_6$, a contradiction. Hence, our claim holds. Next, we assert

$$\liminf_{n\to\infty} \frac{y_n}{n^{p-1}} = 0.$$

Otherwise, there exists $n_7 \geq n_6$ and $\gamma > 0$ such that $n \geq n_7$ implies $\frac{y_n}{n^{p-1}} > \gamma > 0$. As $\lim_{n\to\infty} \sigma(n) = \infty$, we can find $n_8 \geq n_7$ such that $\sigma(n) \geq n_7$ for all $n \geq n_8$. Then $\frac{y_{\sigma(n)}}{(\sigma(n))^{p-1}} > \gamma$ for all $n \geq n_8$. Due to (H_2), we have $\sigma(n) > \mu n$ for all $n \geq n_8$. Consequently, for all $n \geq n_8$, we have $y_{\sigma(n)} > \gamma(\mu n)^{p-1}$. Hence, $\frac{y_{\sigma(n)}}{n^{p-1}} > \gamma\mu^{p-1} > 0$ for all $n \geq n_8$, a contradiction. Thus, our assertion that $\liminf_{n\to\infty} \frac{y_n}{n^{p-1}} = 0$ holds. Since $p \geq 1$, due to Lemma 2.5, we can choose $B > 0$ such that $w_n > Bn^{p-1}$ for all $n \geq n_9 \geq n_8$. Thus,

$$\liminf_{n\to\infty} \frac{y_n}{w_n} = 0. \tag{3.8}$$

Set

$$p_n^* := p_n \frac{w_{\tau(n)}}{w_n} \quad \text{for } n \geq n_9.$$

It is clear from (H_8), and $\lim_{n\to\infty} w_n = \infty$ that

$$\lim_{n\to\infty} \frac{F_n}{w_n} = 0.$$

Then we have

$$\begin{aligned} 1 &= \lim_{n\to\infty} \left[\frac{w_n}{w_n}\right] \\ &= \lim_{n\to\infty} \left[\frac{y_n - p_n y_{\tau(n)} - F_n}{w_n}\right] \\ &= \lim_{n\to\infty} \left[\frac{y_n}{w_n} - p_n^* \frac{y_{\tau(n)}}{w_{\tau(n)}} - \frac{F_n}{w_n}\right] \\ &= \lim_{n\to\infty} \left[\frac{y_n}{w_n} - p_n^* \frac{y_{\tau(n)}}{w_{\tau(n)}}\right]. \end{aligned} \tag{3.9}$$

Since $\{w_n\}$ is an increasing sequence, we have $\dfrac{w_{\tau(n)}}{w_n} < 1$ for all $n \geq n_9$. If $\{p_n\}$ is in (A_1), then $0 \leq p_n^* < p_n \leq b < 1$ for all $n \geq n_9$. However, if $\{p_n\}$ is in (A_2), then $0 \geq p_n^* \geq p_n \geq -b > -1$ for all $n \geq n_9$. Hence, it is clear that if $\{p_n\}$ satisfies (A_1) or (A_2), then so does $\{p_n^*\}$. Hence, use of Lemma 2.1 yields, due to (3.8), that

$$\lim_{n\to\infty} \left[\frac{y_n}{w_n} - p_n^* \frac{y_{\tau(n)}}{w_{\tau(n)}}\right] = 0,$$

a contradiction to (3.9). Hence the unbounded solution $\{y_n\}$ cannot be eventually positive. Next, if $\{y_n\}$ is an eventually negative solution of (1.1) for all $n \geq n_1$, then we set $x_n := -y_n$ for $n \geq n_0$ to obtain $x_n > 0$ for all $n \geq n_1$. Thus, (1.1) reduces to

$$\Delta^m\big(x_n - p_n x_{\tau(n)}\big) + q_n \widetilde{G}(x_{\sigma(n)}) = \widetilde{f}_n \quad \text{for } n \geq n_0 \tag{3.10}$$

where

$$\{\widetilde{f}_n\} := \{-f_n\} \quad \text{and} \quad \widetilde{G}(u) := -G(-u) \quad \text{for } u \in \mathbb{R}. \tag{3.11}$$

Further,

$$\{\widetilde{F}_n\} := \{-F_n\} \quad \text{implies} \quad \{\Delta^m \widetilde{F}_n\} = \{\widetilde{f}_n\}. \tag{3.12}$$

In view of the above facts, it can be easily verified that the following conditions hold.

($\overline{\text{H}}_0$) $\widetilde{G}$ is nondecreasing and $u\widetilde{G}(u) > 0$ for all real $u \neq 0$,

($\overline{\text{H}}_3$) $\liminf_{|u|\to\infty} \dfrac{\widetilde{G}(u)}{u} \geq \delta > 0$,

($\overline{\text{H}}_8$) There exists a bounded sequence $\{\widetilde{F}_n\}$ such that $\{\Delta^m \widetilde{F}_n\} = \{\widetilde{f}_n\}$.

Proceeding as in the proof for the previous case, we obtain a contradiction. Hence, $\{y_n\}$ is oscillatory and the proof is complete. □

The following example illustrates Theorem 3.1.

Example 3.2. The neutral equation

$$\Delta^3\left(y_n - \frac{1}{2}y_{n-1}\right) + 135y_{n-2} = 0 \quad \text{for } n \geq 1 \tag{3.13}$$

satisfies all the conditions of Theorem 3.1. Hence, all the unbounded solutions are oscillatory. As such, $\{y_n\} = \{(-2)^n\}$ is an unbounded solution, which oscillates. But the results of [14, 16] cannot be applied to this equation, because $G(u) = u$ for $u \in \mathbb{R}$ is linear.

Theorem 3.3. *Let $m \geq 2$. Suppose that $\{p_n\}$ satisfies one of the conditions (A_1)–(A_4). If (H_0), (H_6) and (H_7) hold, then every bounded solution of* (1.1) *oscillates or tends to zero as $n \to \infty$.*

Proof. Let $\{y_n\}$ be a bounded solution of (1.1). If it oscillates, then there is nothing to prove. If it does not oscillate, then either $y_n > 0$ or $y_n < 0$ for all $n \geq n_1$ for some $n_1 \geq n_0$. Suppose $y_n > 0$ for all $n \geq n_1$. There exists an integer $n_2 \geq n_1$ such that $y_n, y_{\tau(n)} > 0$ and $y_{\sigma(n)} > 0$ for all $n \geq n_2$. Set $\{z_n\}$ and $\{w_n\}$ as in (3.1) and (3.2) respectively, to obtain (3.3). Then $w_n, \Delta w_n, \ldots, \Delta^{m-1}w_n$ are monotonic and of single sign for all $n \geq n_2 \geq n_1$. Since $\{y_n\}$ is bounded, $\{z_n\}$ and $\{w_n\}$ are bounded too. Using (H_7) and the monotonic nature of $\{w_n\}$, we obtain $\lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = \lambda$, which exists and is finite. Then applying Lemma 2.4 to (3.3), we obtain

$$w_n = \lambda + \frac{(-1)^{m-1}}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}q_i G(y_{\sigma(i)}) \tag{3.14}$$

for all $n \geq n_2$. Thus,

$$\frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}q_i G(y_{\sigma(i)}) < \infty \tag{3.15}$$

for all $n \geq n_2$. Using Lemma 2.7 and Remark 2.8 in the above inequality, we obtain

$$\sum_{i=n}^{\infty} i^{m-1} q_i G(y_{\sigma(i)}) < \infty \tag{3.16}$$

for all $n \geq n_2$. The above inequality due to (H_6) yields $\liminf_{n\to\infty} G(y_{\sigma(n)}) = 0$. Since $\lim_{n\to\infty} \sigma(n) = \infty$, it can be easily shown that $\liminf_{n\to\infty} G(y_n) = 0$. This implies due to ($H_0$) that $\liminf_{n\to\infty} y_n = 0$. From Lemma 2.1, it follows that $\lim_{n\to\infty} z_n = 0$. If $\{p_n\}$ is in (A_1), then

$$\begin{aligned}
0 = \lim_{n\to\infty} z_n &= \limsup_{n\to\infty}\big(y_n - p_n y_{\tau(n)}\big)\\
&\geq \limsup_{n\to\infty} y_n + \liminf_{n\to\infty}\big(-p_n y_{\tau(n)}\big)\\
&\geq (1-b)\limsup_{n\to\infty} y_n.
\end{aligned}$$

This implies $\limsup_{n\to\infty} y_n = 0$. Hence, $y_n \to 0$ as $n \to \infty$. If $\{p_n\}$ is in (A$_2$) or (A$_3$), then since $y_n \le z_n$ for all $n \ge n_2$, it follows that $y_n \to 0$ as $n \to \infty$. If $\{p_n\}$ satisfies (A$_4$), then $z_n \le y_n - b_2 y_{\tau(n)}$ for all $n \ge n_2$. Hence, it follows that

$$\begin{aligned} 0 = \lim_{n\to\infty} z_n &\le \liminf_{n\to\infty} \big(y_n - b_2 y_{\tau(n)}\big) \\ &\le \limsup_{n\to\infty} y_n + \liminf_{n\to\infty} \big(- b_2 y_{\tau(n)}\big) \\ &= (1-b_2) \limsup_{n\to\infty} y_n. \end{aligned}$$

Then $\limsup_{n\to\infty} y_n = 0$. Thus, $\lim_{n\to\infty} y_n = 0$.

If $\{y_n\}$ is eventually negative, then we may proceed with $x_n := -y_n$ defined for $n \ge n_0$ as in the proof of Theorem 3.1 and note that $\{y_n\}$ is an eventually solution of (3.10) with (3.11) and (3.12). Moreover, the condition ($\overline{\mathrm{H}}_0$) along with the following one holds.

($\overline{\mathrm{H}}_7$) There exists a sequence $\{\widetilde{F}_n\}$ such that $\{\Delta^m \widetilde{F}_n\} = \{\widetilde{f}_n\}$ and $\lim_{n\to\infty} \widetilde{F}_n = 0$.

Then proceeding as above, we prove $\lim_{n\to\infty} y_n = 0$. Thus, the proof is complete. □

Remark 3.4. Theorem 3.3 holds when G is linear, superlinear, or sublinear.

Next, we give a few examples to establish the significance of our results.

Example 3.5. Consider the neutral equation

$$\Delta^m \left(y_n - \frac{1}{2} y_{n-1} \right) + n^{-m} y_{n-2}^{\alpha} = n^{-m} 2^{\alpha(2-n)} \tag{3.17}$$

for $n \ge 1$, where $m \ge 2$ and α is a positive rational, being the quotient of two odd integers. Here, $\{p_n\} = \{1/2\}$ satisfies (A$_1$) and $q_n = n^{-m}$, $f_n = n^{-m} 2^{\alpha(2-n)}$ for $n \ge 1$. It is clear that $\sum_{i=1}^{\infty} i^{m-1} f_i < \infty$. Hence by Remark 2.9, it follows that

$$F_n = \frac{(-1)^m}{(m-1)!} \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} i^{-m} 2^{\alpha(2-i)} \quad \text{for } n \ge 1.$$

Obviously, $F_n < \infty$ for all $n \ge 1$. Hence, the equation (3.17) satisfies all the conditions of Theorem 3.3. Hence, every bounded nonoscillatory solution tends to zero at infinity. In particular $\{y_n\} = \{2^{-n}\}$ is a solution of (3.17), which tends to zero at infinity.

If $\alpha \ge 1$, then (3.17) does not come under the purview of the results in [14, 16], hence those results fail to deliver any conclusion. Further, even if $\alpha < 1$, then $m \ge 2$ implies $m - \alpha m + \alpha > 1$. This further implies (1.6) does not hold. Hence, Theorem 1.2 cannot be applied to (3.17). Thus, Theorem 3.3 along with Theorem 3.1 of this paper improves and generalizes Theorem 1.2.

Example 3.6. Consider the neutral equation

$$\Delta^m\left(y_n+\frac{1}{2}y_{n-1}\right)+n^{-m}y_{n-2}^{\alpha}=(-1)^m2^{-n-m+1}+n^{-m}2^{\alpha(2-n)} \quad \text{for } n\geq 1, \tag{3.18}$$

where $m \geq 2$, α is a positive rational, which is the quotient of two odd integers. Here, $\{p_n\}=\{-1/2\}$ satisfies (A_2) and $q_n=n^{-m}$, $f_n=(-1)^m2^{-n-m+1}+n^{-m}2^{\alpha(2-n)}$ for $n\geq 1$. Easily, we can verify that $\sum_{i=1}^{\infty} i^{m-1}f_i<\infty$ and the equation (3.18) satisfies all the conditions of Theorem 3.3 for (A_2). Hence, $\{y_n\}=\{2^{-n}\}$ is a solution of (3.18), which asymptotically tends to zero. If $\alpha\geq 1$, then results of [14, 16] cannot be applied to (3.18). Further, if $\alpha<1$, then neither Theorem 1.1(ii) nor [14, Corollary 3] be applied because (1.6) does not hold. Thus, Theorem 3.3 along with Theorem 3.1 of this paper improves and generalizes Theorem 1.1(ii) and [14, Corollary 3].

Example 3.7. Consider the equation

$$\Delta^4y_n+\frac{1}{n^4}y_{n-1}^{\alpha}=\frac{1}{2^{n+4}}+\frac{2^{\alpha(1-n)}}{n^4} \quad \text{for } n\geq 1, \tag{3.19}$$

where α is a positive rational, being the quotient of two odd integers. Here, $\{p_n\}=\{0\}$ satisfies (A_1) and (A_2) and $q_n=1/n^4$, $f_n=1/2^{n+4}+2^{\alpha(1-n)}/n^4$ for $n\geq 1$. It is easy to verify that $\sum_{i=1}^{\infty} i^3 f_i<\infty$ and equation (3.19) satisfies all the conditions of Theorem 3.3 for (A_2). Hence, $\{y_n\}=\{2^{-n}\}$ is a solution of (3.19), which tends to zero at infinity. If $\alpha\geq 1$, then the results of the papers [14, 16] cannot be applied to (3.19). Further, even if $\alpha<1$, then Theorem 1.1(i) cannot be applied, because (1.7) does not hold. Thus, Theorem 3.3 along with Theorem 3.1 of this paper improves and generalizes Theorem 1.1(i) and [14, Corollary 3].

4 Sufficient Conditions – II

In this section, we find sufficient conditions for oscillation of (1.1) when $\{p_n\}$ satisfies one of the conditions (A_5)–(A_7). With the help of a counter example, we proved in the introduction that some results of [11] with (A_5) and (A_7) are inaccurate. Further, Theorem 1.5 is true only for sublinear equations. Here an attempt is made to extend the results to linear and superlinear equations and correct the results of [11].

Theorem 4.1. *Suppose that $m\geq 2$, and that (A_6) holds. Assume that $\sigma(\tau(n))=\tau(\sigma(n))$ for all large n. Let (H_0)–(H_3) and (H_7) hold. Further, assume that*

(H_9) $G(-u)=-G(u)$ *for all* $u\in\mathbb{R}$,

(H_{10}) *For all real* $u, v > 0$*, there exists a scalar* $\beta > 0$ *such that* $G(u)G(v) \geq G(uv)$ *and* $G(u) + G(v) \geq \beta G(u + v)$*,*

(H_{11}) $\sum_{i=n_1}^{\infty} i^{m-2} q_i^* = \infty$*, where for* $n \geq n_1$*,* $q_n^* := \min\{q_n, q_{\tau(n)}\}$*.*

Then every solution of (1.1) *oscillates or tends to zero as* $n \to \infty$*.*

Proof. Let $\{y_n\}$ be an eventually positive solution of (1.1) with $y_n, y_{\tau(n)}, y_{\sigma(n)} > 0$ for all $n \geq n_1 \geq n_0$. Then set $\{z_n\}$ and $\{w_n\}$ as in (3.1) and (3.2) respectively to get (3.3) for all $n \geq n_2 \geq n_1$. Hence, $w_n, \Delta w_n, \Delta^2 w_n, \dots, \Delta^{m-1} w_n$ are monotonic for all $n \geq n_3 \geq n_2$. Consequently, from (H$_7$) it follows that

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} z_n = \lambda, \text{ where } -\infty \leq \lambda \leq \infty. \tag{4.1}$$

If $\lambda < 0$, then $z_n < 0$ for all large n, a contradiction. If $\lambda = 0$, then $y_n \leq z_n$ for all large n implies $\lim_{n\to\infty} y_n = 0$. If $\lambda > 0$, then $w_n > 0$ for all $n \geq n_3$. Then from Lemma 2.2, it follows that there exists an integer p such that $m-1 \geq p \geq 0$ and $(m-p)$ is odd, and for all $n \geq n_4 \geq n_3$, we have $\Delta^j w_n > 0$ for $j = 0, 1, \dots, p$ and $(-1)^{m+j-1}\Delta^j w_n > 0$ for $j = p+1, p+2, \dots, m-1$. Hence, $\lim_{n\to\infty} \Delta^p w_n = l$ exists (finite) and $\lim_{n\to\infty} \Delta^j w_n = 0$ for $j = p+1, p+2, \dots, m-1$. Note that $0 < \lambda < \infty$ implies $p = 0$, but $\lambda = \infty$ implies $p \geq 1$ such that $(m-p)$ is odd. Applying Lemma 2.4 to (3.3), we obtain (3.5) and consequently (3.6) follows. In view of Lemma 2.7 and Remark 2.8, we obtain

$$\sum_{i=n_5}^{\infty} i^{m-p-1} q_i G(y_{\sigma(i)}) < \infty \tag{4.2}$$

for some fixed $n_5 \geq n_4$. Note that, since $\{\tau(n)\}$ is monotonic increasing, it's inverse function $\{\tau^{-1}(n)\}$ exists such that $\tau^{-1}(\tau(n)) = n$ for all $n \geq n_5$. Since $q_n \geq q^*_{\tau^{-1}(n)}$ for all $n \geq n_5$, it follows that

$$\sum_{i=n_5}^{\infty} i^{m-p-1} q^*_{\tau^{-1}(i)} G(y_{\sigma(i)}) < \infty.$$

Then replacing n by $\tau(n)$ in the above inequality and multiplying by the scalar $G(b_2)$, we obtain

$$G(b_2) \sum_{i=n_6}^{\infty} (\tau(i))^{m-p-1} q_i^* G(y_{\sigma(\tau(i))}) < \infty,$$

where $n_6 \geq \tau^{-1}(n_5)$. By (H$_1$), $\tau(n)/n > \kappa > 0$ and $p_n \geq -b_2$ for all $n \geq n_6$. Then due to (H$_0$), we obtain

$$\sum_{i=n_6}^{\infty} i^{m-p-1} q_i^* G(-p_{\sigma(i)}) G(y_{\sigma(\tau(i))}) < \infty.$$

This with the use of (H_{10}) yields

$$\sum_{i=n_6}^{\infty} i^{m-p-1} q_i^* G\big(-p_{\sigma(i)} y_{\sigma(\tau(i))}\big) < \infty.$$

Since $\sigma(\tau(n)) = \tau(\sigma(n))$ for all $n \geq n_6$, the above inequality takes the form

$$\sum_{i=n_6}^{\infty} i^{m-p-1} q_i^* G\big(-p_{\sigma(i)} y_{\tau(\sigma(i))}\big) < \infty. \tag{4.3}$$

From (4.2) and the fact that $q_n \geq q_n^*$ for all $n \geq n_6$, we obtain

$$\sum_{i=n_6}^{\infty} i^{m-p-1} q_i^* G(y_{\sigma(i)}) < \infty. \tag{4.4}$$

Further, using (H_{10}), (4.3) and (4.4), one may get

$$\sum_{i=n_6}^{\infty} i^{m-p-1} q_i^* G(z_{\sigma(i)}) < \infty. \tag{4.5}$$

If $p = 0$, then (H_{11}) and (4.5) implies $\liminf_{n\to\infty} \big(nG(z_{\sigma(n)})\big) = 0$. Applying the assumption $\lim_{n\to\infty} \sigma(n) = \infty$ and (H_0), we obtain $\lim_{n\to\infty} z_n = 0$, a contradiction. If $p > 0$, then by Lemma 2.5, there exists $A > 0$ such that $w_n > An^{p-1}$ for all $n \geq n_7 \geq n_6$. For any $\varepsilon > 0$, using (H_7) we obtain $z_n \geq w_n - \varepsilon$ for all $n \geq n_8 \geq n_7$. Thus, due to Remark 2.6, we can find $A > B > 0$ such that

$$z_n > Bn^{p-1} \quad \text{for all } n \geq n_9 \geq n_8. \tag{4.6}$$

By (H_2), we have $\sigma(n)/n > \mu > 0$ for all $n \geq n_9$. Using this, (4.6) and (H_3) we obtain

$$\begin{aligned}\sum_{i=n_9}^{\infty} i^{m-p-1} q_i^* G(z_{\sigma(i)}) &\geq B\delta \sum_{i=n_9}^{\infty} i^{m-p-1} q_i^* (\sigma(i))^{p-1} \\ &\geq \delta B \mu^{p-1} \sum_{i=n_9}^{\infty} i^{m-2} q_i^* = \infty,\end{aligned}$$

by (H_{11}), a contradiction due to (4.5). Hence the proof for the case where $\{y_n\}$ is eventually positive is complete.

If $\{y_n\}$ is eventually negative, then we may proceed with $\{x_n\} := \{-y_n\}$ as in the proof of Theorem 3.1 and note that $\{x_n\}$ is an eventually positive solution of (3.10) with (3.11) and (3.12). Further, we note that (H_9) implies $G = \widetilde{G}$. In view of this, it is easy to verify that the conditions ($\overline{H}_0$) and ($\overline{H}_3$) along with the following two conditions hold.

($\overline{\text{H}}_9$) $\widetilde{G}(-u) = -\widetilde{G}(u)$ for all $u \in \mathbb{R}$,

($\overline{\text{H}}_{10}$) For all real $u, v > 0$, there exists a scalar $\beta > 0$ such that $\widetilde{G}(u)\widetilde{G}(v) \geq \widetilde{G}(uv)$ and $\widetilde{G}(u) + \widetilde{G}(v) \geq \beta\widetilde{G}(u+v)$.

Also, it is not difficult to see that ($\overline{\text{H}}_7$) holds. Then proceeding as above, in the proof for the case where $\{y_n\}$ eventually negative, we prove that $\lim_{n\to\infty} y_n = 0$ and complete the proof of the theorem. □

Remark 4.2. The prototype of the function G satisfying (H_0), (H_3), (H_9) and (H_{10}) is $G(u) = (\beta + |u|^{\mu})|u|^{\lambda}\text{sgn}(u)$ for $u \in \mathbb{R}$, where $\lambda > 0$, $\mu > 0$, $\lambda + \mu \geq 1$ and $\beta \geq 1$ are reals. For verification, we may take help of the well-known inequality (see [7, p. 292])

$$u^p + v^p \geq \begin{cases} (u+v)^p, & 0 \leq p < 1, \\ 2^{1-p}(u+v)^p, & p \geq 1. \end{cases}$$

Remark 4.3. For $m \geq 2$, the condition (1.9) implies (H_{11}). Further the condition (1.9) implies (H_5). However, if $\{q_n\}$ is monotonic, then both (1.9) and (H_5) are equivalent. Indeed, if $\{q_n\}$ is decreasing, then $\{q_n^*\} = \{q_n\}$. Hence, the equivalence of (1.9) and (H_5) is immediate. On the other hand if $\{q_n\}$ is increasing, then assume that (H_5) holds. Then $\{q_n^*\} = \{q_{\tau(n)}\}$. Hence, $\sum_{i=n_1}^{\infty} q_i^* = \sum_{i=n_1}^{\infty} q_{\tau(i)}^* = \infty$ since $\sum_{i=\tau(n_1)}^{\infty} q_i = \infty$. Thus (1.9) and ($\text{H}_5$) are equivalent, when $\{q_n\}$ is monotonic.

Lemma 4.4. *Suppose that $\{p_n\}$ satisfies the condition (A_7). Further, assume that there exists a positive integer k such that $\tau(n) = n - k$ for all large n. Let (H_0), (H_2), (H_3), (H_6) and (H_7) hold. Then for every nonoscillatory solution $\{y_n\}$ of* (1.1) *with $\{z_n\}$ and $\{w_n\}$ as defined in* (3.1) *and* (3.2) *respectively, either $\lim_{n\to\infty} w_n = 0$ or $\lim_{n\to\infty} w_n = -\infty$.*

Proof. Let $\{y_n\}$ be an eventually positive solution of (1.1) with $y_n, y_{\tau(n)}, y_{\sigma(n)} > 0$ for all $n \geq n_1 \geq n_0$. Then for $n \geq n_1$, we obtain (3.3). Hence, we learn that $w_n, \Delta w_n, \Delta^2 w_n, \ldots, \Delta^{m-1} w_n$ are monotonic for all $n \geq n_2 \geq n_1$. Then (4.1) holds, i.e., $\lim_{n\to\infty} w_n = \lambda$, where $-\infty \leq \lambda \leq \infty$. By the method of contradiction, we now show that $\lambda \neq \infty$. Assume, if possible that $\lambda = \infty$. Then $w_n > 0$ and $\Delta w_n > 0$ for all $n \geq n_2$. Due to (3.3) and Lemma 2.2, it follows that there exist $n_3 \geq n_2$ and an integer p with $m - 1 \geq p \geq 0$ and $(m - p)$ is odd, such that $n \geq n_3$ implies

$$\begin{aligned} \Delta^j w_n &> 0 \quad \text{for } j = 0, 1, 2, \ldots, p, \\ (-1)^{m+j-1}\Delta^j w_n &> 0 \quad \text{for } j = p+1, p+2, \ldots, m-1. \end{aligned} \tag{4.7}$$

Hence, $\lim_{n\to\infty} \Delta^p w_n = l$ exists and $\lim_{n\to\infty} \Delta^j w_n = 0$ for $j = p+1, p+2, \ldots, m-1$. If $p = 0$, then $0 \leq \lambda < \infty$, a contradiction. Hence, $m - 1 \geq p \geq 1$. Applying Lemma 2.4

to (3.3), we obtain (3.5). Consequently, (3.6) and then (3.7) follows due to Lemma 2.7 and Remark 2.8. From this, it follows, due to (H_6), that $\liminf_{n\to\infty}(G(y_{\sigma(n)})/n^p) = 0$. Hence, $\liminf_{n\to\infty}(y_{\sigma(n)}/n^p) = 0$, by ($H_0$) and ($H_3$). As $\lim_{n\to\infty}\sigma(n) = \infty$ and by (H_2), $\sigma(n) > \mu n$ for all large n, we obtain $\liminf_{n\to\infty}\frac{y_n}{n^p} = 0$. Due to Lemma 2.5, we can find $M_0 > 0$ such that $w_n > M_0 n^{p-1}$ for all $n \geq n_4 \geq n_3$. For any $\varepsilon > 0$, from (3.2) it follows due to (H_7) that $z_n \geq w_n - \varepsilon$ for all large n. From this, it follows, again by Remark 2.6 that there exists M_1 with $M_0 > M_1 > 0$, and $y_n - p_n y_{\tau(n)} > M_1 n^{p-1}$ for all $n \geq n_5 > n_4$. That is

$$y_n > y_{\tau(n)} + M_1 n^{p-1} \quad \text{for all } n \geq n_5 \tag{4.8}$$

due to (A_7). Let

$$n_6 \geq \max\{(p-2)k/3, n_5\}, \quad M := \min\{y_n : n_6 \leq n \leq n_6 + k\}$$

and

$$0 < \beta < \min\left\{\frac{M}{(n_6+k)^p}, \frac{M_1}{2pk}\right\}.$$

Define

$$A(n) := \begin{cases} (M_1 - p\beta k)n^{p-1} + \beta\sum_{i=2}^{p}(-1)^i\binom{p}{i}k^i n^{p-i}, & p \geq 2 \\ M_1 - \beta k, & p = 1 \end{cases}$$

for $n \geq n_6$. If p is odd, then we may write

$$\begin{aligned}\sum_{i=2}^{p}(-1)^i\binom{p}{i}k^i n^{p-i} =& \left[\binom{p}{2}k^2n^{p-2} - \binom{p}{3}k^3n^{p-3}\right] \\ &+ \left[\binom{p}{4}k^4n^{p-4} - \binom{p}{5}k^5n^{p-5}\right] \\ &+ \cdots + \left[\binom{p}{p-1}k^{p-1}n - \binom{p}{p}k^p\right],\end{aligned}$$

to obtain

$$\sum_{i=2}^{p}(-1)^i\binom{p}{i}k^i n^{p-i} > 0,$$

because

$$\binom{p}{i}k^i n^{p-i} > \binom{p}{i+1}k^{i+1}n^{p-i-1},$$

if and only if

$$n > k\binom{p}{i+1}\Big/\binom{p}{i} = \frac{(p-i)k}{i+1}$$

for $i = 2, 4, \ldots, p-1$. Further, $n \geq n_6$ implies that

$$n \geq n_6 > \frac{(p-2)k}{3} > \frac{(p-4)k}{5} > \cdots > \frac{k}{p}.$$

If p is even, then we put the terms in pair as above with the last single positive term $(-1)^p\binom{p}{p}k^p$. Thus, $A(n) > 0$ for all $n \geq n_6$. Since $y_n \geq M$ for all n with $n_6 \leq n \leq n_6 + k$ and $M > \beta(n_6+k)^p$, then $y_n > \beta n^p$ for all n with $n_6 \leq n \leq n_6 + k$. Since $\tau(n) = n - k$, then $n_6 + k \leq n \leq n_6 + 2k$ implies $n_6 \leq \tau(n) \leq n_6 + k$. Using (4.8), we obtain, for all $n_6 + k \leq n \leq n_6 + 2k$,

$$\begin{aligned} y_n >& y_{\tau(n)} + M_1 n^{p-1} > \beta(\tau(n))^p + M_1 n^{p-1} \\ =& \beta(n-k)^p + M_1 n^{p-1} > \beta n^p, \end{aligned}$$

because, for $p \geq 2$,

$$\begin{aligned} \beta n^p < A(n) + \beta n^p =& (M_1 - p\beta k)n^{p-1} + \beta\big[(n-k)^p - n^p + pkn^{p-1}\big] + \beta n^p \\ =& M_1 n^{p-1} + \beta(n-k)^p, \end{aligned}$$

and for $p = 1$, $A(n) + \beta n = M_1 + \beta(n-k) > \beta n$. Proceeding as above we have $y_n > \beta n^p$ for all $n \geq n_6$. Hence,

$$\liminf_{n\to\infty} \frac{y_n}{n^p} \geq \beta > 0,$$

a contradiction. Thus, $\lambda \neq \infty$. If $\lambda \neq -\infty$, then λ is finite. This implies that $(-1)^{m+j}\Delta^j w_n < 0$ for $j = 1, 2, \ldots, m-1$, and $\lim_{n\to\infty} \Delta^j w_n = 0$ for $j = 1, 2, \ldots, m-1$. Then applying Lemma 2.4 to (3.3), we obtain (3.14). Consequently (3.16) holds. From this it follows, due to (H_6), that $\liminf_{n\to\infty} G(y_n) = 0$, and hence $\liminf_{n\to\infty} y_n = 0$ by (H_0). Then application of Lemma 2.1 yields $\lim_{n\to\infty} z_n = 0$. Thus, $\lim_{n\to\infty} w_n = 0$ by (4.1). The proof for the case when $\{y_n\}$ is eventually negative is similar. □

Theorem 4.5. *Suppose that* $\{p_n\}$ *satisfies* (A_7)*, and* m *is odd. Assume* (H_0), (H_2), (H_3) *and* (H_6)*. Then every solution of the equation*

$$\Delta^m\big(y_n - p_n y_{n-k}\big) + q_n G(y_{\sigma(n)}) = 0 \quad \textit{for } n \geq n_0 \tag{4.9}$$

oscillates or tends to $\pm\infty$ *as* $n \to \infty$.

Proof. Assume that $\{y_n\}$ is any nonoscillatory positive solution of (4.9) such that $n \geq n_1 \geq n_0$, $y_n, y_{\tau(n)}, y_{\sigma(n)} > 0$. Then setting $\{z_n\}$ as in (3.1), we obtain

$$\Delta^m z_n = -q_n G(y_{\sigma(n)}) \leq 0 \quad \text{for all } n \geq n_1. \tag{4.10}$$

Then applying the Lemma 4.4 for $\{f_n\} = \{0\}$, we have $\lim_{n\to\infty} z_n = 0$ or $\lim_{n\to\infty} z_n = -\infty$. If the former holds, i.e., $\lim_{n\to\infty} z_n = 0$, then from Lemma 2.4 and (4.10), we find

$$z_n = \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}q_i G(y_{\sigma(i)}) \quad \text{for all } n \ge n_2 \tag{4.11}$$

since m is odd. Thus,

$$\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}q_i G(y_{\sigma(i)}) < \infty \tag{4.12}$$

for all $n \ge n_3 \ge n_2$. Note that $z_n > 0$ for all large n, since m is odd. This implies $\liminf_{n\to\infty} y_n > 0$ due to (A$_7$). Then there exists γ such that $y_n > \gamma > 0$ for all $n \ge n_4 \ge n_3$. Hence, for al $n \ge n_4$, we have

$$\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}q_i G(y_{\sigma(i)}) > G(\gamma)\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}q_i = \infty,$$

a contradiction to (4.12) by (H$_6$). If $\lim_{n\to\infty} z_n = -\infty$, then from (A$_7$) and (3.1) it follows that $y_{n-k} \ge -z_n/b_2$ for all large n. This implies $\lim_{n\to\infty} y_n = \infty$. Similarly, we prove $\lim_{n\to\infty} y_n = -\infty$, when $y_n < 0$ for all large n. Thus, the proof is complete. □

Corollary 4.6. *Suppose that $\{p_n\}$ satisfies* (*A*$_7$)*, and m is odd. Assume* (*H*$_0$)*,* (*H*$_2$)*,* (*H*$_3$) *and* (*H*$_6$)*. Then every bounded solution of the equation* (4.9) *oscillates.*

Proof. It follows directly from Theorem 4.5. □

Theorem 4.7. *Suppose that m is odd. Assume* (*H*$_0$)*,* (*H*$_2$)*,* (*H*$_3$) *and* (*H*$_6$)*. Then every solution of the equation*

$$\Delta^m\big(y_n - y_{n-k}\big) + q_n G(y_{\sigma(n)}) = 0 \quad \textit{for } n \ge n_0$$

oscillates.

Proof. Applying Lemma 4.4, we obtain $\lim_{n\to\infty} z_n = 0$ or $\lim_{n\to\infty} z_n = -\infty$. If the latter holds, then $z_n < 0$ for all large n. This implies $\{y_n\}$ is bounded. Consequently, $\{y_n\}$ is bounded, a contradiction. If the former holds, then proceeding as in the proof of Theorem 4.5, we get another contradiction. Hence, the proof is complete. □

For our next result we need the following hypothesis.

(H$_{12}$) Suppose that for every subsequence $\{q_{n_j}\}$ of $\{q_n\}$, we have

$$\sum_{j=0}^{\infty}(n_j)^{m-1}q_{n_j} = \infty,$$

or equivalently $\liminf_{n\to\infty}(n^{m-1}q_n) > 0$ (see Remark 4.8 below).

Remark 4.8. Consider the following two statements.

(S$_1$) $\sum_{j=1}^{\infty} q_{n_j} = \infty$ for every subsequence $\{q_{n_j}\}$ of $\{q_n\}$,

(S$_2$) $\liminf_{n\to\infty} q_n > 0$.

It is mentioned in [11, pp. 95] that (S$_1$) is weaker than (S$_2$), i.e., (S$_2$) implies (S$_1$) but not the other way around. To support their claim, the authors have presented the counter example $\{q_n\} = \{1/n\}$, which is in fact, not correct. It is because we can find a subsequence $\{n_j\} = \{j^2\}$ of $\{n\}$ such that the subsequence $\{q_{n_j}\}$ does not satisfy (S$_1$). We now claim that (S$_1$) is equivalent to (S$_2$). It is obvious that (S$_2$) implies (S$_1$). Conversely, suppose that (S$_2$) does not hold. Then, we can find an increasing divergent subsequence $\{n_j\}$ of positive integers such that $\lim_{j\to\infty} q_{n_j} = 0$. Without the loss of any generality, we may assume that $\{n_j\}$ satisfies $q_{n_j} \le 1/j^2$ for all $j \in \mathbb{N}$. Then clearly, $\sum_{j=1}^{\infty} q_{n_j} \le \sum_{j=1}^{\infty} 1/j^2 = \pi^2/6 < \infty$. This proves our claim. Therefore, (S$_1$) holds if and only if (S$_2$) holds.

Theorem 4.9. *Suppose that $\{p_n\}$ satisfies the condition (A_7). Further, assume that there exists a positive integer k such that $\tau(n) = n - k$ for all large n. Let (H_0), (H_2), (H_3), (H_7) and (H_{12}) hold. Then*

(*i*) *every bounded solution of* (1.1) *oscillates or tends to zero as $n \to \infty$,*

(*ii*) *every unbounded solution of* (1.1) *oscillates or tends to $\pm\infty$ as $n \to \infty$.*

Proof. Clearly (H$_{12}$) implies (H$_6$). Now, let us prove (i) and assume $\{y_n\}$ be any nonoscillatory positive solution of (1.1) which is bounded. We have to prove that $\lim_{n\to\infty} y_n = 0$. Set $\{z_n\}$ and $\{w_n\}$ as in (3.1) and (3.2) respectively to get (3.3). Since (H$_{12}$) implies (H$_6$), we apply Lemma 4.4 to get $\lim_{n\to\infty} w_n = 0$ or $\lim_{n\to\infty} w_n = -\infty$. Since $\{y_n\}$ is bounded, $\{w_n\}$ is bounded, and hence $\lim_{n\to\infty} w_n = -\infty$ is not possible. Thus, $\lim_{n\to\infty} w_n = 0$. Then we apply Lemma 2.4 to (3.3) to get (3.14). Consequently, (3.15) and (3.16) follows. Then we apply (H$_6$) to get $\liminf_{n\to\infty} G(y_{\sigma(n)}) = 0$. This implies $\liminf_{n\to\infty} y_{\sigma(n)} = 0$, because of (H$_0$). Then applying the condition, $\lim_{n\to\infty} \sigma(n) = \infty$, we obtain $\liminf_{n\to\infty} y_n = 0$. Suppose $\limsup_{n\to\infty} y_n = \omega > 0$. Then we can find a subsequence such that $y_{\tau(n_j)} \ge \eta > 0$ for all $j \ge n_1$. Hence,

$$\sum_{j=n_1}^{\infty} (n_j)^{m-1} q_{n_j} G(y_{\tau(n_j)}) \ge G(\eta) \sum_{j=n_1}^{\infty} (n_j)^{m-1} q_{n_j} = \infty,$$

a contradiction to (3.16). The proof for the case where $y_n < 0$ for all large n is similar.

Next, let us prove (ii) and assume $\{y_n\}$ be an unbounded positive solution of (1.1). Then we proceed as in case (i) above, apply Lemma 4.4 to obtain $\lim_{n\to\infty} w_n = 0$ or $\lim_{n\to\infty} w_n = -\infty$. In this case, we claim $\lim_{n\to\infty} w_n = 0$ cannot hold. Otherwise, as in the proof for the case (i), we prove (3.16) holds. Since $\{y_n\}$ is unbounded, we can find a subsequence such that $y_{\tau(n_j)} \geq \zeta > 0$ for all $j \geq n_1$. Hence

$$\sum_{j=n_1}^{\infty} (n_j)^{m-1} q_{n_j} G(y_{\tau(n_j)}) \geq G(\zeta) \sum_{j=n_1}^{\infty} (n_j)^{m-1} q_{n_j} = \infty,$$

a contradiction to (3.16). Thus, $\lim_{n\to\infty} w_n = -\infty$. We observe that (4.1) holds because of (H_7). Hence, $\lim_{n\to\infty} z_n = -\infty$. From ($A_7$) and (3.1) it follows that $y_{\tau(n)} \geq -z_n/b_2$ for all large n. This implies $\lim_{n\to\infty} y_n = \infty$. The proof for the case, $y_n < 0$ for large n, is similar. □

Theorem 4.10. *Suppose that $\{p_n\}$ satisfies the condition (A_5). Assume that there exists a positive integer k such that $\tau(n) = n - k$ for all large n. Let (H_0), (H_2)–(H_4), (H_7) and (H_{12}) hold. Then*

(*i*) *every bounded solution of* (1.1) *oscillates or tends to zero as $n \to \infty$,*

(*ii*) *every unbounded solution of* (1.1) *oscillates or tends to $\pm\infty$ as $n \to \infty$.*

Proof. Suppose $\{y_n\}$ is an eventually positive solution of (1.1). Note that Lemma 2.1 can be applied for (A_5) and $\{p_n\}$ is in (A_5) implies $\{p_n^*\}$ is in (A_5). Then using (H_4) and proceeding as in the proof of Theorem 3.1, we prove $\lim_{n\to\infty} w_n = \infty$, is not possible. Then using (H_{12}) and proceeding as in the proof of Theorem 4.9, we get the desired result. □

5 Necessary Conditions

In this section, we would like to find necessary conditions for every solution of (1.1) to oscillate or tend to zero as $n \to \infty$. For the purpose, we require the following fixed point theorem.

Lemma 5.1 (Krasnoselskii's fixed point theorem [5]). *Let Y be a Banach space. Let S be a bounded closed convex subset of Y and let A, B be maps of S into Y such that $Ax + By \in S$ for every pair of $x, y \in S$. If A is a contraction and B is completely continuous, then the equation*

$$Ay + By = y$$

has a solution in S.

Definition 5.2 (see [4, Definition 3.2, pp. 196]). A set of sequences in l^∞ is uniformly Cauchy (or equi-Cauchy) if for every $\varepsilon > 0$ there exists an integer n_1 such that

$$|y_i - y_j| < \varepsilon$$

whenever $i, j \geq n_1$ for every $\{y_n\}$ in S.

Theorem 5.3 (see [4, Theorem 3.3, pp. 196]). *A bounded uniformly Cauchy subset S of l^∞ is relatively compact.*

Theorem 5.4. *Suppose that $\{p_n\}$ satisfies* (A_1) *or* (A_2). *Let* (H_8) *hold. If every solution of* (1.1) *oscillates or tends to zero as $n \to \infty$, then* (H_6) *holds.*

Proof. Suppose that $\{p_n\}$ satisfies (A_1). Assume for the sake of contradiction that (H_6) does not hold. Then

$$\sum_{i=n_0}^{\infty} i^{m-1} q_i < \infty. \tag{5.1}$$

Hence, all we need to show is the existence of a bounded solution $\{y_n\}$ of (1.1) with $\liminf_{n\to\infty} y_n > 0$. From ($H_8$), we find a positive constant k and a positive integer $n_1 \geq n_0$ such that

$$|F_n| < k \quad \text{for all } n \geq n_1. \tag{5.2}$$

Choose two positive constants L and c such that $L \geq 5k/(1-b)$ and $c \leq k$. Since $G \in C(\mathbb{R}, \mathbb{R})$, let

$$\mu := \max\big\{|G(u)| : c \leq u \leq L\big\}. \tag{5.3}$$

Then using (5.1) and Remark 2.8, for $\varepsilon > 0$, one can fix $n_3 > n_2$ such that $n \geq n_3$ implies

$$\mu \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} q_i < \varepsilon. \tag{5.4}$$

For (A_1) let us take $\varepsilon \leq k$. Choose $n_5 \geq n_3$ such that $n_4 := \min\{\tau(n_5), \sigma(n_5)\} \geq n_3$. Let $Y = \ell_\infty^{N_0}$, Banach space of real bounded sequences $y = \{y_n\}$ with $y_1 = y_2 = \cdots = y_{n_4}$ and supremum norm

$$\|y\| := \sup\{|y_n| : n \geq n_4\}.$$

Define

$$S = \big\{y \in Y : c \leq y_n \leq L \text{ for all } n \geq n_4\big\}. \tag{5.5}$$

Clearly, S is a bounded closed and convex subset of Y. Now, we define two operators A and $B : S \to Y$ as follows. For $y = \{y_n\} \in S$, define

$$(Ay)_n = \begin{cases} (Ay)_{n_5}, & n_4 \leq n \leq n_5 \\ p_n y_{\tau(n)} + F_n + \lambda, & n \geq n_5, \end{cases} \tag{5.6}$$

where $\lambda := 3k$, and

$$(By)_n = \begin{cases} (By)_{n_5}, & n_4 \le n \le n_5 \\ \dfrac{(-1)^{m-1}}{(m-1)!} \displaystyle\sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} q_i G(y_{\sigma(i)}), & n \ge n_5. \end{cases} \tag{5.7}$$

First we show that if $x = \{x_n\}, y = \{y_n\} \in S$, then $Ax + By \in S$. Hence, for any $\{x_n\}, \{y_n\} \in S$ and all $n \ge n_5$, we obtain

$$\begin{aligned}(Ax)_n + (By)_n \le & p_n x_{\tau(n)} + 3k \\ & + \frac{1}{(m-1)!} \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} q_i |G(y_{\sigma(i)})| + |F_n| \\ \le & bL + 3k + k + k \le L.\end{aligned}$$

On the other hand

$$\begin{aligned}(Ax)_n + (By)_n \ge & 3k - \frac{1}{(m-1)!} \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} q_i |G(y_{\sigma(i)})| - |F_n| \\ \ge & 3k - k - k \ge c.\end{aligned}$$

Hence

$$c \le (Ax)_n + (By)_n \le L \quad \text{for all } n \ge n_5.$$

Thus, we proved that $Ax + By \in S$ for any $x, y \in S$. Next, we show that A is a contraction on S. In fact for $x, y \in S$ and all $n \ge n_5$ we have

$$\|(Ax)_n - (Ay)_n\| \le |p_n||x_{\tau(n)} - y_{\tau(n)}| \le b||x - y||.$$

This implies A is a contraction because $0 < b < 1$. Next, we show that B is completely continuous. For this as a first step we show that B is continuous. Suppose $\{y_n^l\}$ is a sequence of points in S (with l taken from the index set) which converges to $\{y_n\}$ in S as $l \to \infty$. Since S is closed $y \in S$. For $n \ge N_1$ we have

$$|(By^l)_n - (By)_n| \le \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} q_i |G(y^l_{\sigma(i)}) - G(y_{\sigma(i)})|.$$

Since G are continuous, therefore

$$|G(y^l_{\sigma(i)}) - G(y_{\sigma(i)})| \to 0 \quad \text{as } l \to \infty.$$

Hence, B is continuous. Next what remains to show is BS is relatively compact. Using the result [4, Theorem 3.3], i.e., Theorem 5.3 we need only show that BS is uniformly

Cauchy (see Definition 5.2). Let $\{y_n\}$ be a sequence in S. From (5.1) and Remark 2.8, it follows that for $\varepsilon > 0$, there exists $n_6 \geq n_5$ such that for all $n \geq n_6$,

$$\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}q_i|G(y_{\sigma(i)})| < \frac{\varepsilon}{2}.$$

Then, for any $i \geq j \geq n_6$, we have

$$|(By)_i - (By)_j| < \varepsilon.$$

Thus, BS is uniformly Cauchy. Hence, it is relatively compact. Then by Lemma 5.1, we can find $y^0 = \{y_n^0\}$ in S such that $Ay^0 + By^0 = y^0$. Clearly, $\{y_n^0\}$ is a bounded, positive solution of (1.1) with limit infimum greater than or equal to $c > 0$. Thus, the proof is complete. If $\{p_n\}$ satisfies (A_2), then the proof is similar, only thing we have to do is to suitably fix c, L, ε and λ. In this regard, first decrement b if necessary, so that $b < 1/5$. Then select L, c, ε and λ such that $5k \leq L < k/b$, $0 < c \leq k - bL$, $0 < \varepsilon \leq k$ and $\lambda := 3k$. The mappings A and B are defined similarly, as defined for (A_1). Then proceeding as above we complete the proof. □

Remark 5.5. Theorem 5.4 improves [11, Theorems 4.1 and 4.2], where there are restrictions on m and on the bounds of $\{F_n\}$. Further, it generalizes and extends the necessary part of [10, Theorem 2.3]. In all these results of [10, 11] the authors require (H_0) and the condition that G is Lipschitzian in intervals of the form $[a, b]$ unlike in Theorem 5.4. Further, Theorem 5.4 holds even if $\{q_n\}$ changes sign. In that case we have to replace $\{q_n\}$ by $\{|q_n|\}$ in (H_6).

Theorem 5.6. *Suppose that $\{p_n\}$ satisfies (A_3) or (A_4). Let (H_8) hold. If every solution of* (1.1) *oscillates or converges to zero as $n \to \infty$, then (H_6) holds.*

Proof. Suppose that $\{p_n\}$ satisfies (A_4). The proof is similar to the proof of Theorem 5.4 with the following changes in the parameters c, L, λ and ε. Choose $L := k(2b_1 + 3b_2)/b_1(b_1-1)$ and $0 < c \leq k/b_1$. Then $L > c > 0$ and assume $\varepsilon = k$. We may define the mappings A and B as

$$(Ay)_n := \begin{cases} Ay_{n_5}, & n_4 \leq n \leq n_5, \\ \dfrac{y_{\tau^{-1}(n)}}{p_{\tau^{-1}(n)}} + \dfrac{\lambda}{p_{\tau^{-1}(n)}} + \dfrac{F_{\tau^{-1}(n)}}{p_{\tau^{-1}(n)}}, & n \geq n_5, \end{cases}$$

where $\lambda := 3b_2k/b_1$, and

$$(By)_n := \begin{cases} By_{n_5}, & n_4 \leq n \leq n_5 \\ \dfrac{(-1)^m}{(m-1)!p_{\tau^{-1}(n)}} \displaystyle\sum_{i=\tau^{-1}(n)}^{\infty}(i-\tau^{-1}(n)+m-1)^{m-1}q_iG(y_{\sigma(i)}), & n \geq n_5. \end{cases}$$

The function τ^{-1} used in the definition of the operators A and B, is the inverse function of τ, which exists because $\{\tau(n)\}$ is increasing, with $\tau^{-1}(\tau(n)) = n$ for all $n \geq n_0$. Then proceeding as in the proof of Theorem 5.4, we find a positive bounded solution with limit infimum not less than $c > 0$. If $\{p_n\}$ satisfies (A_3), then the proof is similar to the proof for the case (A_4), hence we leave it for the reader to guess and find the values of $c, L, \varepsilon, \lambda$ and complete the proof. □

6 Final Comments

In view of Theorems 3.3, 5.4 and 5.6, the following result follows as a corollary.

Corollary 6.1. *Suppose that any one of the conditions* (A_1)–(A_4) *holds. Then under the assumptions* (H_0) *and* (H_7) *every bounded solution of* (1.1) *oscillates or asymptotically tends to zero if and only if* (H_6) *holds.*

We conclude this paper with two open problems for further research.

Open Problem 6.2. Can we prove Theorem 4.1 under a condition weaker than (H_{11})?

Open Problem 6.3. Can we prove Theorem 4.9 with the assumption (H_6) in place of (H_{12})? Or with any other condition weaker than (H_{12})?

References

[1] R. P. Agarwal, M. M. S. Manuel and E. Thandapani. Oscillatory and nonoscillatory behavior of second order neutral delay difference equations. *Math. Comput. Modelling*, 24(1):5–11, 1996.

[2] R. P. Agarwal, S. R. Grace. Oscillation of higher order nonlinear difference equations of neutral type. *Appl. Math. Lett.*, 12(8):77–83, 1999.

[3] R. P. Agarwal. *Difference Equations and Inequalities: Theory, Methods, and Applications*. Marcel Dekker Inc., New York, 2000.

[4] S. S. Cheng and W. T. Patula. An existence theorem for a nonlinear difference equation. *Nonlinear Anal.*, 20(3):193–203, 1993.

[5] L. H. Erbe, Q. K. Kong and B. G. Zhang. *Oscillation Theory for Functional Differential Equations*. Marcel Dekker Inc., New York, 1995.

[6] I. Gyori and G. Ladas. *Oscillation Theory of Delay Differential Equations with Applications*. Clarendon Press, Oxford, 1991.

[7] T. H. Hilderbrandt. *Introduction to the Theory of Integration*. Academic Press, New York, 1963.

[8] W. G. Kelley and A. C. Peterson. *Difference Equations: An Introduction with Applications*. Academic Press, New-York, 1991.

[9] S. C. Mallik and S. Arora. *Mathematical Analysis*. New Age International(P) Ltd. Publishers, New Delhi, 2001.

[10] N. Parhi and A. K. Tripathy. Oscillation of forced nonlinear neutral delay difference equations of first order. *Czech. Math. J.*, 53(1):83–101 (128), 2003.

[11] N. Parhi and A. K. Tripathy. Oscillation of a class of nonlinear neutral difference equations of higher order. *J. Math. Anal. Appl.*, 284(2):756–774, 2003.

[12] R. N. Rath, S. K. Rath and B. L. S. Barik. Nonoscillation criteria for solutions of higher-order functional difference equations of neutral type. *Int. J. Difference Equations*, 3(2):289–304, 2008.

[13] E. Thandapani, P. Sundaram, J. R. Graef and P. W. Spikes. Asymptotic behaviour and oscillation of solutions of neutral delay difference equations of arbitrary order. *Math. Slovaca*, 47:539–551, 1997.

[14] E. Thandapani, M. M. S. Manuel, J. R. Graef and P. W. Spikes. Oscillation of a higher order neutral difference equation with a forcing term. *Int. J. Math. Math. Sci.*, 22(1):147–154, 1999.

[15] E. Thandapani and K. Mahalingam. Necessary and sufficient conditions for oscillation of second order neutral difference equations. *Tamkang J. Math.*, 34(2):137–145, 2003.

[16] M. K. Yıldız, Ö. Öcalan. Oscillation results for higher order nonlinear neutral delay difference equations. *Appl. Math. Lett.*, 20(3):243–247, 2007.

[17] Y. Zhou and Y. Q. Huang. Existence for nonoscillatory solutions of higher-order nonlinear neutral difference equations. *J. Math. Anal. Appl.*, 280(1):63–76, 2003.