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Definiteness of quadratic functionals for Hamiltonian and symplectic systems: A survey

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Abstract

In this paper we provide a survey of characterizations of the nonnegativity and positivity of quadratic functionals arising in the theory of linear Hamiltonian and symplectic systems. We study these functionals on traditional continuous time domain (under and without controllability), on discrete domain, and on time scale domain which unifies and extends both previous types. For each case we distinguish functionals with zero, separated, and jointly varying endpoints. The presented conditions are formulated in terms of the properties of a special conjoined basis of the considered linear system. It is now easy to compare all the results – between continuous, discrete, and time scale cases, between the zero, separated, and jointly varying endpoints, and between the nonnegativity and positivity.

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1 Introduction

In this paper we survey conditions characterizing nonnegativity and positivity of quadratic functionals corresponding to linear Hamiltonian and symplectic systems. These conditions are formulated in terms of the properties of special solution(s) of the given linear system. More specifically, we will discuss

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- the *linear Hamiltonian differential systems*

$$x' = A(t)\,x + B(t)\,u, \quad u' = C(t)\,x - A^T(t)\,u, \quad t \in [a,b], \tag{S$_c$}$$

with the quadratic functional

$$\mathcal{F}_c(x,u) := \int_a^b \{\, x^T(t)\,C(t)\,x(t) + u^T(t)\,B(t)\,u(t)\}\,\mathrm{d}t, \tag{1.1}$$

- the *discrete symplectic systems*

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k, \quad k \in [0,N]_{\mathbb{N}} := \{0,\ldots,N\}, \tag{S$_d$}$$

with the discrete quadratic functional

$$\mathcal{F}_d(x,u) := \sum_{k=0}^{N} \{x_k^T \mathcal{C}_k^T \mathcal{A}_k x_k + 2 x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{D}_k^T \mathcal{B}_k u_k\}, \tag{1.2}$$

- and the *time scale symplectic* (or *Hamiltonian*) *systems*

$$x^\Delta = \mathcal{A}(t)\,x + \mathcal{B}(t)\,u, \quad u^\Delta = \mathcal{C}(t)\,x + \mathcal{D}(t)\,u, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \tag{S$_t$}$$

with the time scale quadratic functional

$$\mathcal{F}_t(x,u) := \int_a^b \{x^T \mathcal{C}^T (I + \mu \mathcal{A})\,x + 2\mu x^T \mathcal{C}^T \mathcal{B} u + u^T (I + \mu \mathcal{D}^T)\,\mathcal{B} u\}(t)\,\Delta t. \tag{1.3}$$

Systems (S$_c$), (S$_d$), and (S$_t$) will be commonly referred to as (S$_i$) for $i \in \{c,d,t\}$. As it is well known, see e.g., [29], system (S$_t$) covers both systems (S$_c$) and (S$_d$) as special cases.

Quadratic functionals in (1.1), (1.2), (1.3) arise as second variations in nonlinear calculus of variations and optimal control problems, see e.g., [2, 16, 17, 37, 41, 53] for the continuous time theory, e.g., [12,23,28] for the discrete time theory, and e.g., [5,27, 33, 35] for the time scales theory.

In the literature there are several conditions used in order to characterize the definiteness of quadratic functionals. These are the *conjoined bases* conditions [6, 25, 28, 39, 41, 43, 47, 51, 52], *conjugate points theory* [6, 13, 25, 31, 41, 43, 44, 54, 57], *coupled points theory* [24, 26, 30, 47, 50–52, 55, 56], *Riccati equations* or *inequalities* [6, 19, 20, 25, 31, 32, 34, 40, 42, 43, 47, 52, 53], *perturbations* [21, 22, 32, 45–47], and *eigenvalue problems* [1,9,37,49]. In the present paper we focus on the first notion, namely the conditions in terms of the *natural conjoined basis* (or *principal solution*) of the given linear system. We consider quadratic functionals with zero, separated, and jointly varying endpoints, gradually increasing their generality.

The paper is organized as follows. In the next section we introduce the quadratic functionals for the three systems (S_c), (S_d), (S_t) and the corresponding boundary conditions. Then in Section 3 we recall the notions of conjoined bases of systems (S_i), $i \in \{c, d, t\}$. In Sections 4, 5, 6 we present characterizations of the nonnegativity and positivity of these quadratic functionals for the continuous time (under and without controllability), discrete time, and time scales, respectively, and discuss their mutual relation. It is then easy to compare the corresponding results for the zero, separated, and jointly varying endpoints.

In this paper we shall not repeat the standard time scale notation and terminology, which can be found e.g., in [10, 11, 27, 29]. In particular, $\sigma(t)$ and $\rho(t)$ are the forward and backward jumps at the point t, and $\mu(t) := \sigma(t) - t$ is the graininess at t.

2 Quadratic Functionals

In this section we introduce in details the quadratic functionals discussed in this paper. We distinguish three types of boundary conditions (in the order of generality), namely

- the *zero endpoints*
$$x(a) = 0 = x(b), \tag{2.1}$$
- the *separated endpoints*
$$x(a) \in \operatorname{Im} R_a, \quad x(b) \in \operatorname{Im} R_b, \tag{2.2}$$
- and the *jointly varying endpoints*
$$\begin{pmatrix} x(a) \\ x(b) \end{pmatrix} \in \operatorname{Im} R. \tag{2.3}$$

In (2.2) the matrices R_a, $R_b \in \mathbb{R}^{n\times n}$, while in (2.3) the matrix $R \in \mathbb{R}^{2n\times 2n}$.

The quadratic functionals for separated endpoints have the form

$$\mathcal{G}_i(x,u) := x^T(a)\,\Gamma_a\,x(a) + x^T(b)\,\Gamma_b\,x(b) + \mathcal{F}_i(x,u), \quad i \in \{c, d, t\}, \tag{2.4}$$

depending on the continuous or discrete or time scale setting, where Γ_a, $\Gamma_b \in \mathbb{R}^{n\times n}$ are given symmetric matrices. The quadratic functionals with jointly varying endpoints have the form

$$\mathcal{H}_i(x,u) := \begin{pmatrix} x(a) \\ x(b) \end{pmatrix}^T \Gamma \begin{pmatrix} x(a) \\ x(b) \end{pmatrix} + \mathcal{F}_i(x,u), \quad i \in \{c, d, t\}, \tag{2.5}$$

where $\Gamma \in \mathbb{R}^{2n\times 2n}$ is a given symmetric matrix.

Note that when the matrices $R = \operatorname{diag}\{R_a, R_b\}$ and $\Gamma = \operatorname{diag}\{\Gamma_a, \Gamma_b\}$ have $n \times n$ block diagonal structure, the boundary conditions (2.3) and the functionals $\mathcal{H}_i$ reduce to the separated boundary conditions (2.2) and to the functionals $\mathcal{G}_i$. Furthermore, when $R = 0 = \Gamma$, both boundary conditions (2.3) and (2.2) reduce to the zero boundary conditions in (2.1), and the functionals $\mathcal{H}_i$ and $\mathcal{G}_i$ reduce to the functional $\mathcal{F}_i$.

We say that a pair (x,u) is *admissible* for the quadratic functional $\mathcal{F}_i$, $\mathcal{G}_i$, $\mathcal{H}_i$, if (x,u) satisfies the first equation of the linear system (S_i) and the corresponding boundary conditions (2.1), (2.2), (2.3), respectively. In the continuous time case we also assume that $x \in \mathrm{C}^1_{\mathrm{p}}$ (piecewise continuously differentiable) on $[a,b]$ and $u \in \mathrm{C}_{\mathrm{p}}$ (piecewise continuous) on $[a,b]$, in the discrete case x is defined on $[0,N+1]_{\mathbb{N}}$ and u is defined on $[0,N]_{\mathbb{N}}$, while in the time scale setting we assume that $x \in \mathrm{C}^1_{\mathrm{prd}}$ (piecewise rd-continuously Δ-differentiable) on $[a,b]_{\mathbb{T}}$ and $u \in \mathrm{C}_{\mathrm{prd}}$ (piecewise rd-continuous) on $[a,\rho(b)]_{\mathbb{T}}$.

The functionals $\mathcal{F}_i$, $\mathcal{G}_i$, $\mathcal{H}_i$ are *nonnegative* if they take nonnegative values on all corresponding admissible pairs (x,u), while the functionals $\mathcal{F}_i$, $\mathcal{G}_i$, $\mathcal{H}_i$ are *positive* if they take positive values on all corresponding admissible (x,u) with $x \not\equiv 0$

Remark 2.1. The boundary conditions for jointly varying endpoints in (2.3) are in some literature in the form

$$\begin{pmatrix} -x(a) \\ x(b) \end{pmatrix} \in \operatorname{Im} R^T,$$

for example in [14, 18, 32, 37, 39]. However, a simple transformation can be used to obtain the boundary conditions in the form (2.3).

3 Conjoined Bases

We assume that the coefficients in systems (S_i), $i \in \{c,d,t\}$, satisfy the following standing hypotheses:

- for system (S_c), the matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot) \in \mathbb{R}^{n\times n}$ belong to C_{p} on $[a,b]$, and $B(t)$ and $C(t)$ are symmetric on $[a,b]$,

- for system (S_d), the matrices $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k \in \mathbb{R}^{n\times n}$, $k \in [0,N]_{\mathbb{N}}$, are such that the $2n\times 2n$ matrix $\mathcal{S}_k := \left(\begin{smallmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{smallmatrix}\right)$ is symplectic, i.e.,

$$\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J} \quad \text{for all } k \in [0,N]_{\mathbb{N}}, \text{ where } \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

- for system (S_t), the matrices $\mathcal{A}(\cdot)$, $\mathcal{B}(\cdot)$, $\mathcal{C}(\cdot)$, $\mathcal{D}(\cdot) \in \mathbb{R}^{n\times n}$ belong to $\mathrm{C}_{\mathrm{prd}}$ on $[a,\rho(b)]_{\mathbb{T}}$, and the $2n\times 2n$ matrix $\mathcal{S}(t) := \left(\begin{smallmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{smallmatrix}\right)$ satisfies

$$\mathcal{S}^T(t)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\,\mathcal{S}^T(t)\,\mathcal{J}\mathcal{S}(t) = 0 \quad \text{for all } t \in [a,\rho(b)]_{\mathbb{T}}.$$

The above hypotheses imply that the solutions of systems (S_i) globally exist and are uniquely determined by the initial conditions at any given point of the considered interval.

A matrix solution (X, U) of (S_i) is a *conjoined basis* if the matrix $X^T(t)\,U(t)$ is symmetric and $\operatorname{rank}(X^T(t), U^T(t)) = n$ at some (and hence for all) t. Given a separated boundary conditions as in (2.2), a *natural conjoined basis* (X_a, U_a) of (S_i) is determined by the initial conditions

$$X_a(a) = R_a, \quad R_a^T U_a(a) = R_a^T\, \Gamma_a\, R_a.$$

Given the zero boundary conditions as in (2.1), we shall use the *principal solution* $(\hat{X}, \hat{U})$ of system (S_i) determined by the initial conditions

$$\hat{X}(a) = 0, \quad \hat{U}(a) = I, \tag{3.1}$$

which is in this context a natural conjoined basis. For the jointly varying endpoints we shall use the principal solution $(\hat{X}, \hat{U})$ of (S_i) as well as another solution $(\bar{X}, \bar{U})$ of (S_i) given by the initial conditions

$$\bar{X}(a) = I, \quad \bar{U}(a) = 0. \tag{3.2}$$

Note that $(\bar{X}, \bar{U})$ and $(\hat{X}, \hat{U})$ are normalized conjoined bases of (S_i) according to the standard terminology, i.e., $\bar{X}^T\hat{U} - \bar{U}^T\hat{X} = I$. For the case of jointly varying endpoints we will also need the following $2n \times 2n$ matrices

$$X_*(t) := \begin{pmatrix} 0 & I \\ \hat{X}(t) & \bar{X}(t) \end{pmatrix}, \quad U_*(t) := \begin{pmatrix} -I & 0 \\ \hat{U}(t) & \bar{U}(t) \end{pmatrix}. \tag{3.3}$$

Note that in some papers the authors use instead of $(\bar{X}, \bar{U})$ an alternative conjoined basis $(\tilde{X}, \tilde{U})$ which is given by $\tilde{X}(a) = -I$ and $\tilde{U}(a) = 0$, see e.g., [32, 37, 39]. However, the present approach via $(\bar{X}, \bar{U})$ is more convenient especially when dealing with Riccati matrix equations [20, 34, 45].

4 Continuous Time Theory

In this section we consider the continuous time linear Hamiltonian system (S_c) and the quadratic functionals $\mathcal{F}_c$, $\mathcal{G}_c$, and $\mathcal{H}_c$ defined in (1.1), $(2.4)_c$, and $(2.5)_c$, respectively. We present two sets of results, one set under a controllability assumption and one set without controllability. These results basically come from [37, 39, 41, 48]. For an easy comparison we first present the results pertaining the nonnegativity for zero, separated, and jointly varying endpoints, and then the corresponding result pertaining the positivity.

4.1 Continuous Time Theory under Controllability

Recall that the pair (A,B) is *controllable* (or completely controllable or identically normal) if $u' = -A^T(t)\,u$ and $B(t)\,u = 0$ on some nondegenerate subinterval $\mathcal{I}$ of $[a,b]$ always implies that $u(\cdot) \equiv 0$ on $\mathcal{I}$. If (X,U) is a conjoined basis of system (S$_c$), then a point $t_0 \in [a,b]$ where $X(t_0)$ is singular is called a *focal point* of (X,U). It is well known that under $B(t) \geq 0$ on $[a,b]$, the pair (A,B) is controllable if and only if the focal points of every conjoined basis of (S$_c$) are isolated in $[a,b]$, see [37, Theorem 4.1.3].

Next we present the results on the nonnegativity. Recall that $(\hat{X},\hat{U})$ and (X_a,U_a) are the principal solution and the natural conjoined basis of system (S$_c$), and (X_*,U_*) is defined by (3.3).

Theorem 4.1 (Nonnegativity, zero endpoints). *Assume that (A,B) is controllable. Then the quadratic functional $\mathcal{F}_c$ in* (1.1) *is nonnegative if and only if*

(*i*) $B(t) \geq 0$ *on* $[a,b]$,

(*ii*) *the matrix $\hat{X}(t)$ is invertible for all $t \in (a,b)$.*

Proof. See e.g., [37, Remark 2.4.2]. □

Theorem 4.2 (Nonnegativity, separated endpoints). *Assume that (A,B) is controllable. Then the quadratic functional $\mathcal{G}_c$ in* (2.4)$_c$ *is nonnegative if and only if*

(*i*) $B(t) \geq 0$ *on* $[a,b]$,

(*ii*) *the matrix $X_a(t)$ is invertible for all $t \in (a,b)$,*

(*iii*) *the matrix* $\Gamma_b + U_a(b)\,X_a^{\dagger}(b) \geq 0$ *on* $\operatorname{Im} R_b$,

(*iv*) $\operatorname{Im} R_b \subseteq \operatorname{Im} X_a(b)$.

Proof. See [48, Theorem 6.3], [39, Remark 2(iii)], and also [37, Remark 2.4.2]. □

Theorem 4.3 (Nonnegativity, jointly varying endpoints). *Assume that (A,B) is controllable. Then the quadratic functional $\mathcal{H}_c$ in* (2.5)$_c$ *is nonnegative if and only if*

(*i*) $B(t) \geq 0$ *on* $[a,b]$,

(*ii*) *the matrix $\hat{X}(t)$ is invertible for all $t \in (a,b)$,*

(*iii*) *the matrix* $\Gamma + U_*(b)\,X_*^{\dagger}(b) \geq 0$ *on* $\operatorname{Im} R$,

(*iv*) $\operatorname{Im} R \subseteq \operatorname{Im} X_*(b)$.

Proof. The sufficiency of conditions (i)–(iv) follows from [39, Remark 5] and from (a more general) Theorem 4.10. The necessity of conditions (i)–(iv) can be established by a similar technique as in the proof of [48, Theorem 6.3]. □

The corresponding results for the positivity have the following form.

Theorem 4.4 (Positivity, zero endpoints). *Assume that* (A, B) *is controllable. Then the quadratic functional* $\mathcal{F}_c$ *in* (1.1) *is positive if and only if*

(*i*) $B(t) \geq 0$ *for all* $t \in [a, b]$,

(*ii*) *the matrix* $\hat{X}(t)$ *is invertible for all* $t \in (a, b]$.

Proof. See [43, pp. 284–285] or a special case of [37, Theorem 2.4.2], which is in fact Theorem 4.5 below. □

Theorem 4.5 (Positivity, separated endpoints). *Assume that* (A, B) *is controllable. Then the quadratic functional* $\mathcal{G}_c$ *in* (2.4)$_c$ *is positive if and only if*

(*i*) $B(t) \geq 0$ *for all* $t \in [a, b]$,

(*ii*) *the matrix* $X_a(t)$ *is invertible for all* $t \in (a, b]$,

(*iii*) *the matrix* $\Gamma_b + U_a(b)\, X_a^{-1}(b) > 0$ *on* $\operatorname{Im} R_b$.

Proof. See [37, Theorem 2.4.2] or [48, Theorem 6.2]. □

Theorem 4.6 (Positivity, jointly varying endpoints). *Assume that* (A, B) *is controllable. Then the quadratic functional* $\mathcal{H}_c$ *in* (2.4)$_c$ *is positive if and only if*

(*i*) $B(t) \geq 0$ *for all* $t \in [a, b]$,

(*ii*) *the matrix* $\hat{X}(t)$ *is invertible for all* $t \in (a, b]$,

(*iii*) *the matrix* $\Gamma + U_*(b)\, X_*^{-1}(b) > 0$ *on* $\operatorname{Im} R$.

Proof. See [37, Theorem 2.4.1]. □

Remark 4.7. (i) The invertibility condition (ii) in Theorems 4.1–4.3 means that the principal solution $(\hat{X}, \hat{U})$ or the natural conjoined basis (X_a, U_a) has *no focal points* in the interval (a, b). The invertibility condition (ii) in Theorems 4.4–4.6 means that $(\hat{X}, \hat{U})$ or (X_a, U_a) has *no focal points* in $(a, b]$.

(ii) As it is noted in [37, Remark 2.4.2], the controllability assumption in Theorems 4.2 and 4.5 can be dropped if the matrix R_a is invertible, i.e., when the initial endpoint is free. In this case the matrix $X_a(t)$ is invertible on $[a, b)$ for the nonnegativity, resp. on $[a, b]$ for the positivity.

(iii) Note that the statement of Theorem 4.3 is new.

(iv) The gap between the nonnegativity and positivity in the corresponding results in Theorems 4.1 and 4.4, Theorems 4.2 and 4.5, and Theorems 4.3 and 4.6 is as small as possible. Namely, the corresponding invertibility conditions (ii) differ only at $t = b$, and the final inequalities in (iii) are strict in the case of positivity.

(v) The invertibility of the matrix $X_*(b)$ in Theorem 4.6 follows (or is in fact equivalent with) the invertibility of the matrix $\hat{X}(b)$, as can be easily seen from its definition in (3.3).

4.2 Continuous Time Theory without Controllability

If the controllability assumption is removed, conjoined bases (X,U) of (S$_c$) may have $X(t)$ singular on a nondegenerate interval. In this case the inverse X^{-1} should be replaced by the Moore–Penrose pseudoinverse $X^{\dagger}$, see [3, 4, 39] for its properties. In particular, the result of [39, Lemma 6] shows that $X^{\dagger} \in \mathrm{C}^1_{\mathrm{p}}$ provided $X \in \mathrm{C}^1_{\mathrm{p}}$ and the kernel of $X(\cdot)$ is constant. In addition, if (X,U) is a conjoined basis of (S$_c$), then $B(t) \geq 0$ on $[a,b]$ implies that the kernel of $X(\cdot)$ is *piecewise constant* on $[a,b]$, by [39, Theorem 3]. In this case the invertibility conditions (ii) in Theorems 4.1, 4.2, and 4.3 (nonnegativity) are replaced by a certain *image condition* involving admissible pairs (x,u), while the invertibility conditions (ii) in Theorems 4.4, 4.5, and 4.6 (positivity) are replaced by a certain *kernel condition* for $\hat{X}(\cdot)$ or $X_a(\cdot)$, see the results below.

Recall that the conjoined bases $(\hat{X},\hat{U})$ and $(\bar{X},\bar{U})$ of (S$_c$) and (X_*,U_*) are given by (3.1), (3.2), and (3.3), respectively.

Theorem 4.8 (Nonnegativity, zero endpoints). *The quadratic functional $\mathcal{F}_c$ in* (1.1) *is nonnegative if and only if*

(*i*) $B(t) \geq 0$ *for all* $t \in [a,b]$,

(*ii*) *the image condition* $x(t) \in \operatorname{Im} \hat{X}(t)$ *holds for all* $t \in [a,b]$ *and for all admissible* (x,u) *satisfying zero endpoints* (2.1).

Proof. See the special case of [39, Theorem 2], which is Theorem 4.9 below. □

Theorem 4.9 (Nonnegativity, separated endpoints). *The quadratic functional $\mathcal{G}_c$ in* (2.4)$_c$ *is nonnegative if and only if*

(*i*) $B(t) \geq 0$ *for all* $t \in [a,b]$,

(*ii*) *the image condition* $x(t) \in \operatorname{Im} X_a(t)$ *holds for all* $t \in [a,b]$ *and for all admissible* (x,u) *satisfying separated endpoints* (2.2),

(*iii*) *the matrix* $\Gamma_b + U_a(b)\, X_a^{\dagger}(b) \geq 0$ *on* $\operatorname{Im} R_b \cap \operatorname{Im} X_a(b)$.

Proof. See [39, Theorem 2]. □

Theorem 4.10 (Nonnegativity, jointly varying endpoints). *The quadratic functional $\mathcal{H}_c$ in* (2.5)$_c$ *is nonnegative if and only if*

(*i*) $B(t) \geq 0$ *for all* $t \in [a,b]$,

(*ii*) *the image condition* $x(t) - \bar{X}(t)\, x(a) \in \operatorname{Im} \hat{X}(t)$ *holds for all* $t \in [a,b]$ *and for all admissible* (x,u) *satisfying jointly varying endpoints* (2.3),

(*iii*) *the matrix* $\Gamma + U_*(b)\, X_*^{\dagger}(b) \geq 0$ *on* $\operatorname{Im} R \cap \operatorname{Im} X_*(b)$.

Proof. See [39, Corollary 3]. □

The corresponding results for the positivity now follow.

Theorem 4.11 (Positivity, zero endpoints). *The quadratic functional* $\mathcal{F}_c$ *in* (1.1) *is positive if and only if*

(*i*) $B(t) \geq 0$ *for all* $t \in [a, b]$,

(*ii*) $\operatorname{Ker} \hat{X}(t) \subseteq \operatorname{Ker} \hat{X}(\tau)$ *for all* $t, \tau \in [a, b]$, $\tau \leq t$.

Proof. See a special case of [39, Theorem 1], which is Theorem 4.12 below. □

Theorem 4.12 (Positivity, separated endpoints). *The quadratic functional* $\mathcal{G}_c$ *in* (2.4)$_c$ *is positive if and only if*

(*i*) $B(t) \geq 0$ *for all* $t \in [a, b]$,

(*ii*) $\operatorname{Ker} X_a(t) \subseteq \operatorname{Ker} X_a(\tau)$ *for all* $t, \tau \in [a, b]$, $\tau \leq t$,

(*iii*) *the matrix* $\Gamma_b + U_a(b)\, X_a^{\dagger}(b) > 0$ *on* $\operatorname{Im} R_b \cap \operatorname{Im} X_a(b)$.

Proof. See [39, Theorem 1]. □

Theorem 4.13 (Positivity, jointly varying endpoints). *The quadratic functional* $\mathcal{H}_c$ *in* (2.4)$_c$ *is positive if and only if*

(*i*) $B(t) \geq 0$ *for all* $t \in [a, b]$,

(*ii*) $\operatorname{Ker} \hat{X}(t) \subseteq \operatorname{Ker} \hat{X}(\tau)$ *for all* $t, \tau \in [a, b]$, $\tau \leq t$,

(*iii*) *the matrix* $\Gamma + U_*(b)\, X_*^{\dagger}(b) > 0$ *on* $\operatorname{Im} R \cap \operatorname{Im} X_*(b)$.

Proof. See [39, Corollary 2]. □

Remark 4.14. (i) The matrix $\hat{X}(\cdot)$ or $X_a(\cdot)$ in Theorems 4.8–4.13 can be singular on the whole interval $[a, b]$.

(ii) The kernel condition (ii) in Theorems 4.11–4.13 means that the principal solution $(\hat{X}, \hat{U})$ or the natural conjoined basis (X_a, U_a) has *no generalized focal points* in $(a, b]$, see [39, Definition 1].

(iii) Note that the kernel condition (ii) in Theorems 4.11–4.13 implies the corresponding image condition (ii) in Theorems 4.8–4.10, respectively, but not vice versa. This follows from [39, Corollary 4].

(iv) The gap between the nonnegativity and positivity in the corresponding results in Theorems 4.8 and 4.11, Theorems 4.9 and 4.12, and Theorems 4.10 and 4.13 is as small as possible. Namely, the final inequalities in (iii) are strict in the case of positivity.

(v) Under the controllability assumption, the image condition (ii) in Theorems 4.8–4.10 is equivalent to the invertibility condition (ii) and inclusion (iv) in Theorems 4.1–4.3, respectively. This is discussed in [39, Remark 2(iii)].

5 Discrete Time Theory

In this section we present discrete time results which are parallel to the continuous time results from Section 5. Hence, we consider the discrete symplectic system (S$_d$) and the quadratic functionals $\mathcal{F}_d$, $\mathcal{G}_d$, and $\mathcal{H}_d$ defined in (1.2), (2.4)$_d$, and (2.5)$_d$, respectively.

In the discrete time setting all the points are naturally isolated. Therefore it is expected that in this case the controllability assumption shall not be needed. In this respect the discrete time results in this section correspond to the results in Subsection 4.2. However, it is interesting to note that the continuous time results without the controllability assumption (in Subsection 4.2) were *motivated* by the discrete time results from this section.

In order to avoid double indices, we denote in this section the natural conjoined basis (X_a, U_a) of system (S$_d$) by (X, U), and the matrices (X_*, U_*) defined in (3.3) by (X^*, U^*). For the natural conjoined basis (X, U) of (S$_d$) we define the following matrices

$$P_k := X_k X_{k+1}^\dagger \mathcal{B}_k, \quad M_k := (I - X_{k+1} X_{k+1}^\dagger)\, \mathcal{B}_k, \quad T_k := I - M_k^\dagger M_k.$$

For the case of zero and jointly varying endpoints these matrices are defined through the principal solution $(\hat{X}, \hat{U})$ of (S$_d$) and in this case they will also be denoted with hat, i.e.,

$$\hat{P}_k := \hat{X}_k \hat{X}_{k+1}^\dagger \mathcal{B}_k, \quad \hat{M}_k := (I - \hat{X}_{k+1} \hat{X}_{k+1}^\dagger)\, \mathcal{B}_k, \quad \hat{T}_k := I - \hat{M}_k^\dagger \hat{M}_k.$$

Since in the discrete case the time interval is $[0, N+1]_{\mathbb{N}}$, i.e., the endpoints are $a = 0$ and $b = N+1$, we will denote in this section the matrices Γ_a, Γ_b, R_a, and R_b by Γ_0, Γ_{N+1}, R_0, and R_{N+1}, respectively.

First we state the results for the nonnegativity. Recall that the conjoined basis $(\bar{X}, \bar{U})$ of (S$_d$) and $(X^*, U^*) := (X_*, U_*)$ are given by (3.2) and (3.3).

Theorem 5.1 (Nonnegativity, zero endpoints). *The quadratic functional $\mathcal{F}_d$ in* (1.2) *is nonnegative if and only if*

(*i*) $\hat{T}_k \hat{P}_k \hat{T}_k \geq 0$ *for all* $k \in [0, N]_{\mathbb{N}}$,

(*ii*) *the image condition* $x_k \in \operatorname{Im} \hat{X}_k$ *holds for all* $k \in [0, N+1]_{\mathbb{N}}$ *and for all admissible* (x, u) *satisfying zero endpoints* (2.1).

Proof. See [8, Theorem 1.1]. □

Theorem 5.2 (Nonnegativity, separated endpoints). *The quadratic functional $\mathcal{G}_d$ in* (2.4)$_d$ *is nonnegative if and only if*

(*i*) $T_k P_k T_k \geq 0$ *for all* $k \in [0, N]_{\mathbb{N}}$,

(*ii*) *the image condition* $x_k \in \operatorname{Im} X_k$ *holds for all* $k \in [0, N+1]_{\mathbb{N}}$ *and for all admissible* (x, u) *satisfying separated endpoints* (2.2), (*iii*) *the matrix* $\Gamma_{N+1} + U_{N+1}X_{N+1}^{\dagger} \geq 0$ *on* $\operatorname{Im} R_{N+1} \cap \operatorname{Im} X_{N+1}$.

Proof. See [7, Theorem 2]. □

Theorem 5.3 (Nonnegativity, jointly varying endpoints). *The quadratic functional* $\mathcal{H}_d$ *in* (2.5)$_d$ *is nonnegative if and only if*

(*i*) $\hat{T}_k\hat{P}_k\hat{T}_k \geq 0$ *for all* $k \in [0, N]_{\mathbb{N}}$,

(*ii*) *the image condition* $x_k - \bar{X}_k x_0 \in \operatorname{Im} \hat{X}_k$ *holds for all* $k \in [0, N+1]_{\mathbb{N}}$ *and for all admissible* (x, u) *satisfying jointly varying endpoints* (2.3),

(*iii*) *the matrix* $\Gamma + U_{N+1}^*(X_{N+1}^*)^{\dagger} \geq 0$ *on* $\operatorname{Im} R \cap \operatorname{Im} X_{N+1}^*$.

Proof. See [19, Theorem 2]. □

Now we state the corresponding results for the positivity.

Theorem 5.4 (Positivity, zero endpoints). *The quadratic functional* $\mathcal{F}_d$ *in* (1.2) *is positive if and only if*

(*i*) $\hat{P}_k \geq 0$ *for all* $k \in [0, N]_{\mathbb{N}}$,

(*ii*) $\operatorname{Ker} \hat{X}_{k+1} \subseteq \operatorname{Ker} \hat{X}_k$ *for all* $k \in [0, N]_{\mathbb{N}}$.

Proof. See [6, Theorem 1]. □

Theorem 5.5 (Positivity, separated endpoints). *The quadratic functional* $\mathcal{G}_d$ *in* (2.4)$_d$ *is positive if and only if*

(*i*) $P_k \geq 0$ *for all* $k \in [0, N]_{\mathbb{N}}$,

(*ii*) $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ *for all* $k \in [0, N]_{\mathbb{N}}$,

(*iii*) *the matrix* $\Gamma_{N+1} + U_{N+1}X_{N+1}^{\dagger} > 0$ *on* $\operatorname{Im} R_{N+1} \cap \operatorname{Im} X_{N+1}$.

Proof. See [25, Theorem 5]. □

Theorem 5.6 (Positivity, jointly varying endpoints). *The quadratic functional* $\mathcal{H}_d$ *in* (2.4)$_d$ *is positive if and only if*

(*i*) $\hat{P} \geq 0$ *for all* $k \in [0, N]_{\mathbb{N}}$,

(*ii*) $\operatorname{Ker} \hat{X}_{k+1} \subseteq \operatorname{Ker} \hat{X}_k$ *for all* $k \in [0, N]_{\mathbb{N}}$,

(*iii*) *the matrix* $\Gamma + U_{N+1}^*(X_{N+1}^*)^{\dagger} > 0$ *on* $\operatorname{Im} R \cap \operatorname{Im} X_{N+1}^*$.

Proof. See [25, Theorem 10]. □

Remark 5.7. (i) Historically, the results on the positivity in Theorems 5.4–5.6 were derived much earlier than the results for the nonnegativity in Theorems 5.1–5.3.

(ii) All the results in this section are nicely reviewed in [45].

(iii) The P-condition (i) together with the kernel condition (ii) in Theorems 5.4–5.6 mean that the principal solution $(\hat{X}, \hat{U})$ or the natural conjoined basis (X, U) has *no focal points* in $(0, N+1]_{\mathbb{N}}$, see [6, Definition 3]. This is different from the continuous time where the notion of no focal points does not involve the Legendre condition $B(t) \geq 0$ on $[a, b]$, see Remark 4.7(i) and Remark 4.14(ii).

(iv) The matrix $T_k P_k T_k$ is always symmetric, see [38, Lemma 1]. Furthermore, whenever $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ holds, then P_k is symmetric, see [6, Lemma 3].

(v) The kernel condition (ii) in Theorems 5.4–5.6 implies the corresponding image condition (ii) in Theorems 5.1–5.3, respectively, but not vice versa. This is contained in [6, Remark 1(v)].

(vi) We can now see how the continuous image condition (ii) in Theorems 4.8–4.10 was motivated by the discrete image condition (ii) from Theorems 5.1–5.3. Also, the continuous kernel condition (ii) in Theorems 4.11–4.13 was motivated by the discrete kernel condition (ii) from Theorems 5.4–5.6, even though the latter was discovered some seven years earlier.

6 Time Scale Theory

In this section we present results unifying the corresponding statements from the continuous and discrete time theories, as well as extending them to arbitrary time scales. Hence, we consider the time scale symplectic system (S_t) and the quadratic functionals $\mathcal{F}_t$, $\mathcal{G}_t$, and $\mathcal{H}_t$ defined in (1.3), (2.4)$_t$, and (2.5)$_t$, respectively. We state these time scale results without any controllability assumption, although historically some of them were obtained earlier under certain controllability on time scale intervals $[a, s]_{\mathbb{T}}$ where s is a dense point, see [14, 15, 18].

Similarly to the continuous time case we now have that the kernel of $X(\cdot)$ is *piecewise constant* for any conjoined basis (X, U) of (S_t), provided the functional $\mathcal{F}_t$ is nonnegative (which is, however, a stronger condition than the Legendre condition $B(t) \geq 0$ on $[a, b]$ required in the continuous time case), see [29, Theorem 8.1]. As in the discrete case, we shall use for time scales the Moore–Penrose generalized inverse $X^{\dagger}$. Analogously to the continuous case, the differentiability of $X^{\dagger}$ is proven in [29, Lemma 2.1] under the assumption that $X \in \mathrm{C}^1_{\mathrm{prd}}$ and the kernel of $X(\cdot)$ is constant.

For the natural conjoined basis (X_a, U_a) of (S_t) we define the following matrices

$$P_a(t) := X_a(t)[X_a^{\sigma}(t)]^{\dagger}\mathcal{B}(t), \quad M_a(t) := \{[I - X_a^{\sigma}(X_a^{\sigma})^{\dagger}]\mathcal{B}\}(t),$$
$$T_a(t) := I - M_a^{\dagger}(t)\, M_a(t).$$

Here $f^{\sigma}(t)$ stands for the composition $f\big(\sigma(t)\big)$. For the case of zero and jointly varying endpoints these matrices are defined through the principal solution $(\hat{X}, \hat{U})$ of (S_t), and in this case we set

$$\hat{P}(t) := \hat{X}(t)[\hat{X}^\sigma(t)]^\dagger \mathcal{B}(t),\ \hat{M}(t) := \{[I - \hat{X}^\sigma(\hat{X}^\sigma)^\dagger]\mathcal{B}\}(t),\ \hat{T}(t) := I - \hat{M}^\dagger(t)\,\hat{M}(t).$$

The main results on the nonnegativity are as follows. Recall that the conjoined basis $(\bar{X}, \bar{U})$ of (S_t) and (X_*, U_*) are given by (3.2) and (3.3).

Theorem 6.1 (Nonnegativity, zero endpoints). *The quadratic functional $\mathcal{F}_t$ in* (1.3) *is nonnegative if and only if $\hat{X}(\cdot)$ has piecewise constant kernel on $[a, b]_{\mathbb{T}}$ and*

(*i*) $\hat{T}(t)\,\hat{P}(t)\,\hat{T}(t) \geq 0$ *for all $k \in [a, \rho(b)]_{\mathbb{T}}$,*

(*ii*) *the image condition $x(t) \in \operatorname{Im} \hat{X}(t)$ holds for all $t \in [a, b]_{\mathbb{T}}$ and for all admissible (x, u) satisfying zero endpoints* (2.1).

Proof. See a special case of [29, Theorem 4.2], which is Theorem 6.2 below. □

Theorem 6.2 (Nonnegativity, separated endpoints). *The quadratic functional $\mathcal{G}_t$ in* (2.4)$_t$ *is nonnegative if and only if $X_a(\cdot)$ has piecewise constant kernel on $[a, b]_{\mathbb{T}}$ and*

(*i*) $T_a(t)\,P_a(t)\,T_a(t) \geq 0$ *for all $k \in [a, \rho(b)]_{\mathbb{T}}$,*

(*ii*) *the image condition $x(t) \in \operatorname{Im} X_a(t)$ holds for all $t \in [a, b]_{\mathbb{T}}$ and for all admissible (x, u) satisfying separated endpoints* (2.2),

(*iii*) *the matrix $\Gamma_b + U_a(b)\,X_a^\dagger(b) \geq 0$ on $\operatorname{Im} R_b \cap \operatorname{Im} X_a(b)$.*

Proof. See [29, Theorem 4.2]. □

Theorem 6.3 (Nonnegativity, jointly varying endpoints). *The quadratic functional $\mathcal{H}_t$ in* (2.5)$_t$ *is nonnegative if and only if $\hat{X}(\cdot)$ has piecewise constant kernel on $[a, b]_{\mathbb{T}}$ and*

(*i*) $\hat{T}(t)\,\hat{P}(t)\,\hat{T}(t) \geq 0$ *for all $k \in [a, \rho(b)]_{\mathbb{T}}$,*

(*ii*) *the image condition $x(t) - \bar{X}(t)\,x(a) \in \operatorname{Im} \hat{X}(t)$ holds for all $t \in [a, b]_{\mathbb{T}}$ and for all admissible (x, u) satisfying jointly varying endpoints* (2.3),

(*iii*) *the matrix $\Gamma + U_*(b)\,X_*^\dagger(b) \geq 0$ on $\operatorname{Im} R \cap \operatorname{Im} X_*(b)$.*

Proof. See [32, Theorem 4.2]. □

The results for the positivity are presented in the following.

Theorem 6.4 (Positivity, zero endpoints). *The quadratic functional $\mathcal{F}_t$ in* (1.3) *is positive if and only if*

(*i*) $\hat{P}(t) \geq 0$ *for all $k \in [a, \rho(b)]_{\mathbb{T}}$,* (*ii*) $\operatorname{Ker} \hat{X}(t) \subseteq \operatorname{Ker} \hat{X}(\tau)$ *for all* $t, \tau \in [a,b]_{\mathbb{T}}$, $\tau \le t$.

Proof. See a special case of [29, Theorem 4.1], which is Theorem 6.5 below. □

Theorem 6.5 (Positivity, separated endpoints). *The quadratic functional* $\mathcal{G}_t$ *in* (2.4)$_t$ *is positive if and only if*

(*i*) $P_a(t) \ge 0$ *for all* $k \in [a, \rho(b)]_{\mathbb{T}}$,

(*ii*) $\operatorname{Ker} X_a(t) \subseteq \operatorname{Ker} X_a(\tau)$ *for all* $t, \tau \in [a,b]_{\mathbb{T}}$, $\tau \le t$,

(*iii*) *the matrix* $\Gamma_b + U_a(b)\, X_a^{\dagger}(b) > 0$ *on* $\operatorname{Im} R_b \cap \operatorname{Im} X_a(b)$.

Proof. See [29, Theorem 4.1]. □

Theorem 6.6 (Positivity, jointly varying endpoints). *The quadratic functional* $\mathcal{H}_t$ *in* (2.5)$_t$ *is positive if and only if*

(*i*) $\hat{P}(t) \ge 0$ *for all* $k \in [a,b]_{\mathbb{T}}$,

(*ii*) $\operatorname{Ker} \hat{X}(t) \subseteq \operatorname{Ker} \hat{X}(\tau)$ *for all* $t, \tau \in [a,b]_{\mathbb{T}}$, $\tau \le t$,

(*iii*) *the matrix* $\Gamma + U_*(b)\, X_*^{\dagger}(b) > 0$ *on* $\operatorname{Im} R \cap \operatorname{Im} X_*(b)$.

Proof. See [32, Theorem 4.1]. □

Remark 6.7. (i) It is very easy to see how the time scale results unify and extend the corresponding results from the continuous and discrete cases in Sections 4 and 5.

(ii) The P-condition (i) together with the kernel condition (ii) in Theorems 6.4–6.6 mean that the principal solution $(\hat{X}, \hat{U})$ or the natural conjoined basis (X_a, U_a) has *no generalized focal points* in $(a,b]_{\mathbb{T}}$, see [29, Definition 4.1]. This notion now unifies the continuous and discrete time notions discussed in Remark 4.14(ii) and Remark 5.7(iii).

(iii) The P-condition (i) in Theorems 6.1–6.6 can be formulated in an equivalent form in terms of the symmetric matrix

$$\mathcal{P}(t) := \big\{\mathcal{B} + \mu\, [\mathcal{D}^T - \mathcal{B}^T U^{\sigma} (X^{\sigma})^{\dagger}]\, \mathcal{B}\big\}(t),$$

as it is discussed in [36, Remark 3.5]. The above matrix $\mathcal{P}(t)$ reduces to the matrix $\mathcal{B}(t) = B(t)$ in the continuous time case, so that the P-condition (i) in Theorems 6.1–6.6 corresponds to the Legendre condition (i) in Theorems 4.1–4.6 and Theorems 4.8–4.13.

(iv) The image condition (ii) in Theorems 6.1–6.3 unifies the continuous and discrete image conditions (ii) from Theorems 4.8–4.10 and Theorems 5.1–5.3. The kernel condition (ii) in Theorems 6.4–6.6 unifies the continuous and discrete kernel conditions (ii) from Theorems 4.11–4.13 and Theorems 5.4–5.6.

(v) The kernel condition (ii) in Theorems 6.4–6.6 implies the corresponding image condition (ii) in Theorems 6.1–6.3, respectively, but not vice versa. This is established in [29, Proposition 5.2]. The kernel condition (ii) in Theorems 6.4–6.6 also implies that the kernel of $\hat{X}(\cdot)$ or $X_a(\cdot)$ is piecewise constant on $[a,b]_{\mathbb{T}}$, since the kernel can actually decrease in $[a,b]_{\mathbb{T}}$ at most n times.

Remark 6.8. All the characterizations of the nonnegativity and positivity of quadratic functionals in this paper may contain another item, namely the statements regarding the solvability of the corresponding continuous, discrete, and time scale Riccati matrix equations. An overview of such results can be found in [34].

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