Stability and Hopf Bifurcation for a Discrete Disease Spreading Model in Complex Networks

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Abstract

In this paper, a discrete disease spreading model in complex networks with time delay is investigated. The stability of positive equilibrium and existence of Hopf bifurcation are explored by analyzing the associated characteristic equation. Furthermore, a numerical example is given.

1 Introduction

Recently, to describe the effect of the new link-adding probability p in the topological transition of the N-W small world network model, Li proposed a general nonlinear model of disease spreading in [5] as

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = 1 + 2pV(t - \tau) - \mu(1 + 2p)V^2(t - \tau), \tag{1.1}$$

where V is the total influenced volume, μ is a measure of nonlinear interactions in the network and p is the probability of add linkages between pairs of randomly chosen nodes. More details can be seen in [5].

For system (1.1), there is a unique positive equilibrium

$$V^* = \frac{p + \sqrt{p^2 + \mu(1 + 2p)}}{\mu(1 + 2p)}.$$

Stability of the positive equilibrium and Hopf bifurcation behavior were studied when μ and τ were considered as the bifurcation parameters in [5] and [2] respectively. Further, bifurcation control using a time-delayed feedback controller for the complex network model (1.1) was considered in [1]. And other related models can be found in [4, 6].

It has been shown that the dynamics of difference equations can be different from the corresponding differential equations. And it is interesting to investigate the preservability of bifurcation under the numerical discretization. In [2], the existence and properties of Hopf bifurcation for system (1.1) were established. So in this paper, we mainly consider the discrete analogue of system (1.1) obtained by Euler method. By using the bifurcation theory of discrete system in [3], we obtain that the Hopf bifurcation can be preserved under discretization by Euler method.

The remainder of this paper is organized as follows. Stability of positive equilibrium of the corresponding discrete model is investigated in Section 2. In Section 3, formulas for determining the direction of Hopf bifurcation and stability of periodic solutions are derived. Finally, some numerical simulations are performed to illustrate the theoretical results.

2 Stability Analysis

Let $y(t) = V(\tau t)$. Then system (1.1) can be rewritten as

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \tau + 2p\tau y(t-1) - \tau\mu(1+2p)y^2(t-1). \tag{2.1}$$

Next, we consider the step size of the form $h = \dfrac{1}{m}$, where m is a positive integer. Applying Euler's method to this equation yields the difference equation

$$y_{n+1} = y_n + h\tau + 2ph\tau y_{n-m} - h\tau\mu(1+2p)y_{n-m}^2, \tag{2.2}$$

and equation (2.2) has the unique positive equilibrium $y^* = \dfrac{p + \sqrt{p^2 + \mu(1+2p)}}{\mu(1+2p)}$. Set $u_n = y_n - y^*$. Then system (2.2) can be reduced to

$$u_{n+1} = u_n - 2ah\tau u_{n-m} - h\tau\mu(1+2p)u_{n-m}^2,$$

where $a = \sqrt{p^2 + \mu(1+2p)}$. We rewrite the equation as

$$y_{n+1} = y_n - 2ah\tau y_{n-m} - h\tau\mu(1+2p)y_{n-m}^2. \tag{2.3}$$

Denote $Y_n = (y_n, y_{n-1}, \ldots, y_{n-m})^T$. We can rewrite (2.3) in the form

$$Y_{n+1} = F(Y_n, \tau), \tag{2.4}$$

where $F = F(F_0, F_1, \ldots, F_m)^T$ and

$$F_k = \begin{cases} y_n - 2ah\tau y_{n-m} - h\tau\mu(1+2p)y_{n-m}^2 & k = 0, \\ y_{n-k+1} & 1 \le k \le m. \end{cases}$$

The linear part of equation (2.3) is

$$Y_{n+1} = AY_n, \tag{2.5}$$

where $A = \begin{bmatrix} 1 & 0 & \ldots & 0 & -2ah\tau \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$ is an $(m+1) \times (m+1)$ matrix. It is easy to

obtain that the characteristic equation of (2.5) is

$$z^{m+1} - z^m + 2ah\tau = 0. \tag{2.6}$$

Then the stability of the positive equilibrium y^* depends on the distribution of the zeros of equation (2.6). When $\tau = 0$, equation (2.6) can be reduced to $z^{m+1} - z^m = 0$, which has a root $z = 0$ of multiplicity m and a simple root $z = 1$. By direct computation, we can get $\left. \dfrac{\mathrm{d}|z|^2}{\mathrm{d}\tau} \right|_{z=1,\tau=0} = \left[z\dfrac{\mathrm{d}\bar{z}}{\mathrm{d}\tau} + \bar{z}\dfrac{\mathrm{d}z}{\mathrm{d}\tau} \right]_{z=1,\tau=0} = -4ah < 0.$

Proposition 2.1 (see [7]). *Suppose that $\hat{B} \subset \mathbb{R}$ is a bounded, closed, and connected set, $f(\lambda, \tau) = \lambda^m + p_1(\tau)\lambda^{m-1} + p_2(\tau)\lambda^{m-2} + \cdots + p_m(\tau)$ is continuous in $(\lambda, \tau) \in C \times \hat{B}$, τ is a parameter, $\tau \in \hat{B}$. Then as τ varies, the sum of the order of the zeros of $f(\lambda, \tau)$ out of the unit circle $\{ \lambda \in C : |\lambda| > 1 \}$ can change only if a zero appears on or cross the unit circle.*

From Proposition 2.1, we have the following results.

Lemma 2.2. *There exists a $\tau' > 0$ such that for $0 < \tau < \tau'$, all roots of equation (2.6) have modulus less than one.*

Assume that equation (2.6) has a root with modulus one. Let $0 < \omega^* < \pi$ and $e^{i\omega^*}$ be a root of equation (2.6) when $\tau = \tau^*$. Then $e^{i(m+1)\omega^*} - e^{im\omega^*} + 2ah\tau^* = 0$. Separating the real and imaginary parts, we have

$$\begin{cases} \sin(m+1)\omega^* - \sin m\omega^* = 0, \\ \cos(m+1)\omega^* - \cos m\omega^* = -2ah\tau^*, \end{cases} \tag{2.7}$$

and thus, $\cos\omega^* = 1 - 2a^2h^2\tau^{*2}$. Obviously, if h is sufficiently small, then equation (2.6) has roots with modulus one such that

$$\begin{cases} \cos\omega^* = 1 - 2a^2h^2\tau^{*2}, \\ \tau^* = \dfrac{\cos m\omega^* - \cos(m+1)\omega^*}{2ah}, \\ h = \dfrac{1}{m}. \end{cases} \tag{2.8}$$

Lemma 2.3. *If h is sufficiently small and τ^* and ω^* satisfy (2.8), then*

$$d_h = \left.\frac{d|z|^2}{d\tau}\right|_{\tau=\tau^*,\omega=\omega^*} > 0.$$

Proof. From (2.6), we have

$$
\begin{aligned}
d_h &= \left[z\frac{\mathrm{d}\bar{z}}{\mathrm{d}\tau} + \bar{z}\frac{\mathrm{d}z}{\mathrm{d}\tau}\right]\Bigg|_{\tau=\tau^*,\omega=\omega^*} \\
&= \left[\frac{-2ahz}{(m+1)\bar{z}^m - m\bar{z}^{m-1}} + \frac{-2ah\bar{z}}{(m+1)z^m - mz^{m-1}}\right]\Bigg|_{\tau=\tau^*,\omega=\omega^*} \\
&= -4ah\frac{(m+1)\cos(m+1)\omega^* - m\cos m\omega^*}{D},
\end{aligned}
$$

where

$$D = [(m+1)\cos m\omega^* - m\cos(m-1)\omega^*]^2 + [(m+1)\sin m\omega^* - m\sin(m-1)\omega^*]^2.$$

From (2.7), we have $\cos(m+1)\omega^* = -ah\tau$. Then

$$
\begin{aligned}
\cos(m+1)\omega^* + m\cos(m+1)\omega^* - m\cos m\omega^* &= \cos(m+1)\omega^* - 2a\tau \\
&= -a\tau(h+2) < 0.
\end{aligned}
$$

Thus, $d_h > 0$. This completes the proof. □

By Proposition 2.1 and Lemmas 2.2 and 2.3, we have the following result on stability and Hopf bifurcation for difference equation (2.2).

Theorem 2.4. *If the step size h is sufficiently small, then there exists an infinite sequence of values of the time delay parameter $\tau_0 < \tau_1 < \ldots < \tau_j < \ldots$, such that the positive equilibrium y^* of equation (2.2) is asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. Furthermore, Hopf bifurcation occurs at y^* when $\tau = \tau_j$, $j = 0, 1, 2, \ldots$, where τ_j satisfies (2.8).*

3   Direction and Stability of Hopf Bifurcation

In the previous section, we obtained conditions under which a family of periodic solutions bifurcate from the positive equilibrium at the critical value τ. Further, it is novel and interesting to determine the direction, stability and period of these periodic solutions. Following the ideas of Y. A. Kuznetsov [3], we derive the explicit formulae for determining the properties of the Hopf bifurcation at the critical value τ^* by using the normal form and the center manifold theory.

Set $\tau = \tau_0 + \eta$, $\eta \in \mathbb{R}$. Then $\eta = 0$ is a Hopf bifurcation value for equation (2.2). Equation (2.2) at y^* is

$$y_{n+1} = y_n - 2ah\tau_0 y_{n-m} - 2h\mu\tau_0(1 + 2p)y_{n-m}^2. \tag{3.1}$$

So we can write (3.1) as

$$Y_{n+1} = AY_n + \frac{1}{2}B(Y_n, Y_n),$$

where $B(Y_n, Y_n) = (b_0(Y_n, Y_n), 0, \ldots, 0)$ and $b_0(\phi, \psi) = -2h\tau\mu(1 + 2p)\phi_m\psi_m$. Let $q \in \mathbb{C}^{m+1}$ be a complex eigenvector of A corresponding $e^{i\omega^*}$. Then

$$Aq = e^{i\omega^*}, \qquad A\bar{q} = e^{-i\omega^*}\bar{q}. \tag{3.2}$$

We also introduce an adjoint eigenvector $q^* \in \mathbb{C}^{m+1}$ having the properties

$$A^T q^* = e^{-i\omega^*}q^*, \qquad A^T\bar{q}^* = e^{-i\omega^*}\bar{q}^*, \tag{3.3}$$

and satisfying the normalization $\langle q^*, q \rangle = 1$, where $\langle q^*, q \rangle = \sum_{i=0}^{m} \bar{q}_i^* q_i$. By direct computation, we have the following result.

Lemma 3.1. *Let* $q = (q_0, q_1, \ldots, q_m)^T$ *be the eigenvector of* A *corresponding to the eigenvalue* $e^{i\omega^*}$*, and* $q^* = (q_0^*, q_1^*, \ldots, q_m^*)$ *be the eigenvector of* A^T *corresponding to the eigenvalue* $e^{-i\omega^*}$*. Then*

$$q = p(e^{i\omega^*}) = (1, e^{-i\omega^*}, \ldots, e^{-i(m-1)\omega^*}, \alpha e^{-im\omega^*})^T,$$

$$q^* = K(1, \alpha e^{im\omega^*}, \ldots, \alpha e^{2i\omega^*}, e^{i\omega^*})^T,$$

where $\alpha = -2ah\tau$ *and* $K = (1 + m\alpha e^{i(m+1)\omega^*})^{-1}$.

The critical real eigenspace T_{centre} corresponding to $e^{\pm i\omega^*}$ is two dimensional and is spanned by $\{\text{Re}(q), \text{Im}(q)\}$. The real eigenspace T_{stable} corresponding to all eigenvalues of A other than $e^{\pm i\omega^*}$ is $n - 2$ dimensional. All vectors $x \in \mathbb{R}^{m+1}$ can be decomposed as

$$x = \nu q + \bar{\nu}\bar{q} + y,$$

where $\nu \in \mathbb{C}$, $\nu q + \bar{\nu}\bar{q} \in T_{\text{centre}}$ and $y \in T_{\text{stable}}$. The complex variable ν can be viewed as a new coordinate on T_{centre}, so we have

$$\nu = \langle q^*, x \rangle, \quad y = x - \langle q^*, x \rangle q - \langle \bar{q}^*, x \rangle \bar{q}.$$

In this coordinate, the system at $\tau = \tau^*$ has the form

$$\nu \mapsto e^{i\omega^*}\nu + \langle q^*, F(\nu q + \bar{\nu}\bar{q} + y) \rangle,$$

$$y \mapsto Ay + F(\nu q + \bar{\nu}\bar{q} + y) - \langle q^*, F(\nu q + \bar{\nu}\bar{q} + y) \rangle q - \langle \bar{q}^*, F(\nu q + \bar{\nu}\bar{q} + y) \rangle \bar{q}.$$

Using Taylor expansions,

$$
\begin{cases}
\nu \mapsto z\nu + \dfrac{1}{2}g_{20}\nu^2 + g_{11}\nu\bar{\nu} + \dfrac{1}{2}g_{02}\bar{\nu}^2 + \dfrac{1}{2}G_{21}\nu^2\bar{\nu} + \langle G_{10}, y\rangle\nu + \langle G_{01}, y\rangle\bar{\nu}, \\
y \mapsto A_0 y + \dfrac{1}{2}H_{20}\nu^2 + H_{11}\nu\bar{\nu} + \dfrac{1}{2}H_{02}\bar{\nu}^2 + O(|\nu|)^3,
\end{cases}
\tag{3.4}
$$

where $g_{ij} \in C$, $G_{10}, G_{01} \in C^{m+1}$, and

$$
g_{20} = \langle q^*, B(q,q)\rangle, \quad g_{11} = \langle q^*, B(q,\bar{q})\rangle, \quad g_{02} = \langle q^*, B(\bar{q},\bar{q})\rangle,
$$

$$
G_{21} = \langle q^*, C(q,q,\bar{q})\rangle, \langle G_{10}, y\rangle = \langle q^*, B(q,y)\rangle, \langle G_{01}, y\rangle = \langle q^*, B(\bar{q},y)\rangle,
$$

$$
H_{20} = B(q,q) - \langle q^*, B(q,q)\rangle q - \langle \bar{q}^*, B(q,q)\rangle\bar{q},
$$

$$
H_{11} = B(q,\bar{q}) - \langle q^*, B(q,\bar{q})\rangle q - \langle \bar{q}^*, B(q,\bar{q})\rangle\bar{q},
$$

$$
H_{02} = B(\bar{q},\bar{q}) - \langle q^*, B(q,\bar{q})\rangle q - \langle \bar{q}^*, B(\bar{q},\bar{q})\rangle\bar{q}.
$$

Now, we seek the center manifold which has the representation

$$
y = V(\nu,\bar{\nu}) = \frac{1}{2}w_{20}\nu^2 + w_{11}\nu\bar{\nu} + \frac{1}{2}w_{02}\bar{\nu}^2 + O(|\nu|^3),
\tag{3.5}
$$

where $\langle q^*, w_{ij}\rangle = 0$. Substituting of (3.5) into (3.4), we can get

$$
w_{20} = (z^2 I - A)^{-1}H_{20}, \quad w_{11} = (I - A)^{-1}H_{11}, \quad w_{02} = (\bar{z}^2 I - A)^{-1}H_{02}.
$$

The Taylor coefficients can be expressed by

$$
\begin{aligned}
g_{21} = {} & \langle q^*, C(q,q,\bar{q})\rangle - 2\langle q^*, B(q,(I-A)^{-1}B(q,\bar{q}))\rangle \\
& + \langle q^*, B(\bar{q},(e^{2i\omega^*}I - A)^{-1}B(q,q))\rangle \\
& - \frac{e^{-i\omega^*}(1 - 2e^{i\omega^*})}{1 - e^{i\omega^*}}\langle q^*, B(q,q)\rangle\langle q^*, B(q,\bar{q})\rangle \\
& - \frac{2}{1 - e^{-i\omega^*}}|\langle q^*, B(\bar{q},\bar{q})\rangle|^2 - \frac{e^{i\omega}}{e^{3i\omega} - 1}|\langle q^*, B(\bar{q},\bar{q})\rangle|^2.
\end{aligned}
$$

Using normal form transformations, we may transform away the quadratic terms and all cubic terms except the $\nu^2\bar{\nu}-$term, in the Taylor expansion of the system given in equation (3.4). This will leave us with the system

$$
\nu \mapsto \mu\nu + c_1\nu^2\bar{\nu} + O(|\nu|^4).
\tag{3.6}
$$

The sign of c_1 therefore determines properties of the bifurcation. The computations result in the following expression for c_1:

$$
c_1 = \frac{g_{20}g_{11}(1 - 2z^2)}{2(z^2 - z)} + \frac{|g_{11}|^2}{1 - \bar{z}} + \frac{|g_{02}|^2}{2(z^2 - \bar{z})} + \frac{g_{21}}{2},
$$

which gives, for the critical value of c_1,

$$c_1(\tau^*) = \frac{g_{20}g_{11}(1 - 2e^{i\omega^*})}{2(e^{2i\omega^*} - e^{i\omega^*})} + \frac{|g_{11}|^2}{1 - e^{-i\omega^*}} + \frac{|g_{02}|^2}{2(e^{2i\omega^*} - e^{-i\omega^*})} + \frac{g_{21}}{2}.$$

Therefore, we have the following theorem about the properties of Hopf bifurcation.

Theorem 3.2. *The direction and stability of Hopf bifurcation in system* (2.2) *are determined by the sign of* $l = -\dfrac{Re[e^{-i\omega^*}c_1(\tau^*)]}{d_h}$*: if* $l > 0$ *(*< 0*), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for* $\tau > \tau^*$ *(*$\tau < \tau^*$*); the sign of* $Re[e^{-i\omega^*}c_1(\tau^*)]$ *determines the stability of the periodic solutions: the periodic solutions are orbitally stable (unstable) if* $Re[e^{-i\omega^*}c_1(\tau^*)] < 0$ *(*> 0*).*

4   Numerical Simulation

In this section, we consider the following discrete small-world networks:

$$y_{n+1} = y_n + h\tau + 0.2h\tau y_{n-m} - 1.5h\tau(1 + 0.2)y_{n-m}^2. \tag{4.1}$$

The unique positive equilibrium is $y^* = 0.8030$ and $\tau_0 = 0.5838$ is the first Hopf bifurcation value. According to the algorithms in the previous section, we can get $l = 1.0164$, which means that the bifurcation is supercritical. Moreover, the bifurcating periodic solutions are orbitally stable. Figures 4.1 and 4.2 are about difference equation (4.1) when the step size is $h = 1/10$. From Figure 4.1, the positive equilibrium y^* is asymptotically stable when $\tau = 0.4 < 0.5838$. From Figure 4.2, we can see that y^* is unstable and the system undergoes a periodic solution.

Acknowledgements

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Figure 4.1: The equilibrium of (4.1) is asymptotically stable when $\tau = 0.4$
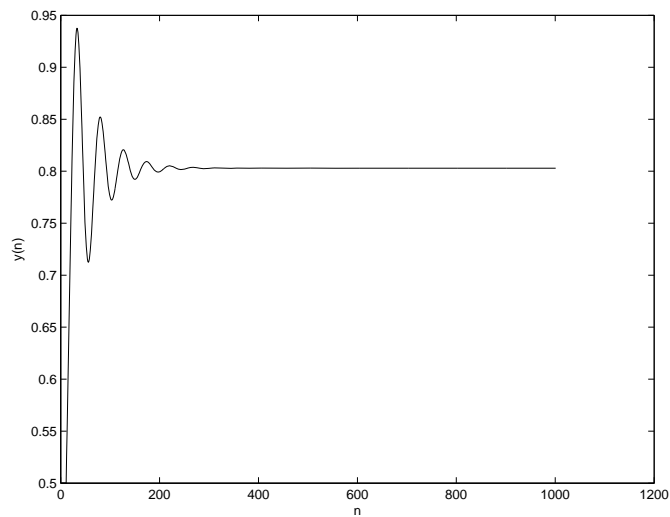
Figure 4.2: System (4.1) has an attracting invariant closed curve for $\tau = 0.6 > 0.5838$
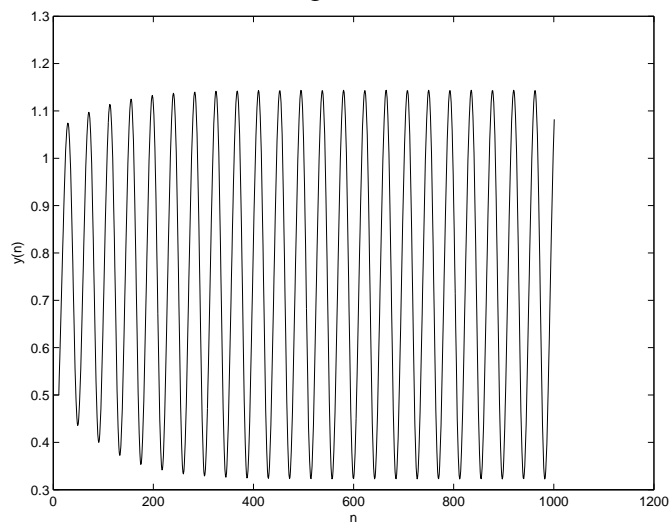
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