

Nonoscillation Criteria for Solutions of Higher-order Functional Difference Equations of Neutral Type

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Abstract

In this paper, we find necessary conditions for every solution of the neutral functional difference equation

$$\Delta^{m-1}(r_n \Delta(y_n - p_n y_{\tau(n)})) + q_n G(y_{\sigma(n)}) = f_n$$

to oscillate or to tend to zero as $n \rightarrow \infty$, where Δ is the forward difference operator, given by $\Delta x_n = x_{n+1} - x_n$; and p_n, q_n, r_n are sequences of real numbers with $q_n \geq 0, r_n > 0$. It is supposed that $\tau(n)$ and $\sigma(n)$ are increasing sequences of integers such that they are less than n and approach $+\infty$ as $n \rightarrow \infty$. Moreover, it will be assumed that m is a positive integer ≥ 2 and that $G \in C(\mathbb{R}, \mathbb{R})$. The results of this paper improve, generalize and extend some recent results.

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1 Introduction

In this paper, we find necessary conditions so that every solution of the neutral functional difference equation

$$\Delta^{m-1}(r_n \Delta(y_n - p_n y_{\tau(n)})) + q_n G(y_{\sigma(n)}) = f_n \quad (1.1)$$

oscillates or tends to zero as $n \rightarrow \infty$, where Δ is the forward difference operator, given by $\Delta y_n = y_{n+1} - y_n$; and $\{p_n\}$, $\{f_n\}$, $\{q_n\}$, and $\{r_n\}$ are sequences of real numbers with $q_n \geq 0$ and $r_n > 0$. It is supposed that $G \in C(\mathbb{R}, \mathbb{R})$ and $m \geq 2$ is a positive integer. Moreover, it will be assumed that $\tau(n)$ and $\sigma(n)$ are increasing sequences of integers, such that they are less than n and approach ∞ as $n \rightarrow \infty$. Some of the following conditions will be assumed later in this article.

(H₀) G is Lipschitzian in every interval of form $[a, b]$ with $0 < a < b$.

(H₁) $xG(x) > 0$ for $x \neq 0$ and G is nondecreasing.

$$(H_2) \sum_{n=1}^{\infty} n^{m-2} q_n = \infty.$$

$$(H_3) \sum_{n=1}^{\infty} 1/r_n = \infty.$$

$$(H_4) \sum_{n=1}^{\infty} 1/r_n < \infty.$$

$$(H_5) \sum_{n=1}^{\infty} \frac{1}{r_n} \sum_{i=n}^{\infty} (i - n + 1)^{m-2} q_i = \infty.$$

(H₆) There exists a bounded sequence $\{F_n\}$ such that $\Delta^{m-1} F_n = f_n$.

(H₇) There exists a convergent sequence $\{F_n\}$ such that $\lim_{n \rightarrow \infty} F_n = 0$ and $\Delta^m F_n = f_n$.

During the last two decades, many authors all over the world have taken keen interest in studying oscillatory behaviour of solutions of neutral difference equations, due to its important applications in the field of science and computers. For recent results and references of the subject, see the monograph by Agarwal [3], the papers [1, 2, 4, 5], [8–10] and [12–17] and the references cited there in. It seems that (1.1) is not studied much. If we put $r_n \equiv 1$, $\tau(n) = n - m$ and $\sigma(n) = n - k$ in (1.1), where m, k are positive integers, then we obtain

$$\Delta^m(y_n - p_n y_{n-m}) + q_n G(y_{n-k}) = f_n. \quad (1.2)$$

In the literature we find that most of the results (see [2,9,10,13,14]) coincide with those of (1.2). In these papers the authors required the conditions (H_0) , (H_1) and (H_7) for their results. However, in this work, we prove our results under (H_6) , which is weaker than (H_7) , and successfully remove the restrictions (H_0) and (H_1) . Different ranges of $\{p_n\}$ are considered in this paper including the rare ones like $p_n = \pm 1$. This paper improves, extends/generalizes some recent results.

Let n_0 be a fixed nonnegative integer. Let $\rho = \min\{\tau(n_0), \sigma(n_0)\}$. By a solution of (1.1) we mean a real sequence $\{y_n\}$ which is defined for all integers $n \geq \rho$ and satisfies (1.1) for $n \geq n_0$. Clearly if the initial condition

$$y_n = a_n \quad \text{for } \rho \leq n \leq n_0 \tag{1.3}$$

is given, then the equation (1.1) has a unique solution satisfying the given initial condition (1.3). A solution $\{y_n\}$ of (1.1) is said to be oscillatory if for every positive integer $n_0 > 0$, there exists $n \geq n_0$ such that $y_n y_{n+1} \leq 0$; otherwise $\{y_n\}$ is said to be nonoscillatory.

We would like to present the following useful remarks.

Remark 1.1. (i) Since $r_n > 0$, only one of (H_3) and (H_4) holds but not both.

(ii) If (H_3) holds, then (H_2) implies (H_5) but not conversely.

(iii) If (H_4) holds, then (H_5) implies (H_2) but not conversely.

2 Positive Solutions – I

In this section we assume that there exist positive real numbers b, c and d such that the sequence $\{p_n\}$ satisfies one of the following conditions.

$$(A_1) \quad 0 \leq p_n \leq b < 1.$$

$$(A_2) \quad -1 < -b \leq p_n \leq 0.$$

$$(A_3) \quad -d \leq p_n \leq -c < -1.$$

$$(A_4) \quad 1 < c \leq p_n \leq d.$$

For our purpose we need the following definition and the subsequent results from [4, 6, 11].

Definition 2.1. [4, Definition 3.2, pp 196] A set of sequences in l^∞ is uniformly Cauchy (or equi Cauchy) if for every $\epsilon > 0$ there exists an integer N such that

$$|x_i - x_j| < \epsilon$$

whenever $i, j > N$ for every $x = \{x_k\}$ in S .

Theorem 2.2. [4, Theorem 3.3, pp 196] A bounded uniformly Cauchy subset S of l^∞ is relatively compact.

Lemma 2.3 (Krasnoselskii's Fixed Point Theorem [6]). Let X be a Banach space and S be a bounded closed convex subset of X . Let A, B be operators from S to X such that $Ax + By \in S$ for every pair of $x, y \in S$. If A is a contraction and B is completely continuous, then the equation

$$Ax + Bx = x$$

has a solution in S .

Lemma 2.4. [11] If $\sum u_n$ and $\sum v_n$ are two positive term series with $\lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n} \right) = l$, where l is a nonzero finite number, then the two series converge or diverge together. If $l = 0$, then $\sum v_n$ is convergent implies $\sum u_n$ is convergent. If $l = \infty$, then $\sum v_n$ is divergent implies $\sum u_n$ is divergent.

Remark 2.5. We may recall the well-known factorial function

$$n^{(r)} = (n-1)(n-2) \cdots (n-r+1)$$

if $r \leq n$; otherwise it is zero. Since $(n-r+1)^r < n^{(r)} < n^r$, it follows from Lemma 2.4 that (H_5) implies and is implied by the condition

$$\sum_{n=1}^{\infty} \frac{1}{r_n} \sum_{i=n}^{\infty} (i-n+m-2)^{(m-2)} q_i = \infty,$$

and further (H_2) implies and is implied by the condition

$$\sum_{i=n}^{\infty} (i-n+m-2)^{(m-2)} = \infty.$$

Now, our first result is as follows.

Theorem 2.6. Let (A_1) , (H_4) and (H_6) hold. If every solution of (1.1) oscillates or tends to zero as $n \rightarrow \infty$, then (H_5) holds.

Proof. We use the contraposition method. Assuming that (H_5) does not hold, we try to find a solution to (1.1) that does not oscillate and does not tend to zero. From the negation of (H_5) , we have

$$\sum_{n=1}^{\infty} \frac{1}{r_n} \sum_{i=n}^{\infty} (i-n+1)^{m-2} q_i < \infty. \quad (2.1)$$

By Remark 2.5, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{r_n} \sum_{i=n}^{\infty} (i - n + m - 2)^{(m-2)} q_i < \infty. \tag{2.2}$$

Using the continuity of G , we set

$$\mu = \max \left\{ |G(x)| : \frac{3(1-b)}{5} \leq x \leq 1 \right\}. \tag{2.3}$$

Then by (H_4) and (H_6) , we obtain

$$\sum_{i=1}^{\infty} \frac{|F_i|}{r_i} < \infty. \tag{2.4}$$

Since (2.2) and (2.4) hold, we can find $n_1 > 0$ such that $n \geq n_1$ implies

$$\frac{\mu}{(m-2)!} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i - j + m - 2)^{(m-2)} q_i < \frac{1-b}{10} \tag{2.5}$$

and

$$\sum_{i=n}^{\infty} \frac{|F_i|}{r_i} < \frac{1-b}{10}. \tag{2.6}$$

Choose N_1 large enough, so that $N_0 \geq n_1$, where $N_0 = \min\{\tau(N_1), \sigma(N_1)\}$. Let $X = l_{\infty}^{N_0}$, the Banach space of bounded real sequences $x = \{x_n\}$, with the supremum norm

$$\|x\| = \sup\{|x_n| : n \geq N_0\}.$$

In this space, we define the closed and convex set

$$S = \left\{ y \in X : \frac{3(1-b)}{5} \leq y_n \leq 1, n \geq N_0 \right\}. \tag{2.7}$$

Now we define two operators A and B , from S to X , such that fixed points of $A + B$ are solutions of (1.1). For $y \in S$, define

$$(Ay)_n = \begin{cases} (Ay)_{N_1}, & N_0 \leq n \leq N_1, \\ p_n y_{\tau(n)} + \frac{4(1-b)}{5}, & n \geq N_1 \end{cases} \tag{2.8}$$

and

$$(By)_n = \begin{cases} (By)_{N_1}, & N_0 \leq n \leq N_1, \\ \frac{(-1)^{m-1}}{(m-2)!} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i - j + m - 2)^{(m-2)} q_i G(y_{\sigma(i)}) \\ - \sum_{i=n}^{\infty} \frac{F_i}{r_i}, & n \geq N_1. \end{cases} \tag{2.9}$$

First we show that if $x, y \in S$, then $Ax + By \in S$. With $x = \{x_n\}$ and $y = \{y_n\}$ in S , and $n \geq N_1$, we obtain

$$(Ax)_n + (By)_n = \frac{(-1)^{m-1}}{(m-2)!} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i G(y_{\sigma(i)}) + p_n x_{\tau(n)} + \frac{4(1-b)}{5} - \sum_{i=n}^{\infty} \frac{F_i}{r_i}.$$

Note that for $i \geq N_1$, the sequence $\{y_{\sigma(i)}\}$ is in S , so that $|G(y_{\sigma(i)})| \leq \mu$. Using (2.5), (2.6), and $0 \leq p_n \leq b < 1$, we have

$$(Ax)_n + (By)_n < b + \frac{4(1-b)}{5} + \frac{1-b}{10} + \frac{1-b}{10} = 1$$

and

$$(Ax)_n + (By)_n > \frac{4(1-b)}{5} - \frac{1-b}{10} - \frac{1-b}{10} = \frac{3}{5}(1-b).$$

Therefore, $3(1-b)/5 < (Ax)_n + (By)_n \leq 1$ so that $Ax + By$ belongs to S for all x, y in S .

Next we show that A is a contraction in S . In fact for x, y in S and $n \geq N_1$,

$$\|(Ax)_n - (Ay)_n\| \leq |p_n| |x_{\tau(n)} - y_{\tau(n)}| \leq b \|x - y\|.$$

This implies A is a contraction, because $0 < b < 1$.

Next we show that B is completely continuous. As a first step we show that B is continuous. Suppose $x^l \equiv \{x_n^l\}$ is a sequence of points in S (with l taken from the index set) which converges (in fact uniformly for $n \geq N_0$) to $x \equiv \{x_n\}$ in S as $l \rightarrow \infty$. Since S is closed, $x \in S$. For $n \geq N_1$ we have

$$|(Bx^l)_n - (Bx)_n| \leq \frac{1}{(m-2)!} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i |G(x_{\sigma(i)}^l) - G(x_{\sigma(i)})|.$$

Since G is continuous, $|G(x_{\sigma(i)}^l) - G(x_{\sigma(i)})|$ approaches zero as $l \rightarrow \infty$ for each $i \geq N_1$. Hence, B is continuous. It remains to show that BS is relatively compact. Using [4, Theorem 3.3] we need only show that BS is uniformly Cauchy. Let $x \equiv \{x_n\}$ be a sequence in S . Using (2.1) and (2.4), for $\epsilon > 0$, there exists $N^* \geq N_1$ such that, for $n \geq N^*$,

$$\sum_{i=n}^{\infty} \left| \frac{F_i}{r_i} \right| + \frac{1}{(m-2)!} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i \mu < \epsilon/2.$$

Then for $n_2 > n_1 \geq N^*$,

$$\begin{aligned} |(Bx)_{n_2} - (Bx)_{n_1}| &< \sum_{i=n_2}^{\infty} \left| \frac{F_i}{r_i} \right| + \frac{1}{(m-2)!} \sum_{j=n_2}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_j \mu \\ &+ \frac{1}{(m-2)!} \sum_{j=n_1}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_j \mu + \sum_{i=n_1}^{\infty} \left| \frac{F_i}{r_i} \right| \\ &< 2\frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus, BS is uniformly cauchy. Hence, it is relatively compact. Then, by Lemma 2.3 there is an x^0 in S such that $Ax^0 + Bx^0 = x^0$; i.e., for $y = x^0$ and $n \geq N_1$, $y_n = (A + B)y_n$. Thus,

$$y_n = p_n y_{\tau(n)} + \frac{4}{5}(1-b) + \frac{(-1)^{m-1}}{(m-2)!} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i G(y_{\sigma(i)}) - \sum_{i=n}^{\infty} \frac{F_i}{r_i}.$$

Applying the forward difference operator Δ , we obtain

$$\Delta(y_n - p_n y_{\tau(n)}) + \frac{(-1)^{m-1}}{(m-2)!} \frac{1}{r_n} \sum_{i=n}^{\infty} (i-n+m-2)^{(m-2)} q_i G(y_{\sigma(i)}) = \frac{F_n}{r_n}.$$

Note that $(n+1)^{(r)} - n^{(r)} = rn^{(r-1)}$ and $n^{(r)} = 0$, if $n < r$. Then multiplying by r_n , and applying Δ for $m-1$ times, with $\Delta^{m-1} F_n = f_n$, we obtain (1.1). Therefore, $(x^0)_n$ is a solution of (1.1) and is bounded below by $3(1-b)/5$. Thus, $(x^0)_n$ is nonoscillatory and does not approach zero as $n \rightarrow \infty$. This completes the proof. \square

Corollary 2.7. *Let (A_1) , (H_4) , (H_6) hold. If every solution of (1.1) oscillates or tends to zero as $n \rightarrow \infty$, then (H_2) holds.*

The proof of this corollary follows from Remark 1.1(iii) and Theorem 2.6.

Theorem 2.8. *Let (A_1) and (H_3) hold. Assume that there exists a positive integer n_0 and a real $\alpha > 0$ such that $i \geq n_0$ implies*

$$r_i > \frac{1}{\alpha} \tag{2.10}$$

and

$$\sum_{i=0}^{\infty} F_i < \infty \quad \text{with} \quad \Delta^{m-1} F_n = f_n. \tag{2.11}$$

If every solution of (1.1) oscillates or tends to zero as $n \rightarrow \infty$, then (H_5) holds.

Proof. Using (2.10) and (2.11), we obtain (2.4) and consequently get (2.6). The rest of the proof is similar to that of Theorem 2.6. \square

Remark 2.9. Condition (2.11) implies (H_6) .

Corollary 2.10. *Let (A_1) , (2.10), (2.11) hold. If every solution of (1.1) oscillates or tends to zero as $n \rightarrow \infty$, then (H_5) holds.*

Proof. By Remark 1.1(i), either (H_3) holds or (H_4) holds exclusively. If (H_4) holds, using condition (2.11) and Theorem 2.6, we get the required solution to (1.1). If (H_3) holds, using Theorem 2.8, we obtain the required solution. \square

Remark 2.11. Corollary 2.10 improves and generalizes [2, Theorem 1] because we have removed the restrictions (H_3) and (H_1) . Furthermore, in their theorem, $p_n \equiv p$, a constant, and m is an even positive integer.

Remark 2.12. In (2.1), if we put $r_n \equiv 1$, then it reduces to

$$\sum_{n=1}^{\infty} \sum_{i=n}^{\infty} (i - n + 1)^{m-2} q_i < \infty. \tag{2.12}$$

(2.12) is required for our next result which follows directly from Corollary 2.10.

Corollary 2.13. *Inequality (2.12) is a sufficient condition for the m th order neutral functional difference equation*

$$\Delta^m(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = f_n \tag{2.13}$$

to have a solution bounded below by a positive constant, under assumptions (A_1) and (2.11).

Remark 2.14. We claim that the condition

$$\sum_{i=1}^{\infty} i^{m-1} q_i < \infty \tag{2.14}$$

implies (2.12). By Lemma 2.4, it follows that (2.14) is equivalent to

$$M_n := \sum_{i=n}^{\infty} (i - n + 1)^{m-1} q_i < \infty \tag{2.15}$$

for $n \geq n_0$. Note that $M_n \rightarrow 0$ as $n \rightarrow \infty$. Further

$$\begin{aligned} \Delta M_n &= M_{n+1} - M_n = \sum_{i=n}^{\infty} \left[(i - n)^{m-1} - (i - n + 1)^{m-1} \right] q_i \\ &= - \sum_{i=n}^{\infty} \left[(i - n + 1)^{m-2} + \binom{m-2}{1} (i - n)^{m-2} \right. \\ &\quad \left. + \binom{m-2}{2} (i - n)^{m-3} + \dots + \binom{m-2}{m-2} (i - n) \right] q_i. \end{aligned}$$

Hence

$$\sum_{i=n}^{\infty} (i - n + 1)^{m-2} q_i < M_n - M_{n+1}.$$

Then, summing from $n = 1$ to $n = k - 1$, we obtain

$$\sum_{n=1}^{k-1} \sum_{i=n}^{\infty} (i - n + 1)^{m-2} q_i < M_1 - M_k.$$

As $k \rightarrow \infty$, we obtain

$$\sum_{n=1}^{\infty} \sum_{i=n}^{\infty} (i - n + 1)^{m-2} q_i < M_1 = \sum_1^{\infty} i^{m-1} q_i < \infty.$$

Hence, our claim holds.

Remark 2.15. Corollary 2.13 improves [13, Theorem 4.2] because in that work the authors assume (H_0) and (H_1) . It may be noted in view of Remark 2.14, that the condition we used, that is (2.12), is weaker than the condition (2.15) assumed in [13].

Next we wish to formulate our results when p_n satisfies (A_2) , (A_3) or (A_4) . Since the proofs are similar to the proofs of the results that have been offered so far, we present only the sketches of the proofs and leave some of them as an exercise for the reader.

Theorem 2.16. *Let (A_2) , (H_4) and (H_6) hold. If every solution of (1.1) oscillates or tends to zero as $n \rightarrow \infty$, then (H_5) holds.*

Proof. We proceed as in the proof of Theorem 2.6, with the following changes:

$$\mu = \max \left\{ |G(x)| : \frac{(1-b)}{10} \leq x \leq 1 \right\}.$$

Assuming (H_4) , (H_6) hold and that (H_5) does not hold, there exist n_1 such for $n \geq n_1$,

$$\frac{\mu}{(m-2)!} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i - j + m - 2)^{(m-2)} q_i < \frac{1-b}{10} \quad \text{and} \quad \sum_{i=n}^{\infty} \frac{|F_i|}{r_i} < \frac{1-b}{10}.$$

Let $S = \left\{ y \in X : \frac{(1-b)}{10} \leq y_n \leq 1, n \geq N_0 \right\}$. Then we define the operators A and B as follows:

$$(Ay)_n = \begin{cases} (Ay)_{N_1}, & N_0 \leq n \leq N_1, \\ p_n y_{\tau(n)} + \frac{7b+3}{10}, & n \geq N_1; \end{cases}$$

$$(By)_n = \begin{cases} (By)_{N_1}, & N_0 \leq n \leq N_1, \\ \frac{(-1)^{m-1}}{(m-2)!} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i G(y_{\sigma(i)}) & \\ - \sum_{i=n}^{\infty} \frac{F_i}{r_i}, & n \geq N_1. \end{cases}$$

Then as in Theorem 2.6 we prove: (i) $Ax + By \in S$, (ii) A is a contraction, and finally (iii) B is completely continuous. Then, by Lemma 2.3, there is a fixed point x_0 in S such that $Ax_0 + Bx_0 = x_0$ which is the required solution that is bounded below by $\frac{1-b}{10}$. □

Theorem 2.17. *Let (A_2) , (H_3) , (2.10), (2.11) hold. If every solution of (1.1) oscillates or tends to zero as $n \rightarrow \infty$, then (H_5) holds.*

The proof of Theorem 2.17 is similar to that of Theorem 2.8.

Definition 2.18. For any positive integer $n \geq n_0$, define

$$\tau_{-1}(n) = \{m : m \text{ is an integer } \geq n \text{ and } \tau(m) = n\}.$$

Remark 2.19. The function τ_{-1} given in Definition 2.18 is the inverse function of $\tau(n)$. Since $\tau(n)$ is increasing, it is one-one. If n is a positive integer greater than or equal to n_0 , then $\tau_{-1}(\tau(n)) = n$.

Theorem 2.20. *Suppose that p_n satisfies (A_3) or (A_4) . Let (H_4) and (H_6) hold. If every solution of (1.1) oscillates or tends to zero as $n \rightarrow \infty$, then (H_5) holds.*

Proof. Suppose that p_n satisfies (A_4) . The proof is similar when p_n satisfies (A_3) . Assume, for the sake of contradiction that (H_5) does not hold. Then we proceed as in the proof of Theorem 2.6, with the following changes:

$$\mu = \max \{|G(x)| : c - 1 \leq x \leq 2d + c\}.$$

Then from (H_4) and (H_6) , one can find $n_1 > 0$ such that for $n \geq n_1$,

$$\frac{\mu}{(m-2)!} \sum_{j=\tau_{-1}(n)}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i < \frac{c(c-1)}{2}$$

and

$$\sum_{i=\tau_{-1}(n)}^{\infty} \frac{F_i}{r_i} < \frac{c(c-1)}{2}.$$

Let

$$S = \{y \in X : c - 1 \leq y_n \leq 2d + c, n \geq N_0\}.$$

Then we define the operators A and B as follows:

$$(Ay)_n = \begin{cases} (Ay)_{N_1}, & N_0 \leq n \leq N_1 \\ \frac{y_{\tau^{-1}(n)}}{p_{\tau^{-1}(n)}} + \frac{2d(c-1)}{p_{\tau^{-1}(n)}}, & n \geq N_1; \end{cases}$$

$$(By)_n = \begin{cases} (By)_{N_1}, & N_0 \leq n \leq N_1, \\ \frac{(-1)^m}{(m-2)! p_{\tau^{-1}(n)}} \sum_{j=\tau^{-1}(n)}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i G(y_{\sigma(i)}) \\ + \frac{1}{p_{\tau^{-1}(n)}} \sum_{i=\tau^{-1}(n)}^{\infty} \frac{F_i}{r_i}, & n \geq N_1. \end{cases}$$

The rest of the proof is similar to that of Theorem 2.6. □

3 Positive Solutions – II

In this section we find positive solutions for (1.1) when $p_n = \pm 1$. We consider the equations

$$\Delta^{m-1} [r_n \Delta(y_n + y_{\tau(n)})] + q_n G(y_{\sigma(n)}) = f_n, \tag{3.1}$$

$$\Delta^{m-1} [r_n \Delta(y_n - y_{\tau(n)})] + q_n G(y_{\sigma(n)}) = f_n. \tag{3.2}$$

For this purpose we need the following result.

Lemma 3.1 (Schauder’s Fixed Point Theorem [7]). *Let S be a closed, convex and nonempty subset of a Banach space X . Let $B : S \rightarrow S$ be a continuous mapping such that $B(S)$ be a relatively compact subset of X . Then B has at least one fixed point in S . This means there is an $x \in S$ such that $Bx = x$.*

Definition 3.2. Define

$$\tau_{-1}^0(n) = n, \quad \tau_{-1}^1(n) = \tau_{-1}(n), \quad \tau_{-1}^2(n) = \tau_{-1}(\tau_{-1}(n)).$$

For any positive integer $i > 2$, we define

$$\tau_{-1}^i(n) = \tau_{-1}(\tau_{-1}^{i-1}(n)).$$

Theorem 3.3. *Suppose (H_4) , (H_6) hold. If every solution of (3.1) oscillates or tends to zero as $n \rightarrow \infty$, then (H_5) holds.*

Proof. Let

$$\mu = \max \{ |G(x)| : 1 \leq x \leq 5 \}.$$

Suppose (H₅) does not hold. Then (2.1) holds. From (H₄) and (H₆) we get (2.4). Hence there exists n_1 such that for $n \geq n_1$

$$\frac{\mu}{(m-2)!} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i < 1 \tag{3.3}$$

and

$$\sum_{i=n}^{\infty} \frac{|F_i|}{r_i} < 1. \tag{3.4}$$

For any real sequence $\{a_n\}$, it is clear that

$$\sum_{l=1}^{\infty} \sum_{i=\tau_{-1}^{2l-1}(n)}^{\tau_{-1}^{2l}(n)-1} a_i < \sum_{i=n}^{\infty} a_i. \tag{3.5}$$

Hence using the above inequality in (3.3) and (3.4), we obtain for $n \geq n_1$

$$\frac{\mu}{(m-2)!} \sum_{l=1}^{\infty} \sum_{j=\tau_{-1}^{2l-1}(n)}^{\tau_{-1}^{2l}(n)-1} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i < 1 \tag{3.6}$$

and

$$\sum_{l=1}^{\infty} \sum_{i=\tau_{-1}^{2l-1}(n)}^{\tau_{-1}^{2l}(n)-1} \frac{F_i}{r_i} < 1. \tag{3.7}$$

Choose N_1 with $N_0 = \min\{\tau(N_1), \sigma(N_1)\} \geq n_1$. Let $S = \{y \in X : 1 \leq y_n \leq 5, n \geq N_0\}$ Define a mapping B from S to X by

$$(By)_n = \begin{cases} (By)_{N_1}, & N_0 \leq n \leq N_1, \\ \frac{(-1)^{m-1}}{(m-2)!} \sum_{l=1}^{\infty} \sum_{j=\tau_{-1}^{2l-1}(n)}^{\tau_{-1}^{2l}(n)-1} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i G(y_{\sigma(i)}) \\ - \sum_{l=1}^{\infty} \sum_{i=\tau_{-1}^{2l-1}(n)}^{\tau_{-1}^{2l}(n)-1} \frac{F_i}{r_i} + 3, & n \geq N_1. \end{cases}$$

Then using (3.6) and (3.7) for $y = y_n \in S$, we have $(By)_n \leq 5$, and $(By)_n \geq 1$. Hence, $By \in S$. Then we proceed as in the proof of Theorem 2.6 and prove that BS

is relatively compact. Then, by Lemma 3.1, there is a fixed point y^0 in S such that $By_n^0 = y_n^0$. Hence,

$$y_n^0 = 3 + \frac{(-1)^{m-1}}{(m-2)!} \sum_{l=1}^{\infty} \sum_{j=\tau_{-1}^{2l-1}(n)}^{\tau_{-1}^{2l}(n)-1} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i G(y_{\sigma(i)}^0) - \sum_{l=1}^{\infty} \sum_{i=\tau_{-1}^{2l-1}(n)}^{\tau_{-1}^{2l}(n)-1} \frac{F_i}{r_i}.$$

In the above, if we replace n by $\tau(n)$ and note that $\tau_{-1}^k(\tau(n)) = \tau_{-1}^{k-1}(n)$ and $\tau_{-1}^0(n) = n$, then we obtain $y_{\tau(n)}^0$. Hence it follows for $n \geq N_1$ that

$$y_n^0 + y_{\tau(n)}^0 = 6 + \frac{(-1)^{m-1}}{(m-2)!} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i G(y_{\sigma(i)}^0) - \sum_{i=n}^{\infty} \frac{F_i}{r_i}.$$

Applying Δ , multiply by r_n , and again applying Δ for $m - 1$ times, we arrive at (3.1). This solution is bounded below by a positive constant, so it does not oscillate and does not tend to zero as $n \rightarrow \infty$. □

Corollary 3.4. *Let (H_4) and (H_6) hold. If every solution of (3.1) oscillates or tends to zero as $n \rightarrow \infty$, then (H_2) holds.*

The proof of Corollary 3.4 follows from Remark 1.1(iii) and Theorem 3.3.

Theorem 3.5. *Let (H_3) , (2.10), (2.11) hold. If every solution of (3.1) oscillates or tends to zero as $n \rightarrow \infty$, then (H_5) holds.*

The proof of Theorem 3.5 is similar to that of Theorem 3.3.

Theorem 3.6. *Suppose that (H_6) holds. For each positive integer $n \geq n_0$ assume that*

$$\sum_{l=1}^{\infty} \sum_{j=\tau_{-1}^l(n)}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i < \infty \tag{3.8}$$

and

$$\sum_{l=1}^{\infty} \sum_{j=\tau_{-1}^l(n)}^{\infty} \frac{1}{r_j} < \infty. \tag{3.9}$$

Then (3.2) has a solution that is bounded below by a positive constant.

Proof. We proceed as in the proof of Theorem 2.6 with the following changes: Let $\mu = \max \{|G(x)| : 1 \leq x \leq 5\}$. Then from (H₆), (3.8) and (3.9), there exists $n_1 > 0$ such that for $n \geq n_1$,

$$\sum_{l=1}^{\infty} \sum_{j=\tau_{-1}^l(n)}^{\infty} \frac{|F_j|}{r_j} < 1, \quad \text{and} \quad \frac{\mu}{(m-2)!} \sum_{l=1}^{\infty} \sum_{j=\tau_{-1}^l(n)}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i < 1.$$

Let $S = \{y \in X : 1 \leq y_n \leq 5, n \geq N_0\}$. Then define the mapping

$$(By)_n = \begin{cases} (By)_{N_1}, & N_0 \leq n \leq N_1, \\ \frac{(-1)^m}{(m-2)!} \sum_{l=1}^{\infty} \sum_{j=\tau_{-1}^l(n)}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i G(y_{\sigma(i)}) \\ + \sum_{l=1}^{\infty} \sum_{j=\tau_{-1}^l(n)}^{\infty} \frac{F_j}{r_j} + 3, & n \geq N_1. \end{cases}$$

Then for $y = \{y_n\} \in S$, we have $(By)_n \leq 5$ and $(By)_n \geq 1$. Hence, $By \in S$. Then using (3.8) and (3.9) we proceed as in the proof of Theorem 2.6 and prove that BS is relatively compact. Then, by Lemma 3.1, there is a fixed point y^0 in S such that $By_n^0 = y_n^0$. Hence,

$$y_n^0 = 3 - \frac{(-1)^{m-1}}{(m-2)!} \sum_{l=1}^{\infty} \sum_{j=\tau_{-1}^l(n)}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i G(y_{\sigma(i)}^0) + \sum_{l=1}^{\infty} \sum_{j=\tau_{-1}^l(n)}^{\infty} \frac{F_j}{r_j}.$$

For $n \geq N_1$, it follows that

$$y_n^0 - y_{\tau(n)}^0 = \frac{(-1)^{m-1}}{(m-2)!} \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (i-j+m-2)^{(m-2)} q_i G(y_{\sigma(i)}^0) - \sum_{j=n}^{\infty} \frac{F_j}{r_j}.$$

Applying Δ , multiplying by r_n , and again applying Δ for $m-1$ times we arrive at (3.2). This solution is bounded below by 2 which is a positive constant. \square

We close this article with an interesting example which illustrates most of our results, whereas the results available in the literature are not applicable to this example.

Example 3.7. Consider the equation

$$\Delta^{m-1}(r_n \Delta(y_n \pm by_{n-1})) + \frac{1}{n^{m+2}} G(y_{n-2}) = 0, \quad n > 0, \tag{3.10}$$

where b is a constant in any one of the ranges of $\{p_n\}$, considered in this paper and $m \geq 2$. Suppose that $r_n = 1$ or $1/n^2$. Let $G(u) = 1 - u^3$. It is not difficult to verify that

$q_n = \frac{1}{n^{m+2}}$ satisfies (2.1), (2.14), and (3.8). The neutral equation (3.10) satisfies the conditions of most of the results of this paper. Hence, it admits a solution, $y_n \equiv 1$, which is bounded below by a positive constant. Since we have no restriction on G , most of the results available in the literature [2, 9, 13] are not applicable to this neutral equation (3.10), because $G(u) = 1 - u^3$ does not satisfy (H_1) or (H_0) which is a requirement in these papers.

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