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Impulsive Stabilization of certain Delay Differential Equations with Piecewise Constant Argument

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Abstract

In this paper, we investigate the impulsive stabilization of certain delay differential equations with piecewise constant argument by using Lyapunov function and analysis methods. Some nonimpulsive systems can be stabilized by imposition of impulsive controls. We also give an example to demonstrate the effectiveness of the proposed control and stabilization methods.

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1 Introduction and Preliminaries

In recent years, impulsive control and impulsive stabilization for delay differential equations have attracted a great deal of attention. It arises naturally from a wide variety of applications such as threshold theory in biology, ecosystems management, and orbital transfer of satellite. Recently, various results for the impulsive stabilization of delay differential equations are obtained via different approaches, for instance, see [2, 4, 8–11]. Impulses can make unstable systems stable and, stable systems can become unstable after impulses effects [1, 3–5, 7, 8]. However, to the best of author's knowledge, there are few works on impulsive stabilization of delay differential equations with piecewise constant argument.

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In this paper, we consider the delay differential equation

$$\begin{cases} (r(t)(x'(t))^\sigma)' + \sum_{i=1}^{N} a_i(t)x^\delta(t-[t]-\tau_i) + p(t)x^\mu(t) = 0, \ t \geq 0, \\ x(t) = \phi(t), -\tau \leq t \leq 0, \ x'(0) = x_0' \end{cases} \tag{1.1}$$

and the corresponding equation with impulses

$$\begin{cases} (r(t)(x'(t))^\sigma)' + \sum_{i=1}^{N} a_i(t)x^\delta(t-[t]-\tau_i) + p(t)x^\mu(t) = 0, & t \neq t_k, \ t \geq 0, \\ x(t) = \phi(t), -\tau \leq t \leq 0, \ \ x'(0) = x_0', & \\ x(t_k) = I_k(x(t_k^-)), \ x'(t_k) = J_k(x'(t_k^-)), & k \in \mathbb{Z}_+, \end{cases} \tag{1.2}$$

where N is any given positive integer, $\tau = \max_{1 \leq i \leq N} \tau_i$, $[t]$ denotes the maximum of the set of integers that are smaller than or equal to t, and $\mathbb{Z}_+$ is the set of all positive integers. The following assumptions will be needed throughout the paper:

(H$_1$) the sequence t_k satisfies $0 \leq t_0 < t_1 < \ldots < t_k < \ldots$, $\lim_{k\to} t_k = \infty$;

(H$_2$) $I_k, J_k : \mathbb{R} \to \mathbb{R}$ are continuous functions and $I_k(0) = J_k(0) = 0$, $k \in \mathbb{Z}_+$;

(H$_3$) $r, a_i, p : [0, \infty) \to \mathbb{R}$ are continuous functions, $i = 1, 2, \ldots, N$;

(H$_4$) $\sigma, \mu, \delta \geq 1$ are constants, $\tau \leq 1$;

(H$_5$) x' denotes the right-hand derivative of x, i.e.,

$$x'(t_k) = x'(t_k^+) = \lim_{h \to 0^+} \frac{x(t_k+h) - x(t_k^+)}{h}.$$

When $a_i = 0$ and $\mu = 1$, (1.2) reduces to the differential equation

$$\begin{cases} (r(t)(x'(t))^\sigma)' + p(t)x(t) = 0, & t \neq t_k, \ t \geq t_0, \\ x(t_0) = x_0, x'(t_0) = x_0', & \\ x(t_k) = I_k(x(t_k^-)), \ x'(t_k) = J_k(x'(t_k^-)), & k \in \mathbb{Z}_+. \end{cases} \tag{1.3}$$

Some properties of (1.3) have been extensively investigated in [6]. Furthermore, if $\sigma = 1$ in (1.3), then it reduces to the differential equation

$$\begin{cases} (r(t)x'(t))' + p(t)x(t) = 0, & t \neq t_k, \ t \geq t_0, \\ x(t_0) = x_0, x'(t_0) = x_0', & \\ x(t_k) = I_k(x(t_k^-)), \ x'(t_k) = J_k(x'(t_k^-)), & k \in \mathbb{Z}_+. \end{cases} \tag{1.4}$$

Oscillatory properties of (1.4) have been discussed in [10, 12]. In the present paper, we deal with the more general equation (1.1). We shall investigate the impulsive stabilization of (1.1) by using Lyapunov function and analysis methods. Some new results are obtained here. It shows that the impulses do contribute to the equation's stability behavior.

In the settings of this paper, according to [9], we can obtain global existence of the solution of (1.1). So, in this paper, we always assume the solutions of (1.1) and (1.2) exist globally.

Definition 1.1. A function $x : [-\tau, a) \to \mathbb{R}$, $a > 0$ is said to be a solution of system (1.2) through $(0, \phi)$ if

(i) x and x' are continuous on $[-\tau, a)\backslash\{t_k; k \in \mathbb{Z}_+\}$ and are right continuous at t_k;

(ii) x satisfies (1.1) almost everywhere in $(-\tau, a)$;

(iii) x and x' fulfill the third equality of (1.2) for each $k \in \mathbb{Z}_+$.

Definition 1.2. The zero solution of (1.1) is said to be exponentially stabilized by impulses, if there exists $\alpha > 0$, sequences $\{t_k\}_{k=1}^{\infty}, I_k, J_k$ satisfying (H$_1$) and (H$_2$) such that for all $\varepsilon > 0$, there exists $\delta > 0$ with the property that, when the solution x of (1.1) through $(0, \phi)$ fulfills

$$\sqrt{\| \phi \|_\tau^2 + (x_0')^2} \le \delta, \tag{1.5}$$

then

$$\sqrt{x^2(t) + (x'(t))^2} \le \varepsilon \exp(-\alpha t), \quad t \ge 0, \tag{1.6}$$

where $\| \phi \|_\tau = \sup\limits_{-\tau \le s \le 0} |\phi(s)|$.

Definition 1.3. The zero solution of (1.1) is said to be exponentially stabilized by periodic impulses if there exists $\alpha > 0$, a sequence $\{t_k\}_{k=1}^{\infty}$ satisfying (H$_1$) and $t_k - t_{k-1} = d$, where $d > 0$ is a constant, I_k, J_k satisfying (A_2) and

$$I_k(x) = I(x), J_k(x) = J(x),\ x \in \mathbb{R},\ k \in \mathbb{Z}_+$$

such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that the solution x of (1.1) through $(0, \phi)$ with (1.5) fulfills (1.6).

Remark 1.4. In the present paper, for convenience, we always suppose $t_0 = 0$.

2 Main Results

In this section we shall establish theorems which provide sufficient conditions for exponential stabilization of (1.1) by impulses.

Theorem 2.1. *Assume that* (*H*$_3$)–(*H*$_5$) *hold. Moreover, suppose*

(*H*$_6$) *there exist constants* $r, a_i, p \ge 0$, $i = 1, 2, \ldots, N$ *such that* $1 \le |r(t)| \le r$, $|p(t)| \le p$, $|a_i(t)| \le a_i$, $t \ge 0$;

(H_7) *we have the inequalitiy*

$$\tau < \left(r \sum_{i=1}^{N} a_i \right)^{-1} \exp\left(- \left(1 + rp + r \sum_{i=1}^{N} a_i \right) \right).$$

Then the zero solution of (1.1) *can be exponentially stabilized by impulses.*

Proof. Since condition (H_7) holds, there exists $\alpha > 0$ such that

$$r \sum_{i=1}^{N} a_i \tau_i \le r\tau \sum_{i=1}^{N} a_i \le \exp(-2\alpha) \exp\left(- \left(1 + rp + r \sum_{i=1}^{N} a_i \right) \right). \tag{2.1}$$

Considering $\tau \le 1$, we may choose a sequence $\{t_k\}_{k=1}^{\infty} = \{n\}_{n=1}^{\infty}$. It is obvious that condition (H_1) is satisfied and $t_{k+1} - t_k = 1$ ($t_0 = 0$). On the other hand, we can choose a sequence $\{\eta_k\}_{k=1}^{\infty}$, $\eta_k \in (0,1)$ that is such that the solution $x(t) = x(t,k,x(k),\phi)$ beginning with $t = k$, and

$$\sqrt{x^2(k) + (x'(k))^2} \le \eta_k$$

satisfies

$$\sqrt{x^2(t) + (x'(t))^2} < 1, \quad t \in [k, k+1). \tag{2.2}$$

Then let

$$|I_k(u)| = d_k|u|, \qquad |J_k(v)| = d_k|v|,$$

$$d_k = \min\left\{ \eta_k \exp(\alpha), \quad \frac{1}{r}\sqrt{T - r\sum_{i=1}^{N} a_i\tau_i} \right\},$$

$$T = \exp(-2\alpha) \exp\left(- \left(1 + rp + r \sum_{i=1}^{N} a_i \right) \right).$$

With what was mentioned above, in view of (2.1), it is obvious that $d_k \ge 0$. For any $\varepsilon \in (0,1)$, let

$$\widehat{\delta} = \min\left\{ \eta_0,\ \varepsilon,\ \frac{\varepsilon \exp(-\alpha)}{\sqrt{1 + r^2 + r\sum_{i=1}^{N} a_i}} \exp\left(-\frac{1}{2} \left(1 + rp + r \sum_{i=1}^{N} a_i \right) \right) \right\}.$$

Next we prove that for each solution $x(t) = x(t,0,\phi)$ of (1.2) with

$$\sqrt{\| \phi \|_\tau^2 + (x_0')^2} \le \widehat{\delta},$$

we have

$$\sqrt{x^2(t)+(x'(t))^2} \le \varepsilon \exp(-\alpha t), \quad t \ge 0.$$

First, for $t \in [0,1)$, we choose the Lyapunov function

$$V(t) = x^2(t) + r^2(t)(x'(t))^{2\sigma} + r\sum_{i=1}^{N} a_i \int_{t-\tau_i}^{0} x^{2\delta}(s)ds.$$

Considering condition (H_6) and (2.2), we have

$$V(t) \ge x^2(t) + (x'(t))^2;$$

$$\begin{aligned} V(t) &\le x^2(t) + r^2(x'(t))^{2\sigma} + r\sum_{i=1}^{N} a_i\tau_i ||\phi||_{\tau_i}^{2\delta} \\ &\le x^2(t) + r^2(x'(t))^{2} + r\sum_{i=1}^{N} a_i\tau_i ||\phi||_{\tau}^{2} \\ &\le \left(1 + r^2 + r\sum_{i=1}^{N} a_i\tau_i\right)\left((x'(t))^2 + ||\phi||_{\tau}^{2}\right); \end{aligned}$$

and when $t \in (0,1)$, if we denote by $V'(t)$ the right upper derivative of $V(t)$ along the solution of (1.2), then in view of (H_4), (2.2) and the fact that $a^2 + b^2 \ge 2|ab|$ for any $a, b \in \mathbb{R}$, we have

$$\begin{aligned} V'(t) &= 2x(t)x'(t) + 2r(t)(x'(t))^{\sigma}\left(r(t)(x'(t))^{\sigma}\right)' - r\sum_{i=1}^{N} a_i x^{2\delta}(t-\tau_i) \\ &= 2x(t)x'(t) + 2r(t)(x'(t))^{\sigma}\left(-\sum_{i=1}^{N} a_i(t)x^{\delta}(t-[t]-\tau_i) - p(t)x^{\mu}(t)\right) \\ &\quad -r\sum_{i=1}^{N} a_i x^{2\delta}(t-\tau_i) \\ &\le x^2(t) + (x'(t))^2 + r\sum_{i=1}^{N} a_i\left(((x'(t))^{2\sigma} + x^{2\delta}(t-\tau_i)\right) + rp\left(((x'(t))^{2\sigma} + x^{2\mu}(t)\right) \\ &\quad -r\sum_{i=1}^{N} a_i x^{2\delta}(t-\tau_i) \\ &= x^2(t) + (x'(t))^2 + r\sum_{i=1}^{N} a_i(x'(t))^{2\sigma} + rp(x'(t))^{2\sigma} + rpx^{2\mu}(t) \\ &\le x^2(t) + (x'(t))^2 + r\sum_{i=1}^{N} a_i(x'(t))^{2} + rp(x'(t))^{2} + rpx^{2}(t) \end{aligned}$$

$$\begin{aligned}&\le \left(1+rp+r\sum_{i=1}^{N}a_i\right)\left(x^2(t)+(x'(t))^2\right)\\&\le \left(1+rp+r\sum_{i=1}^{N}a_i\right)V(t),\end{aligned}$$

which implies that

$$V(t)\le V(0)\exp\left(\left(1+rp+r\sum_{i=1}^{N}a_i\right)t\right),\quad t\in[0,1).$$

So for $t\in[0,1)$, we get

$$\begin{aligned}x^2(t)+(x'(t))^2 &\le V(t)\\&\le V(0)\exp\left(\left(1+rp+r\sum_{i=1}^{N}a_i\right)t\right)\\&\le V(0)\exp\left(1+rp+r\sum_{i=1}^{N}a_i\right)\\&\le \left(1+r^2+r\sum_{i=1}^{N}a_i\tau_i\right)\left((x'(0))^2+||\phi||_\tau^2\right)\exp\left(1+rp+r\sum_{i=1}^{N}a_i\right)\\&\le \left(1+r^2+r\sum_{i=1}^{N}a_i\tau_i\right)\widehat{\delta}^2\exp\left(1+rp+r\sum_{i=1}^{N}a_i\right)\\&\le \varepsilon^2\exp(-2\alpha)\\&\le \varepsilon^2\exp(-2\alpha t),\end{aligned}$$

i.e.,

$$\sqrt{x^2(t)+(x'(t))^2}\le\varepsilon\exp(-\alpha t),\quad t\in[0,1).$$

In particular, we get

$$\sqrt{x^2(1^-)+(x'(1^-))^2}\le\varepsilon\exp(-\alpha).$$

Consequently, it follows that

$$\begin{aligned}\sqrt{x^2(1)+(x'(1))^2}&= d_1\sqrt{x^2(1^-)+(x'(1^-))^2}\\&\le d_1\varepsilon\exp(-\alpha)\\&\le d_1\exp(-\alpha)=\eta_1,\end{aligned}$$

which implies that for $t\in[1,2)$,

$$\sqrt{x^2(t)+(x'(t))^2}<1. \tag{2.3}$$

For $t \in [1, 2)$, we choose the Lyapunov function

$$V(t) = x^2(t) + r^2(t) \cdot (x'(t))^{2\sigma} + r \sum_{i=1}^{N} a_i \int_{t-1-\tau_i}^{0} x^{2\delta}(s)ds.$$

Then in view of (H_4) and (2.3), we can obtain that for $t \in [1, 2)$,

$$\begin{aligned}
& x^2(t) + (x'(t))^2 \leq V(t) \\
\leq\ & V(1) \exp\left(\left(1 + rp + r \sum_{i=1}^{N} a_i\right)(t-1)\right) \\
\leq\ & V(1) \exp\left(1 + rp + r \sum_{i=1}^{N} a_i\right) \\
=\ & \left(x^2(1) + r^2(1)(x'(1))^{2\sigma} + r \sum_{i=1}^{N} a_i \int_{-\tau_i}^{0} x^{2\delta}(s)ds\right) \exp\left(1 + rp + r \sum_{i=1}^{N} a_i\right) \\
\leq\ & \left\{d_1^2\left(x^2(1^-) + r^2(1)(x'(1^-))^{2\sigma}\right) + r \sum_{i=1}^{N} a_i \tau_i ||\phi||_\tau^{2\delta}\right\} \exp\left(1 + rp + r \sum_{i=1}^{N} a_i\right) \\
\leq\ & \left\{d_1^2 r^2\left(x^2(1^-) + (x'(1^-))^{2\sigma}\right) + r \sum_{i=1}^{N} a_i \tau_i ||\phi||_\tau^{2}\right\} \exp\left(1 + rp + r \sum_{i=1}^{N} a_i\right) \\
\leq\ & \left\{d_1^2 r^2 \varepsilon^2 \exp(-2\alpha) + r \sum_{i=1}^{N} a_i \tau_i \hat{\delta}^2\right\} \exp\left(1 + rp + r \sum_{i=1}^{N} a_i\right) \\
\leq\ & \left\{\left(d_1^2 r^2 + r \sum_{i=1}^{N} a_i \tau_i\right) \varepsilon^2 \exp(-2\alpha)\right\} \exp\left(1 + rp + r \sum_{i=1}^{N} a_i\right) \\
\leq\ & T\varepsilon^2 \exp(-2\alpha) \exp\left(1 + rp + r \sum_{i=1}^{N} a_i\right) \\
\leq\ & \varepsilon^2 \exp(-4\alpha) \\
\leq\ & \varepsilon^2 \exp(-2\alpha t).
\end{aligned}$$

Hence,

$$\sqrt{x^2(t) + (x'(t))^2} \leq \varepsilon \exp(-\alpha t), \quad t \in [1, 2).$$

Arguing as before by induction hypothesis, we may prove, in general, that for $k \geq 1$,

$$\sqrt{x^2(t) + (x'(t))^2} \leq \varepsilon \exp(-\alpha t), \quad t \in [k, k+1).$$

Therefore, we finally obtain

$$\sqrt{x^2(t) + (x'(t))^2} \leq \varepsilon \exp(-\alpha t), \quad t \geq 0.$$

The proof is complete. □

Remark 2.2. In Theorem 2.1, we can choose linear functions $I_k(u) = d_k u$, $J_k(v) = d_k v$. In fact, from the procedure of the proof of Theorem 2.1, it is not difficult to find that we only need $I_k(u)$, $J_k(v)$ satisfying $|I_k(u)| \le d_k|u|$, $|J_k(v)| \le d_k|v|$.

Remark 2.3. In the case of $\delta = \mu = \sigma = 1$, assume that the conditions in Theorem 2.1 still hold. Then the zero solution of (1.1) can be exponentially stabilized by periodic impulses.

Proof. Here we also choose the sequence $\{t_k\}_{k=1}^{\infty} = \{n\}_{n=1}^{\infty}$, $t_0 = 0$. Since $\delta = \rho = 1$, we only need to let

$$|I_k(u)| = d \cdot |u|, \quad |J_k(v)| = d \cdot |v|,$$

$$d = \frac{1}{r}\sqrt{T - r\sum_{i=1}^{N} a_i\tau_i}, \quad T = \exp(-2\alpha)\exp\left(-\left(1 + rp + r\sum_{i=1}^{N} a_i\right)\right),$$

$$\tilde{\delta} = \min\left\{\varepsilon, \ \frac{\varepsilon}{\sqrt{1 + r^2 + r\sum_{i=1}^{N} a_i}}\exp(-\alpha)\exp\left(-\frac{1}{2}\left(1 + rp + r\sum_{i=1}^{N} a_i\right)\right)\right\}.$$

Then the rest of the argument is the same as was employed in the proof of Theorem 2.1. Finally we can prove that each solution $x(t) = x(t, 0, \phi)$ of (1.2) with

$$\sqrt{\| \phi \|_\tau^2 + (x'(0))^2} \le \tilde{\delta}$$

also satisfies

$$\sqrt{x^2(t) + (x'(t))^2} \le \varepsilon \exp(-\alpha t), \quad t \ge 0,$$

where $\| \phi \|_\tau = \sup_{-\tau \le s \le 0} |\phi(s)|$. □

3 Applications

We shall give an example to illustrate that the unstable system can be exponentially stabilized by impulses.

Example 3.1. Consider the equation

$$\begin{cases} \left((1 + e^{-t})x'(t)\right)' + 0.125x^3(t - [t] - 0.01) \\ \quad +0.125x^3(t - [t] - 0.02) + p(t)x^4(t) = 0, \ t \ge 0, \\ x(t) = \phi(t) = -\sqrt[3]{\delta}, \ -0.02 \le t \le 0, \\ x'(0) = 1, \end{cases} \tag{3.1}$$

where δ is a positive constant, $p(t) \in \Gamma$, $\Gamma = \{s(t) \in C[0, \mathbb{R}) : |s(t)| \leq \frac{5}{4}\}$. It is not difficult to prove that when $p(t) = 0 \in \Gamma$, $t \geq 0$,

$$x'(t) = \frac{2 + 0.25\delta \cdot t}{1 + e^{-t}} \geq \frac{2 + 0.25\delta \cdot t}{2}, \quad t \geq 0.$$

So once δ is given, it is obvious that $x(t) \to \infty$, $t \to \infty$. Hence the nonimpulsive equations (3.1) is unstable for $p(t) = 0$. However, considering the effect of impulses, we may choose $\alpha = 0.1$, $\tau = 0.02$, $a_i = 0.125$, $i = 1, 2$, $p = \frac{5}{4}$, $r = 1$. Then we have $1 \leq r(t) \leq 2$. It is easy to check that

$$\tau = 0.02 < 2\exp(-4) = \left(r \sum_{i=1}^{2} a_i\right)^{-1} \exp\left(-\left(1 + rp + r\sum_{i=1}^{2} a_i\right)\right).$$

Therefore, the hypotheses in Theorem 2.1 are satisfied and hence the unstable equations (3.1) can be exponentially stabilized by impulses for all $p(t) \in \Gamma$.

Remark 3.2. In Example 3.1, we can find that the solution of (3.1) without impulses effect is unstable. However, the stability is totally controlled by the function $e^{-0.1t}$ under proper impulses effect, which shows that the impulses do contribute to the equation's stability behavior.

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