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On the Reciprocal Difference Equation with Maximum and Periodic Coefficients

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Abstract

We study the nonlinear difference equation

$$x_n = \max\left\{\frac{A_n}{x_{n-1}}, \frac{B_n}{x_{n-2k-1}}\right\}, \quad n \in \mathbb{N}_0,$$

where k is any fixed positive integer and the coefficients A_n, B_n are positive and periodic with the same period 2. The special case when $k = 1$ has been investigated earlier by Mishev, Patula and Voulov. Here we extend their results to the general case.

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1. Introduction

In this paper we investigate the positive solutions of the difference equation

$$x_n = \max\left\{\frac{A_n}{x_{n-1}}, \frac{B_n}{x_{n-2k-1}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where k is any fixed positive integer, $\mathbb{N}_0$ is the set of all nonnegative integers and the coefficients A_n and B_n are positive and periodic with period 2. We further assume that

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the initial values $x_{-1}, x_{-2}, \dots, x_{-2k-1}$ are positive, which means that all solutions of Eq. (1.1) are positive.

The difference equation (1.1) studied in this paper is motivated by a note of G. Ladas [8]. Specifically, consider the following equation, where the coefficients A_i are constants.

$$x_n = \max\left\{\frac{A_1}{x_{n-1}}, \frac{A_2}{x_{n-2}}, \dots, \frac{A_p}{x_{n-p}}\right\}, \quad n \in \mathbb{N}_0. \tag{1.2}$$

In [8], Ladas presented several conjectures about the above equation. His main conjecture is that if the coefficients $A_1, A_2, \dots, A_p$ are nonnegative real numbers, $A_1 + A_2 + \cdots + A_p > 0$, and the initial values $x_{-1}, x_{-2}, \dots, x_{-p}$ are any given positive numbers, then every solution of (1.2) is eventually periodic. Additionally, if the coefficient A_q is dominating; that is, $A_q > \max\{A_j : j \neq q\}$, then every positive solution of Eq. (1.2) is eventually periodic with period $T = 2q$. This conjecture was confirmed for $p = 2$ in [1], for $p = 3$ in [13], for some special cases when $p > 3$ in [12], and for the general case with only two nonzero coefficients in [11]. We call Eq. (1.2) the reciprocal difference equation with maximum, since it is a natural generalization of the simple reciprocal equation

$$x_n = \frac{A_q}{x_{n-q}} \tag{1.4}$$

which we obtain from Eq. (1.2) when A_q is the only nonzero coefficient. Each term of the sequence $\{x_n\}$ defined by Eq. (1.2) satisfies Eq. (1.4) for some q. However, q may depend on the previous terms $x_{n-1}, \dots, x_{n-p}$, so Eq. (1.2) works as a switch.

Further, some of the results obtained for Eq. (1.2) with constant coefficients were generalized for the case of periodic coefficients

$$x_n = \max\left\{\frac{P_n}{x_{n-1}}, \frac{Q_n}{x_{n-2}}\right\} \tag{1.5}$$

and

$$x_n = \max\left\{\frac{P_n}{x_{n-1}}, \frac{Q_n}{x_{n-3}}\right\}, \tag{1.6}$$

where the sequences $\{P_n\}$, $\{Q_n\}$ are periodic with the same period t, that is, $P_{n+t} = P_n$, $Q_{n+t} = Q_n$ for every $n \geq 0$. Eq. (1.5) with $t = 2$ was studied in [3, 14], while the case $t = 3$ was investigated in [2, 6] and, recently, some results for $t > 3$ were obtained in [4, 5, 7]. Eq. (1.6) with $t = 2$ was investigated in [9], while the case $t = 3$ was studied in [10]. Note that Eq. (1.6) is a particular case of Eq. (1.1). Our main results in this paper generalize the results from [9].

2. Preliminaries

Our goal is to study the positive solutions of Eq. (1.1), where k is an arbitrary positive integer and the coefficients A_n, B_n are positive and periodic with period 2. Since A_n and

B_n are periodic, we may assume that they are defined for $n < 0$ as well, that is

$$A_{2s} = A_0,\ A_{2s+1} = A_1,\ B_{2s} = B_0,\ B_{2s+1} = B_1 \ \text{ for every } s \in \mathbb{Z}.$$

Consider the sequence $\{c_n\}_{-2k-1}^{\infty}$ defined by the equations

$$c_{2s-2k-1} = \left(\frac{A_1}{A_0}\right)^s,\ c_{2s-2k} = A_0\left(\frac{A_0}{A_1}\right)^s \quad \text{for every } s \geq 0. \tag{2.1}$$

and observe that for every $n \geq -2k$, $m \geq 1$ we have

$$c_n c_{n-1} = A_n \quad \text{and} \quad c_{n+2m} = \left(\frac{A_n}{A_{n-1}}\right)^m c_n.$$

Then, by the substitution

$$x_n = c_n y_n, \tag{2.2}$$

Eq. (1.1) takes the form

$$y_n = \max\left\{\frac{1}{y_{n-1}}, \frac{Q_n}{y_{n-2k-1}}\right\}, \quad n \in \mathbb{N}_0, \tag{2.3}$$

where

$$Q_n = \frac{B_n}{A_n}\left(\frac{A_{n-1}}{A_n}\right)^k. \tag{2.4}$$

Note that the coefficients Q_n form a period two sequence of positive numbers. That is,

$$Q_n Q_{n+1} = Q^2, \quad \text{where} \quad Q = \sqrt{\frac{B_0 B_1}{A_0 A_1}}.$$

In the special case when $Q_1 = Q_2$, the behavior of the positive solutions of Eq. (2.3) has been studied in [11]. In the remaining case, $Q_1 \neq Q_2$, we will use the substitution

$$y_n y_{n-1} = z_n \tag{2.5}$$

suggested by Ladas [8], which transforms Eq. (2.3) into the difference equation

$$z_n = \max\left\{1, Q_n \prod_{i=1}^{k} \frac{z_{n+1-2i}}{z_{n-2i}}\right\}, \quad n \in \mathbb{N}_0. \tag{2.6}$$

It turns out that every positive solution of Eq. (2.6) must be eventually periodic. In order to prove it, we will need the following lemmas.

Lemma 2.1. Let $\{z_n\}$ be any positive solution of (2.6). Then, the following statements are true:

(i) If $n \geq 0$, then $z_n \geq 1$, $z_n z_{n-2} \dots z_{n-2k} \geq Q_n z_{n-1} \dots z_{n+1-2k}$, and $z_n z_{n-2} \dots z_{n-2k} = Q_n z_{n-1} \dots z_{n+1-2k}$ provided $z_n > 1$.

(ii) If $z_n > 1$ and $z_{n-1} > 1$, then $z_n z_{n-1-2k} = Q^2$ provided $n \geq 1$.

(iii) If $z_n > \max\{1, Q^2\}$, then $z_{n-1} = 1$ provided $n \geq 2k + 1$.

(iv) If $z_n > \max\{Q^2, 1\}$ and $z_{n-2} > 1$, then $z_n z_{n-1-2k} = z_{n-2-2k}$ provided $n \geq 2k+1$.

Proof. Eq. (2.6) immediately implies (i), and the proof of (ii) follows from (i) since $Q_n Q_{n-1} = Q^2$. If we suppose that (iii) does not hold, that is $z_n > \max\{1, Q^2\}$ and $z_{n-1} \neq 1$, for some $n \geq 2k + 1$, then by (i) we obtain $z_{n-1} > 1$ and $z_n z_{n-1-2k} \geq z_n > Q^2$, which contradicts (ii). Hence, (iii) holds. Finally, (i) and (iii) imply (iv) since $Q_n = Q_{n-2}$. ∎

Lemma 2.2. Let $\{z_n\}$ be any positive solution of (2.6) and let $s \in \mathbb{N}$. Then, the following statements are true:

(i) $z_{s+2k} \leq \max\{1\,, Q^2\,, z_{s-2}\}$.

(ii) If $z_{s-1} \leq Q^2$, then $z_{s+2k} z_{s+2k-2} \cdots z_s = Q_s z_{s+2k-1} \cdots z_{s+1}$.

(iii) If $z_s \leq Q^2$ and $z_{s-1} \leq Q^2$, then $z_s z_{s+2k+1} = Q^2$.

Proof. By reiterating Eq. (2.6) and taking into account the periodicity of Q_n, we obtain

$$z_{s+2k} = \max\left\{1\,, \frac{Q_s z_{s+2k-3} \cdots z_{s+1}}{z_{s+2k-2} \cdots z_s} \max\left\{1\,, \frac{Q_{s-1} z_{s+2k-2} \cdots z_s}{z_{s+2k-3} \cdots z_{s-1}}\right\}\right\},$$

which is equivalent to

$$z_{s+2k} = \max\left\{1, \frac{Q_s z_{s+2k-3} \cdots z_{s+1}}{z_{s+2k-2} \cdots z_s}, \frac{Q_{s-1} Q_s}{z_{s-1}}\right\}. \tag{2.7}$$

Since $Q_n Q_{n-1} = Q^2$ for any $n \in \mathbb{N}$, by Lemma 2.1(i), we obtain

$$z_{s+2k} \leq \max\left\{1\,, \frac{z_{s-2}}{z_{s-1}}\,, \frac{Q^2}{z_{s-1}}\right\} \leq \max\left\{1, z_{s-2}, Q^2\right\},$$

which completes the proof of (i).

If $z_{s-1} \leq Q^2$, then Eq. (2.7) yields

$$z_{s+2k} = \max\left\{\frac{Q_{s+2k} z_{s+2k-3} \cdots z_{s+1}}{z_{s+2k-2} \cdots z_s}\,, \frac{Q_{s+2k-1} Q_{s+2k}}{z_{s-1}}\right\},$$

and, taking into account $Q_{s+2k} = Q_s$ and Eq. (2.6) with $n = s + 2k - 1$, we obtain

$$\frac{z_{s+2k} z_{s+2k-2} \cdots z_s}{Q_s z_{s+2k-3} \cdots z_{s+1}} = \max\left\{1\,, \frac{Q_{s+2k-1} z_{s+2k-2} \cdots z_s}{z_{s+2k-3} \cdots z_{s+1} z_{s-1}}\right\} = z_{s+2k-1},$$

which completes the proof of (ii).

Finally, the proof of (iii) follows from (ii) since $Q_s Q_{s+1} = Q^2$. ■

Lemma 2.3. Let $Q_1 Q_2 = Q^2 > 1$ and let $\{z_n\}$ be any positive solution of (2.6). Assume that p is a positive integer such that

$$z_{p+(2k+2)s} > Q^2 \quad \text{for every} \quad s \in \mathbb{N}_0.$$

Then, the following statements hold:

(i) $z_{p-1+(2k+2)s} = 1$ for every $s \geq 1$.

(ii) If r and s_0 are positive integers such that

$$z_{p-m+(2k+2)s} \leq Q^2 \quad \text{provided} \quad s \geq s_0, \ \ m \in \{1, \ldots, r\}, \tag{2.8}$$

then, for every $m \in \{1, \ldots, r\}$, $s \geq s_0 + m$, we have

$$z_{p-m+(2k+2)s} = \begin{cases} 1 & \text{for } m \text{ odd} \\ Q^2 & \text{for } m \text{ even.} \end{cases}$$

Proof. The proof of (i) follows immediately by Lemma 2.1(iii).

From (i), it follows that (ii) is true for $r = 1$. Let $q > 1$ and assume that (ii) is true for $r = q - 1$ and that (2.8) holds with $r = q$. Then, by assumption, for every $m \in \{1, \ldots, q-1\}$, $s \geq s_0 + m$, we have

$$z_{p-m+(2k+2)s} = \begin{cases} 1 & \text{for } m \text{ odd} \\ Q^2 & \text{for } m \text{ even.} \end{cases}$$

On the other hand, for every $s \geq s_0$, (2.8) implies $z_{p-q+(2k+2)s} \leq Q^2$ and $z_{p-q+1+(2k+2)s} \leq Q^2$, which, by Lemma 2.2(iii), yield

$$z_{p-q+(2k+2)(s+1)} \, z_{p-(q-1)+(2k+2)s} = Q^2.$$

Therefore, (ii) is true also for $r = q$ and the proof of (ii) follows by induction. ■

Lemma 2.4. Let $Q_1 Q_2 = Q^2 \leq 1$ and let $\{z_n\}$ be any positive solution of (2.6). Let p and q be positive integers such that $p \geq 4k + 1$ and $q \leq 2k$. Assume that $z_p > 1$, $z_{p+q} > 1$ and

$$z_{p-2k-2+i} = 1 \quad \text{for every} \quad i = 1, \ldots, q-1. \tag{2.9}$$

Then, $z_{p+q} = Q^r z_{p+q-(2k+2)}$, where $r = 1 - (-1)^q$.

Proof. Since $Q^2 \leq 1$, Lemma 2.1(i) and Lemma 2.2(i) imply that

$$1 \leq z_{n+2k+2} \leq z_n \quad \text{for every} \quad n \geq 0, \tag{2.10}$$

which, in view of $p \geq 4k+1$, (2.9) and $z_{p+q} > 1$, yields

$$z_{p+i} = 1 \quad \text{for every} \quad i = 1, \ldots, q-1 \tag{2.11}$$

and $z_{p+q-(2k+2)} > 1$.

If we suppose that $z_{p+q-1-2k} \neq 1$ and take into account the inequality $p+q-2-4k \geq 0$, then, by (2.10) and Lemma 2.1(ii), we obtain that $z_{p+q-1-2k} > 1$, $z_{p+q-2-4k} \geq 1$ and $1 < z_{p+q-1-2k}\, z_{p+q-2-4k} = Q^2 \leq 1$, which is a contradiction. Hence,

$$z_{p+q-1-2k} = 1. \tag{2.12}$$

In the case $q = 2m+1 \leq 2k$, in view of (2.11), (2.12) and Lemma 2.1(i), the inequalities $z_{p+2m+1} > 1$ and $z_p > 1$ imply that

$$\begin{aligned} z_{p+2m+1} &= \frac{Q_{p+2m+1} z_{p+2m} \cdots z_{p+2m+2-2k}}{z_{p+2m-1} \cdots z_{p+2m+1-2k}} = \frac{Q_{p+1} z_p \cdots z_{p+2m+2-2k}}{z_{p-1} \cdots z_{p+2m+1-2k}} \\ &= \frac{Q_{p+1} Q_p z_{p+2m-1-2k} \cdots z_{p+1-2k}}{z_{p+2m-2k} \cdots z_{p-2k}} = \frac{Q^2 z_{p+2m-1-2k}}{z_{p+2m-2k}} \\ &= Q^2 z_{p+2m-1-2k}. \end{aligned}$$

In the case $q = 2m \leq 2k$, in view of (2.11), (2.12) and Lemma 2.1(i), the inequalities $z_{p+2m} > 1$ and $z_p > 1$ imply that

$$\begin{aligned} z_{p+2m} &= \frac{Q_{p+2m} z_{p+2m-1} \cdots z_{p+2m+1-2k}}{z_{p+2m-2} \cdots z_{p+2m-2k}} = \frac{Q_p z_{p-1} \cdots z_{p+2m+1-2k}}{z_p \cdots z_{p+2m-2k}} \\ &= \frac{z_{p+2m-2-2k} \cdots z_{p-2k}}{z_{p+2m-1-2k} \cdots z_{p+1-2k}} = \frac{z_{p+2m-2-2k}}{z_{p+2m-1-2k}} \\ &= z_{p+2m-2-2k}. \end{aligned}$$

The proof is complete. ■

Lemma 2.5. Let $Q_1 Q_2 = Q^2 \neq 1$, and let $\{z_n\}$ be any positive solution of (2.6). Consider the set

$$G = \{p \in \mathbb{N} : z_{p+(2k+2)s} > \max\{1\,,\, Q^2\} \quad \text{for every} \quad s \geq 0\}.$$

Then, the following statements are true:

(i) There exists $n_0 \in \mathbb{N}$ such that

$$z_n \leq \max\{1\,,\, Q^2\} \quad \text{provided} \quad n \geq n_0\,,\ \ n \notin G. \tag{2.13}$$

(ii) For every $m \in \mathbb{N}$, $p \in G$ implies $p + 2m - 1 \notin G$.

(iii) If $p \in G$, then there exists $m_0 \in \mathbb{N}$ such that, for every $m \geq m_0$,

$$z_{p+2m-1} = 1, \qquad z_{p+2m} z_{p+2m-2} \cdots z_{p+2m-2k} = Q_p, \tag{2.14}$$

and

$$Q_p = \max\{Q_1,\ Q_2\} > \max\{1,\ Q^{2k+2}\}. \tag{2.15}$$

(iv) $G = \emptyset$, if and only if $\max\{Q_1,\ Q_2\} \leq \max\{1,\ Q^{2k+2}\}$.

Proof.

(i) Let $D = \{r \in [1, 2k+2] \cap \mathbb{N} : r \notin G\}$. For every $r \in D$, there exists a nonnegative integer s_r such that $z_{r+(2k+2)s_r} \leq \max\{1,\ Q^{2k+2}\}$, which implies by Lemma 2.2(i) that $z_{r+(2k+2)s_r} \leq \max\{1,\ Q^{2k+2}\}$ for every $s \geq s_r$. The number $s_0 = \max\{s_r : r \in D\}$ is well defined, since the set D is finite, and we have

$$z_{r+(2k+2)s_r} \leq \max\{1,\ Q^{2k+2}\} \quad \text{provided} \quad s \geq s_0\,, r \in D.$$

Finally, for every $n \notin G$ such that $n \geq (2k+2)(s_0+1)$, there exist $r \in D$, $s \geq s_0$ such that $n = r + (2k+2)s$ and (2.13) holds with $n_0 = (2k+2)(s_0+1)$. The proof of (i) is complete.

(ii) For the sake of contradiction, suppose that there exist $m \in \mathbb{N}$ and $p \in G$ such that $p + 2m - 1 \in G$. Without loss of generality, we may assume that $(p\,, p + 2m - 1) \cap G = \emptyset$. By Lemma 2.1(iii), G cannot contain two consecutive integers. Therefore, $m > 1$. Taking into account that $n \notin G$ implies $n + (2k+2)s \notin G$ for every $s \in N$, we may also assume $p \geq n_0$, so that (2.13) yields

$$z_{p+r+(2k+2)s} \leq \max\{1\,,\ Q^2\} \ \text{ provided } \ p + r \notin G,\ s \in \mathbb{N}. \tag{2.16}$$

In the case when $Q^2 > 1$, from $p + 2m - 1 \in G$, $(p\,, p + 2m - 1) \cap G = \emptyset$ and (2.16), it follows by Lemma 2.3 that $z_{p+1+(2k+2)s} = Q^2$ for every $s \geq 2m - 1$. On the other hand, for every $s \geq 3$, we have the inequality $z_{p+2k+(2k+2)s} \geq Q^2 > 1$, which follows from $p+2k+2 \in G$ and (2.16) by Lemma 2.3 provided $p+2k \notin G$, or by the definition of G otherwise. Then, for every $s \geq 2m - 1$, Lemma 2.1(iv) implies

$$z_{p+(2k+2)(s+1)} z_{p+1+(2k+2)s} = z_{p+(2k+2)s},$$

which yields $z_{p+(2k+2)(s+1)} Q^2 = z_{p+(2k+2)s}$ and, by induction,

$$z_{p+(2k+2)(s+q)} Q^{2q} = z_{p+(2k+2)s} \quad \text{provided} \quad s \geq 2m - 1,\ q \in \mathbb{N}.$$

Finally, for any fixed $s \geq 2m - 1$, Lemma 2.1(i) implies

$$Q^{2q} \leq z_{p+(2k+2)s} \quad \text{for every} \quad q \in \mathbb{N},$$

which contradicts $Q^2 > 1$.

It remains to consider the case when $Q^2 < 1$. Then, for every $s \in \mathbb{N}$, we have $z_{p+(2k+2)s} > 1$, $z_{p+2m-1+(2k+2)s} > 1$ and, in view of (2.16),

$$z_{p+r+(2k+2)s} = 1 \quad \text{provided } r \in [1, 2m-2] \cap \mathbb{N}.$$

Therefore, by Lemma 2.4,

$$z_{p+2m-1+(2k+2)(s+1)} = Q^2 z_{p+2m-1+(2k+2)s} \quad \text{for every} \quad s \in \mathbb{N},$$

and, by induction, we obtain that, for every $q \in \mathbb{N}$,

$$z_{p+2m-1+(2k+2)(s+q)} = Q^{2q} z_{p+2m-1+(2k+2)s} \quad \text{provided} \quad s \in \mathbb{N}.$$

Finally, for any fixed $s \in \mathbb{N}$, Lemma 2.1(i) implies

$$1 \leq Q^{2q} z_{p+(2k+2)s} \quad \text{for every} \quad q \in \mathbb{N},$$

which contradicts $Q^2 < 1$. The proof of (ii) is complete.

(iii) Let $G \neq \emptyset$ and $p \in G$. In view of (i), there exist positive integers n_0 and m_1 such that (2.13) holds and $p + 2m - 1 \geq n_0$ provided $m \geq m_1$. For every $m \geq m_1$, it follows from (ii) that $p + 2m - 1 \notin G$ and there exists a positive integer $q \geq m$ such that $p + 2q \in G$ and $(p + 2m - 1, p + 2q) \cap G = \emptyset$. Then, $m \leq q \leq k + 1$ and, taking into account (2.13), we obtain the relations

$$z_{p+2m-1+(2k+2)(2k+1)} = 1 \quad \text{provided} \quad m \geq m_1 \tag{2.17}$$

and

$$z_{p+2m+(2k+2)2k} \geq \max\{1\,,\, Q^2\} \quad \text{provided} \quad m \geq m_1, \tag{2.18}$$

which follow by Lemma 2.3 if $Q^2 > 1$ or by Lemma 2.1 otherwise. Therefore, for every $m \geq m_2 = m_1 + (k+1)(2k+1)$, (2.17) and (2.18) imply

$$z_{p+2m-1} = 1 \tag{2.19}$$

and

$$z_{p+2m} \geq \max\{1\,,\, Q^2\}. \tag{2.20}$$

Then, by Lemma 2.2(i), we obtain that $z_{p+2m+2} \leq z_{p+2m-2k}$ provided $m \geq m_2 + k$. Hence, the product $P(m) = z_{p+2m} z_{p+2m-2} \ldots z_{p+2m-2k}$ is a nonincreasing function of m. On the other hand, there exists an integer s_0 such that $(k+1)s_0 \geq m_2 + k$. Since $p \in G$ implies that $z_{p+(2k+2)s} > \max\{1\,,\, Q^2\}$ for every $s \geq s_0$, (2.20) yields

$$P((k+1)s) > \max\{1\,,\, Q^{2k+2}\}$$

and, in view of Lemma 2.1(i) and (2.19), we obtain

$$P((k+1)s) = Q_p z_{p-1+(2k+2)s} z_{p-3+(2k+2)s} \ldots z_{p+(2k+2)s-2k+1} = Q_p.$$

Hence, $Q_p > \max\{1\,, Q^{2k+2}\}$, which implies (2.15). Finally, for every $m \geq (k+1)s_0$ there exists $s \in \mathbb{N}$ such that $(k+1)s \geq m \geq (k+1)s_0$ and, taking into account the monotonicity of $P(m)$, we have

$$Q_p = P((k+1)s) \leq P(m) \leq P((k+1)s_0) = Q_p.$$

Therefore, (2.14) holds for $m \geq (k+1)s_0 \geq m_2 + k$. Since $m_0 = (k+1)s_0$ has the desired property, the proof of (iii) is complete.

(iv) Let $G = \emptyset$. If $Q^2 < 1$, by Lemma 2.2(i), it follows that eventually $z_n = 1$ and, by Lemma 2.1(i), we have $1 \geq Q_n$ for every n. In the case when $Q^2 > 1$, from $G = \emptyset$, by Lemma 2.2, we obtain that eventually $z_{s+2k}z_{s+2k-2}\ldots z_s = Q_s z_{s+2k-1}\ldots z_{s+1}$ and $z_s z_{s+2k+1} = Q^2$. Hence, $Q^{2(2k+1)} \geq z_{s+4k}z_{s+4k-2}\ldots z_s = Q_s Q^{2k}$, which yields $Q^{2k+2} \geq Q_s$. Thus, $G = \emptyset$ implies $\max\{Q_1,\ Q_2\} \leq \max\{1,\ Q^{2k+2}\}$.

Let $\max\{Q_1,\ Q_2\} \leq \max\{1,\ Q^{2k+2}\}$. If we suppose that $G \neq \emptyset$, then by (iii) we obtain (2.15) which contradicts our assumption. The proof is complete. ■

3. Main Results

In this section we present our main results about the equations (2.6), (2.3) and (1.1). Recall that a sequence $\{x_n\}$ is said to be eventually periodic of period T if and only if it eventually becomes periodic of period T, that is, there exists an integer n_0 such that

$$x_{n+T} = x_n \quad \text{for every} \quad n \geq n_0.$$

The minimal number t which may be a period of an eventually periodic sequence x_n is called the prime period of $\{x_n\}$.

The next theorem describes the behavior of all positive solutions of Eq. (2.6) in each one of three mutually exclusive and exhaustive cases, provided $Q_1Q_2 \neq 1$.

Theorem 3.1. Let $Q_1Q_2 = Q^2 \neq 1$. Let $\{z_n\}$ be any positive solution of (2.6). Then the following statements are true:

(i) If $\max\{Q_1,\ Q_2\} \leq 1$, then eventually $z_n = 1$.

(ii) If $1 < \max\{Q_1,\ Q_2\} \leq Q^{2k+2}$, then eventually $z_n z_{n+2k+1} = Q^2$.

(iii) If $\max\{1,\ Q^{2k+2}\} < \max\{Q_1,\ Q_2\} = Q_p$, then eventually $z_{p+2n+1} = 1$ and $z_{p+2n}z_{p+2n-2}\ldots z_{p+2n-2k} = Q_p$.

(iv) $\{z_n\}$ is eventually periodic with (not necessarily prime) period T, where

$$T = \begin{cases} 1 & \text{when } \max\{Q_1,\ Q_2\} \leq 1 \\ 4k+2 & \text{when } 1 < \max\{Q_1,\ Q_2\} \leq Q^{2k+2} \\ 2k+2 & \text{when } \max\{1,\ Q^{2k+2}\} < \max\{Q_1,\ Q_2\}. \end{cases}$$

Proof. If $\max\{1,\ Q^{2k+2}\} \geq \max\{Q_1,\ Q_2\}$, then by Lemma 2.1(i) and Lemma 2.5(i), (iv) we obtain that there exists n_0 such that

$$1 \leq z_n \leq \max\{1\,,\,Q^2\} \quad \text{provided} \quad n \geq n_0,$$

from which (i) follows immediately, while (ii) follows by Lemma 2.2(iii). The proof of (iii) follows from Lemma 2.5(iv), (iii). Finally, the proof of (iv) follows from (i), (ii) and (iii). ■

In the remaining case when $Q_1 Q_2 = 1$, the positive solutions of Eq. (2.6) are eventually periodic with period $2k+2$, which is the same as in Theorem 3.1(iii), but we no longer have the associated identities.

Theorem 3.2. Let $Q_1 Q_2 = 1$. Let $\{z_n\}$ be any positive solution of Eq. (2.6). Then $\{z_n\}$ is eventually periodic with (not necessarily prime) period T, where

$$T = \begin{cases} 1 & \text{when } Q_1 = Q_2 = 1 \text{ and } k = 1 \\ 3 & \text{when } Q_1 = Q_2 = 1 \text{ and } k = 2 \\ 2k+2 & \text{otherwise.} \end{cases}$$

Additionally, there exist prime period T solutions of Eq. (2.6).

Proof. Since $Q_1 Q_2 = Q^2 = 1$, for every fixed $r \in \{1, 2, \ldots, 2k+2\}$, Lemma 2.2(i) implies that the sequence $z_{r+(2k+2)s}$ is nonincreasing. In view of Lemma 2.1(i), if $z_{r+(2k+2)s_r} = 1$ for some $s_r \in \mathbb{N}$, then $z_{r+(2k+2)s} = 1$ for every $s \geq s_r$. Consider the following sets:

$$G = \{r \in \mathbb{N} : z_{r+(2k+2)s} > 1 \quad \text{for every} \quad s \in \mathbb{N}\} \tag{3.1}$$

and

$$D = \{r \in \{1, 2, \ldots, 2k+2\} : r \notin G\}. \tag{3.2}$$

Since the set D is finite, there exists a positive integer s_0 such that, for every $s \geq s_0$,

$$z_{r+(2k+2)s} = 1 \quad \text{provided} \quad r \in D, \quad s \geq s_0.$$

Then, by Lemma 2.4, it follows from $Q^2 = 1$ that

$$z_{r+(2k+2)(s+1)} = z_{r+(2k+2)s} \quad \text{provided} \quad r \in G, \quad s \geq s_0 + 2.$$

Therefore, for every $n \geq (2k+2)(s_0+3)$ we have $z_{n+2k+2} = z_n$.

It remains to consider the case when $k \in \{1\,,2\}$ and $Q_1 = Q_2 = 1$. Since the case $G = \emptyset$ is trivial, we may assume without loss of generality that $p \in G$ for some $p \geq 2k+1$. Then, in view of Lemma 2.1(i), (ii), we obtain $z_p = a > 1 = z_{p-1} = z_{p+1}$. For $k = 1$, this contradicts Eq. (2.6) and so we conclude that $k = 2$. Finally, Eq. (2.3) yields $z_{p+2} = 1$ and $z_{p+3} = a = z_n$. Hence, by induction, $z_{n+3} = z_n$ provided $n \geq p - 1$.

The existence of prime period T solutions of Eq. (2.6) follows by choosing the initial conditions $z_{-1} = \cdots = z_{-2k} = 1$ provided $Q_1 \neq Q_2$, and, alternatively, $z_{-2} = a > 1$ and $z_{-n} = 1$ otherwise, provided $Q_1 = Q_2 = 1$. The proof is complete. ■

Theorem 3.3. Let $Q_1 Q_2 = 1$ and let $\{y_n\}$ be any positive solution of Eq. (2.3). Then, eventually, $y_{n+2k+2} = Q_n y_n$. In the special case, when $Q_1 = Q_2 = k = 1$, we also have $y_{n+2} = y_n$, eventually.

Proof. Since the sequence $\{z_n\}$ defined by the equation

$$z_n = y_n y_{n-1} \tag{3.3}$$

is a positive solution of Eq. (2.6) and $Q_1 Q_2 = Q^2 = 1$, it will be convenient to consider the sets G and D, defined above by (3.1) and (3.2), respectively. Recall that there exists $s_0 \in \mathbb{N}$ such that

$$z_{r+(2k+2)s} = 1 \quad \text{provided} \quad r \in D, \quad s \geq s_0.$$

Therefore, by setting $n_0 = (2k+2)(s_0+1)$, we obtain

$$z_n = 1 \quad \text{provided} \quad n \notin G, \quad n \geq n_0. \tag{3.4}$$

If $G = \emptyset$, then we have $z_n = y_n y_{n-1} = 1$ for every $n \geq n_0$, and, hence, $y_n = y_{n+2} = \cdots = y_{n+2k+2}$. On the other hand, taking into account that $z_n = 1$ for every $n \geq n_0$, from Eq. (2.6) we obtain that $\max\{Q_1, Q_2\} \leq 1$, which implies $Q_1 = Q_2 = 1$ and the proof follows. Note that in the special case, when $Q_1 = Q_2 = k = 1$, by reiterating Eq. (2.6), we obtain $z_n = 1$ for every $n \geq 4$ and, hence, $G = \emptyset$.

So we may assume without loss of generality that $G \neq \emptyset$. Obviously, $r \in G$ implies $r + 2k + 2 \in G$. For every $r \in G$ such that $r \geq 2k$, it follows that $z_{r+1} = 1$ and $z_{r+2k+3} = 1$, since otherwise we obtain a contradiction by Lemma 2.1(iii). Moreover, from $z_{r+2k+2} > 1$ and $z_{r+1} = 1$, taking into account (3.3) and (2.3), we obtain

$$y_{r+2k+2} = \frac{Q_r}{y_{r+1}} = Q_r y_r,$$

which, in view of $z_{r+2k+3} = y_{r+2k+3} y_{r+2k+2} = 1$ and $Q_r Q_{r+1} = 1$, yields

$$y_{r+2k+3} = \frac{1}{y_{r+2k+2}} = \frac{1}{Q_r y_r} = Q_{r+1} y_{r+1}.$$

Hence,

$$y_{n+2k+2} = Q_n y_n \quad \text{provided} \quad \{n, n-1\} \cap G \neq \emptyset, \quad n \geq 2k+1. \tag{3.5}$$

Now we are in position to prove that

$$y_{n+2k+2} = Q_n y_n \quad \text{provided} \quad n \geq 2k + n_0. \tag{3.6}$$

Let $n \geq 2k+n_0$. Since (3.6) follows immediately from (3.5) provided $\{n, n-1\} \cap G \neq \emptyset$, we may assume without loss of generality that $\{n, n-1\} \cap G = \emptyset$. Then, there exists a positive even integer $2p \leq 2k$ such that $(n-2p, n] \cap G = \emptyset$ and $\{n-2p, n-2p-1\} \cap G \neq \emptyset$. Therefore, (3.5) implies that

$$y_{n-2p+2k+2} = Q_n y_{n-2p}.$$

On the other hand, from $(n-2p, n] \cap G = \emptyset$, taking into account (3.4) and the inequalities $n-2p \geq n-2k \geq n_0$, it follows that $z_{n-2p+q} = 1$ for every $q \in \{1, \ldots, 2p\}$, which implies

$$y_{n-2p} = y_n.$$

By Lemma 2.2(i), we also have $(n-2p+2k+2, n+2k+2] \cap G = \emptyset$ and, in a similar fashion, we obtain

$$y_{n-2p+2k+2} = y_{n+2k+2}.$$

Hence, (3.6) holds. The proof is complete. ■

Theorem 3.4. Let $\{x_n\}$ be any positive solution of Eq. (1.1). Let $\{Q_n\}$ be the period 2 sequence defined by (2.4) and denote $Q = \sqrt{Q_1 Q_2}$. Let $T = 2k+2$ when $k > 1 = Q$, and $T = 2$ otherwise. Then, the following statements are true:

(i) If $\max\{Q_1, Q_2\} \leq 1$, then $\{x_n\}$ is eventually periodic with period T provided $A_1 = A_2$, and $\{x_n\}$ is unbounded otherwise.

(ii) If $1 < \max\{Q_1, Q_2\} \leq Q^{2k+2}$, then $\{x_n\}$ is eventually periodic with period 4k+2 provided $B_1 = B_2$, and $\{x_n\}$ is unbounded otherwise.

(iii) If $\max\{1, Q^{2k+2}\} < \max\{Q_1, Q_2\} = Q_p$, for some p, then $\{x_n\}$ is eventually periodic with period 2k+2 provided $B_p = A_{p+1}$, and $\{x_n\}$ is unbounded otherwise.

Proof. Recall that $x_n = c_n y_n$, where the sequence $\{c_n\}$ is defined by (2.1), $\{y_n\}$ is a positive solution of Eq. (2.3) and the sequence $\{z_n\}$, defined by $z_n = y_n y_{n-1}$, is a positive solution of Eq. (2.6). Note that

$$\frac{c_{n+2m}}{c_n} = \left(\frac{A_n}{A_{n-1}}\right)^m \quad \text{for every } n, m \in \mathbb{N}. \tag{3.7}$$

(i) If $Q_1 Q_2 \neq 1$, then by Theorem 3.1, it follows that eventually $z_n = 1$, which implies

$$y_{n+2} = y_n. \tag{3.8}$$

If $Q_1 Q_2 = 1$, then we have $Q_1 = Q_2 = 1$. By Theorem 3.3, we obtain that (3.8) holds eventually, provided $k = 1$, and in the remaining case, when $k > 1 = Q_1 = Q_2$, we have that eventually

$$y_{n+2k+2} = y_n. \tag{3.9}$$

Therefore, in the case when $Q = 1 < k$, from (3.9) and (3.7), we obtain that eventually

$$\frac{x_{n+2k+2}}{x_n} = \frac{c_{n+2k+2}y_{n+2k+2}}{c_n y_n} = \left(\frac{A_n}{A_{n-1}}\right)^{k+1} \tag{3.10}$$

and, otherwise, (3.8) and (3.7) imply that eventually

$$\frac{x_{n+2}}{x_n} = \frac{c_{n+2}y_{n+2}}{c_n y_n} = \frac{A_n}{A_{n-1}}. \tag{3.11}$$

Hence, the proof of (i) follows from (3.10) and (3.11).

(ii) In this case, by Theorem 3.1(ii) , we have that $Q^2 = z_n z_{n+2k+1}$, which implies by Lemma 2.1(i) that $z_n \le Q^2$, eventually. Therefore, by Lemma 2.2(ii) and taking into account that $z_n = y_n y_{n-1}$, we obtain the equation $y_{n+2k}y_{n-1} = Q_n$, which yields

$$\frac{y_{n+4k+2}}{y_n} = \frac{y_{n+4k+2}y_{n+2k+1}}{y_{n+2k+1}y_n} = \frac{Q_n}{Q_{n+1}}. \tag{3.12}$$

Hence, in view of (3.7), it follows that eventually

$$\frac{x_{n+4k+2}}{x_n} = \frac{c_{n+4k+2}y_{n+4k+2}}{c_n y_n} = \left(\frac{A_n}{A_{n-1}}\right)^{k+1} \frac{Q_n}{Q_{n+1}} = \frac{B_n}{B_{n-1}},$$

and the proof of (ii) is complete.

(iii) In the case when $Q_1 Q_2 \ne 1$, by Theorem 3.1, we have that eventually $z_{p+2n+1} = 1$ and $z_{p+2n}z_{p+2n-2}\dots z_{p+2n-2k} = Q_p$, which imply

$$y_{p+2n+1}y_{p+2n} = 1 \quad \text{and} \quad y_{p+2n}y_{p+2n-2k-1} = Q_p. \tag{3.13}$$

Hence, there exists n_0 such that, for every $n \ge n_0$, we have

$$y_{p+2n} = Q_p y_{p+2n-2k-2} \quad \text{and} \quad y_{p+2n+1} = Q_p^{-1} y_{p+2n-2k-1}. \tag{3.14}$$

In the remaining case when $Q_1 Q_2 = 1$, (3.14) follows by Theorem 3.3.

Finally, from (3.14) and (3.7), for every $n \ge n_0$, we obtain

$$\frac{x_{p+2n+2k+2}}{x_{p+2n}} = \frac{c_{p+2n+2k+2}y_{p+2n+2k+2}}{c_{p+2n}y_{p+2n}} = \left(\frac{A_p}{A_{p-1}}\right)^{k+1} Q_p = \frac{B_p}{A_{p-1}}$$

and

$$\frac{x_{p+2n+2k+3}}{x_{p+2n+1}} = \frac{c_{p+2n+2k+3}y_{p+2n+2k+3}}{c_{p+2n+1}y_{p+2n+1}} = \left(\frac{A_{p+1}}{A_p}\right)^{k+1} \frac{1}{Q_p} = \frac{A_{p-1}}{B_p},$$

which completes the proof. ■

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