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On the Sign of Green's Function for Second Order Impulsive Difference Equations

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Abstract

In this paper, we investigate Green's function of second order linear difference equations subject to linear impulse conditions and separated linear boundary conditions. Sufficient conditions that ensure the positiveness of the Green function are established.

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1. Introduction

Differential equations with impulses are a basic tool to study processes that are subjected to abrupt changes in their state. There has been a significant development in the last two decades [3, 4, 8, 13]. Impulsive difference equations have started to be considered quite recently [6, 9, 14–16]. Let $\mathbb{Z}$ denote the set of all integers. For any $l, m \in \mathbb{Z}$ with $l \leq m$, $[l, m]$ will denote the *discrete interval* being the set $\{l, l+1, \ldots, m\}$. Semi-infinite intervals of the form $(-\infty, l]$ and $[l, \infty)$ will denote the discrete sets $\{\ldots, l-2, l-1, l\}$ and $\{l, l+1, l+2, \ldots\}$, respectively. Throughout the paper all intervals will be discrete intervals.

In this paper, we investigate the Green function of the following boundary value problem with impulse (BVPI):

$$-\Delta[p(n-1)\Delta y(n-1)] + q(n)y(n) = h(n), \quad n \in [a, c-1] \cup [c+2, b], \tag{1.1}$$

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$$y(c-1) = d_1 y(c+1), \quad y^{[\Delta]}(c-1) = d_2 y^{[\Delta]}(c+1), \tag{1.2}$$

$$\alpha y(a-1) - \beta y^{[\Delta]}(a-1) = 0, \quad \gamma y(b) + \delta y^{[\Delta]}(b) = 0, \tag{1.3}$$

where $y(n)$ is a desired solution defined for $n \in [a-1, b+1]$; Δ denotes the forward difference operator defined by

$$\Delta y(n) = y(n+1) - y(n);$$

$y^{[\Delta]}(n) = p(n)\Delta y(n)$ denotes the quasi Δ-derivative of $y(n)$; $a, b, c \in \mathbb{Z}$, $a < c < b$, $b - c \geq 2$; the coefficients $p(n)$, $q(n)$ are real-valued functions defined on $[a-1, c-1] \cup [c+1, b]$ and $[a, c-1] \cup [c+2, b]$, respectively, and $p(n) \neq 0$ for all n; $h(n)$ is a real-valued function defined on $[a, c-1] \cup [c+2, b]$; $d_1, d_2, \alpha, \beta, \gamma$, and δ are given real numbers and $d_1 \neq 0$, $d_2 \neq 0$, $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$.

The conditions in (1.3) are boundary conditions, whereas the conditions in (1.2) can be regarded as an impulse phenomenon at c.

In Section 2, we consider the second order linear homogeneous difference equation with impulse

$$-\Delta[p(n-1)\Delta y(n-1)] + q(n)y(n) = 0, \quad n \in (-\infty, c-1] \cup [c+2, \infty),$$

$$y(c-1) = d_1 y(c+1), \quad y^{[\Delta]}(c-1) = d_2 y^{[\Delta]}(c+1).$$

Here, a uniqueness and existence theorem is presented. Next, basic properties of solutions are given and a variation of constants formula for the corresponding nonhomogeneous equation is obtained.

In Section 3, Green's function of the BVPI (1.1)–(1.3) is constructed.

Finally, in Section 4, sign properties of the Green function are investigated.

Note that the problem (1.1)–(1.3) is a discrete analogue of the continuous impulsive BVP considered in [5]. Our preliminary analysis (see below the proof of Theorem 2.1) showed that impulse conditions of the form (1.2) are more adequate for the case of second order difference equations. Sign properties of Green's functions in the case when there is no impulse were investigated for differential equations in [10–12] and for difference equations in [1,2] and other articles.

2. Second Order Linear Difference Equations with Impulse

Let c be an integer and d_1, d_2 be nonzero real numbers. Consider the second order linear homogeneous difference equation with impulse

$$-\Delta[p(n-1)\Delta y(n-1)] + q(n)y(n) = 0, \quad n \in (-\infty, c-1] \cup [c+2, \infty), \tag{2.1}$$

$$y(c-1) = d_1 y(c+1), \quad y^{[\Delta]}(c-1) = d_2 y^{[\Delta]}(c+1), \tag{2.2}$$

where $y(n)$ is a desired solution defined for $n \in \mathbb{Z}$, the coefficients $p(n)$, $q(n)$ are real-valued functions defined on $(-\infty, c-1] \cup [c+1, \infty)$ and $(-\infty, c-1] \cup [c+2, \infty)$, respectively, and $p(n) \neq 0$ for all n.

Using the definition of Δ-derivative we can rewrite problem (2.1), (2.2) in the form

$$\begin{aligned} &-p(n-1)y(n-1)+q_1(n)y(n)-p(n)y(n+1)=0, \\ &n\in(-\infty,c-1]\cup[c+2,\infty), \end{aligned} \tag{2.3}$$

$$\begin{aligned} y(c-1)&=d_1y(c+1), \\ p(c-1)[y(c)-y(c-1)]&=d_2p(c+1)[y(c+2)-y(c+1)], \end{aligned} \tag{2.4}$$

where

$$q_1(n)=q(n)+p(n-1)+p(n), \quad n\in(-\infty,c-1]\cup[c+2,\infty).$$

Theorem 2.1. Let n_0 be a fixed point in $\mathbb{Z}$ and c_0, c_1 be given real numbers. Then problem (2.1), (2.2) has a unique solution $y(n)$, $n\in\mathbb{Z}$, such that

$$y(n_0)=c_0, \quad y^{[\Delta]}(n_0)=c_1. \tag{2.5}$$

Proof. First assume that $n_0\in(-\infty,c-1]$. Solving equation (2.3) on $(-\infty,c-1]$ under the initial conditions (2.5), we find $y(n)$ uniquely for $n\in(-\infty,c]$. Then we find $y(c+1)$ and $y(c+2)$ uniquely from (2.4) and then we solve equation (2.3) uniquely on $[c+2,\infty)$.

Let now $n_0\in[c+1,\infty)$. Solving equation (2.3) on $[c+2,\infty)$ subject to the initial conditions (2.5), we find $y(n)$ uniquely for $n\in[c+1,\infty)$. Then we find $y(c-1)$ and $y(c)$ uniquely from (2.4) and then we solve equation (2.3) uniquely on $(-\infty,c-1]$.

Finally, if $n_0=c$, then we find $y(c)$ and $y(c+1)$ uniquely from the initial conditions (2.5). Then we find $y(c-1)$ and $y(c+2)$ from the impulse conditions (2.4). Next, solving equation (2.3) at first on $(-\infty,c-1]$, we find $y(n)$ uniquely for $n\in(-\infty,c-2]$ and then solving (2.3) on $[c+2,\infty)$, we find $y(n)$ uniquely for $n\in[c+3,\infty)$. ■

Definition 2.2. For two functions $y,z:\mathbb{Z}\to\mathbb{R}$ we define their Wronskian by

$$\begin{aligned} W_n(y,z) &= y(n)z^{[\Delta]}(n)-y^{[\Delta]}(n)z(n) \\ &= p(n)[y(n)z(n+1)-y(n+1)z(n)], \quad n\in\mathbb{Z}. \end{aligned}$$

Theorem 2.3. The Wronskian of any two solutions $y(n)$ and $z(n)$ of problem (2.1), (2.2) is constant on each of the intervals $(-\infty,c-1]$ and $[c+1,\infty)$:

$$W_n(y,z)=\begin{cases} \omega^-, & n\in(-\infty,c-1], \\ \omega^+, & n\in[c+1,\infty). \end{cases} \tag{2.6}$$

In addition,

$$\omega^-=d_1d_2\omega^+ \tag{2.7}$$

and

$$W_c(y,z)=-\frac{p(c)}{p(c-1)}d_2\omega^+. \tag{2.8}$$

Proof. Suppose that $y(n)$ and $z(n)$, where $n \in \mathbb{Z}$, are solutions of (2.1), (2.2). Let us compute the Δ-derivative of $W_n(y,z)$. Using the product rule for Δ-derivative

$$\begin{aligned}\Delta[f(n)g(n)] &= [\Delta f(n)]g(n) + f(n+1)\Delta g(n)\\ &= f(n)\Delta g(n) + [\Delta f(n)]g(n+1),\end{aligned}$$

we have

$$\begin{aligned}\Delta W_n(y,z) &= \Delta\left[y(n)z^{[\Delta]}(n) - y^{[\Delta]}(n)z(n)\right]\\ &= [\Delta y(n)]z^{[\Delta]}(n) + y(n+1)\Delta z^{[\Delta]}(n)\\ &\quad -y^{[\Delta]}(n)\Delta z(n) - [\Delta y^{[\Delta]}(n)]z(n+1)\\ &= y(n+1)\Delta z^{[\Delta]}(n) - [\Delta y^{[\Delta]}(n)]z(n+1).\end{aligned}$$

Further, since $y(n)$ and $z(n)$ are solutions of (2.1), (2.2),

$$\Delta y^{[\Delta]}(n) = q(n+1)y(n+1), \quad n \in (-\infty, c-2] \cup [c+1, \infty),$$

$$\Delta z^{[\Delta]}(n) = q(n+1)z(n+1), \quad n \in (-\infty, c-2] \cup [c+1, \infty).$$

Therefore

$$\Delta W_n(y,z) = 0 \quad \text{for} \quad n \in (-\infty, c-2] \cup [c+1, \infty).$$

The latter implies that $W_n(y,z)$ is constant on $(-\infty, c-1]$ and on $[c+1, \infty)$. Thus we have (2.6), where ω^- and ω^+ are some constants (depending on the solutions $y(n)$ and $z(n)$).

Next using (2.6) and the impulse conditions in (2.2) for $y(n)$ and $z(n)$, we have

$$\begin{aligned}\omega^- &= W_{c-1}(y,z)\\ &= y(c-1)z^{[\Delta]}(c-1) - y^{[\Delta]}(c-1)z(c-1)\\ &= d_1 d_2[y(c+1)z^{[\Delta]}(c+1) - y^{[\Delta]}(c+1)z(c+1)]\\ &= d_1 d_2 W_{c+1}(y,z)\\ &= d_1 d_2 \omega^+\end{aligned}$$

so that (2.7) is established.

Finally, from the impulse conditions in (2.2), we find that

$$y(c) = \left[d_1 - d_2\frac{p(c+1)}{p(c-1)}\right]y(c+1) + d_2\frac{p(c+1)}{p(c-1)}y(c+2). \tag{2.9}$$

Substituting this expression for $y(c)$ and $z(c)$ into

$$W_c(y,z) = p(c)[y(c)z(c+1) - y(c+1)z(c)],$$

we get

$$W_c(y,z) = -d_2\frac{p(c)}{p(c-1)}W_{c+1}(y,z) = -d_2\frac{p(c)}{p(c-1)}\omega^+.$$

Therefore (2.8) is also proved. ■

Corollary 2.4. If $y(n)$ and $z(n)$ are two solutions of (2.1), (2.2), then either $W_n(y,z) = 0$ for all $n \in \mathbb{Z}$ or $W_n(y,z) \neq 0$ for all $n \in \mathbb{Z}$.

By using Theorem 2.1, the following two theorems can be proved in exactly the same way when equation (2.1) does not include any impulse conditions [7].

Theorem 2.5. Any two solutions of (2.1), (2.2) are linearly independent if and only if their Wronskian is not zero.

Theorem 2.6. Problem (2.1), (2.2) has two linearly independent solutions and every solution of (2.1), (2.2) is a linear combination of these solutions.

We say that $y_1(n)$ and $y_2(n)$ form a *fundamental set* of solutions for (2.1), (2.2) provided that they are solutions of (2.1), (2.2) and their Wronskian is not zero.

Let us consider the nonhomogeneous equation

$$-\Delta[p(n-1)\Delta y(n-1)] + q(n)y(n) = h(n), \quad n \in (-\infty, c-1] \cup [c+2, \infty), \tag{2.10}$$

with the impulse conditions

$$y(c-1) = d_1 y(c+1), \quad y^{[\Delta]}(c-1) = d_2 y^{[\Delta]}(c+1), \tag{2.11}$$

where $h(n)$ is a real-valued function defined on $(-\infty, c-1] \cup [c+2, \infty)$. We will extend the function $h(n)$ to the points $n = c$ and $n = c+1$ by setting

$$h(c) = h(c+1) = 0. \tag{2.12}$$

Theorem 2.7. Suppose that $y_1(n)$ and $y_2(n)$ form a fundamental set of solutions of the homogeneous problem (2.1), (2.2). Then a general solution of the corresponding nonhomogeneous problem (2.10), (2.11) is given by

$$y(n) = c_1 y_1(n) + c_2 y_2(n) + y_0(n), \quad n \in \mathbb{Z},$$

where c_1, c_2 are arbitrary constants and

$$y_0(n) = \begin{cases} -\displaystyle\sum_{s=n}^{c} \frac{y_1(n)y_2(s) - y_1(s)y_2(n)}{W_s(y_1, y_2)} h(s), & n \le c, \\ \displaystyle\sum_{s=c+1}^{n} \frac{y_1(n)y_2(s) - y_1(s)y_2(n)}{W_s(y_1, y_2)} h(s), & n \ge c+1. \end{cases} \tag{2.13}$$

Proof. Taking into account (2.12), it is not difficult to verify that the function $y_0(n)$ defined by (2.13) is a particular solution of (2.10), (2.11), namely, $y_0(n)$ satisfies the equation (2.10) and conditions

$$y_0(c-1) = y_0^{[\Delta]}(c-1) = 0, \quad y_0(c+1) = y_0^{[\Delta]}(c+1) = 0.$$

This implies that the statement of the theorem is true. ■

3. Green's Function

Let $a, b, c \in \mathbb{Z}$ be points with $a < c < b$ and $b - c \geq 2$. Consider the following linear boundary value problem with impulse (BVPI):

$$-\Delta[p(n-1)\Delta y(n-1)] + q(n)y(n) = h(n), \quad n \in [a, c-1] \cup [c+2, b], \tag{3.1}$$

$$y(c-1) = d_1 y(c+1), \quad y^{[\Delta]}(c-1) = d_2 y^{[\Delta]}(c+1), \tag{3.2}$$

$$\alpha y(a-1) - \beta y^{[\Delta]}(a-1) = 0, \quad \gamma y(b) + \delta y^{[\Delta]}(b) = 0, \tag{3.3}$$

where $y(n)$ is a desired solution defined for $n \in [a-1, b+1]$; $p(n)$ and $q(n)$, $h(n)$ are real-valued functions defined on $[a-1, c-1] \cup c+1, b]$ and $[a, c-1] \cup [c+2, b]$, respectively, $p(n) \neq 0$ for all n; d_1, d_2 and α, β, γ, δ are given real numbers with $d_1 \neq 0$, $d_2 \neq 0$ and $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$, respectively.

Denote by $\varphi(n)$ and $\psi(n)$ the solutions of the homogeneous problem

$$-\Delta[p(n-1)\Delta y(n-1)] + q(n)y(n) = 0, \quad n \in [a, c-1] \cup [c+2, b], \tag{3.4}$$

$$y(c-1) = d_1 y(c+1), \quad y^{[\Delta]}(c-1) = d_2 y^{[\Delta]}(c+1), \tag{3.5}$$

satisfying the initial conditions

$$\varphi(a-1) = \beta, \quad \varphi^{[\Delta]}(a-1) = \alpha \tag{3.6}$$

and

$$\psi(b) = \delta, \quad \psi^{[\Delta]}(b) = -\gamma, \tag{3.7}$$

respectively. Therefore the first condition in (3.3) is satisfied by $\varphi(n)$, and the second condition is satisfied by $\psi(n)$.

By Theorem 2.3 and the conditions (3.6), (3.7), we have

$$\begin{aligned} W_n(\varphi, \psi) &= \varphi(a-1)\psi^{[\Delta]}(a-1) - \varphi^{[\Delta]}(a-1)\psi(a-1) \\ &= \beta\psi^{[\Delta]}(a-1) - \alpha\psi(a-1) \end{aligned}$$

for $n \in [a-1, c-1]$, and

$$\begin{aligned} W_n(\varphi, \psi) &= \varphi(b)\psi^{[\Delta]}(b) - \varphi^{[\Delta]}(b)\psi(b) \\ &= -\gamma\varphi(b) - \delta\varphi^{[\Delta]}(b) \end{aligned}$$

for $n \in [c+1, b]$. Therefore, taking into account (2.7), we get

$$W_n(\varphi, \psi) = \begin{cases} -d_1 d_2[\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b)], & n \in [a-1, c-1], \\ -\gamma\varphi(b) - \delta\varphi^{[\Delta]}(b), & n \in [c+1, b], \end{cases} \tag{3.8}$$

and also

$$W_n(\varphi, \psi) = \begin{cases} \beta\psi^{[\Delta]}(a-1) - \alpha\psi(a-1), & n \in [a-1, c-1], \\ \dfrac{1}{d_1 d_2}[\beta\psi^{[\Delta]}(a-1) - \alpha\psi(a-1)], & n \in [c+1, b]. \end{cases} \tag{3.9}$$

Notice that, as it follows from (3.8) and (3.9),

$$\beta\psi^{[\Delta]}(a-1) - \alpha\psi(a-1) = -d_1 d_2[\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b)]. \tag{3.10}$$

According to Theorem 2.5, we get from (3.8) that $\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) \neq 0$ if and only if $\varphi(n)$ and $\psi(n)$ are linearly independent. The following theorem describes the condition $\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) \neq 0$ from the other point of view.

Theorem 3.1. $\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) \neq 0$ if and only if the homogeneous problem (3.4), (3.5) has only the trivial soluıtion $y(n) \equiv 0$ satisfying the boundary conditions in (3.3).

Proof. If $\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) = 0$, then by virtue of (3.6), $\varphi(n)$ will be a nontrivial solution of (3.4), (3.5), satisfying the boundary conditions (3.3). Let us now assume that $\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) \neq 0$. Then $\varphi(n)$ and $\psi(n)$ will form a fundamental set of solutions of (3.4), (3.5) and therefore any solution of (3.4), (3.5), (3.3) will have the form

$$y(n) = c_1\varphi(n) + c_2\psi(n),$$

where c_1, c_2 are constants. Substituting this expression of $y(n)$ into boundary conditions in (3.3) and taking into account (3.6) and (3.7), we get

$$c_2[\alpha\psi(a-1) - \beta\psi^{[\Delta]}(a-1)] = 0 \quad \text{and} \quad c_1[\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b)] = 0.$$

Since $\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) \neq 0$ and also $\alpha\psi(a-1) - \beta\psi^{[\Delta]}(a-1) \neq 0$ by (3.10), it follows that $c_1 = c_2 = 0$, that is, the solution $y(n)$ is trivial. ■

Theorem 3.2. If $\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) \neq 0$, then the nonhomogeneous BVPI (3.1)–(3.3) has a unique solution $y(n)$ for which the formula

$$y(n) = \sum_{s=a}^{b} G(n,s)h(s), \quad n \in [a-1, b+1] \tag{3.11}$$

holds, where the function $G(n,s)$ is called the *Green function* of the BVPI (3.1)–(3.3) and defined for $(n,s) \in [a-1, b+1] \times [a-1, b]$ by the formula

$$G(n,s) = -\frac{1}{W_s(\varphi,\psi)}\begin{cases} \varphi(s)\psi(n), & s \leq n, \\ \varphi(n)\psi(s), & n \leq s. \end{cases} \tag{3.12}$$

Proof. Under the condition $\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) \neq 0$, the solutions $\varphi(n)$ and $\psi(n)$ of the homogeneous problem (3.4), (3.5) are linearly independent, and therefore by Theorem 2.7 the general solution of the nonhomogeneous problem (3.1), (3.2) has the form

$$y(n) = \begin{cases} c_1\varphi(n) + c_2\psi(n) - \displaystyle\sum_{s=n}^{c} \frac{y_1(n)y_2(s) - y_1(s)y_2(n)}{W_s(y_1, y_2)} h(s), & n \leq c, \\ c_1\varphi(n) + c_2\psi(n) + \displaystyle\sum_{s=c+1}^{n} \frac{y_1(n)y_2(s) - y_1(s)y_2(n)}{W_s(y_1, y_2)} h(s), & n \geq c+1, \end{cases} \tag{3.13}$$

where c_1 and c_2 are arbitrary constants. Substituting this expression of $y(n)$ into boundary conditions in (3.3), it is not difficult to find that

$$c_1 = -\sum_{s=c+1}^{b} \frac{\psi(s)}{W_s(\varphi, \psi)} h(s), \quad c_2 = -\sum_{s=a}^{c} \frac{\varphi(s)}{W_s(\varphi, \psi)} h(s).$$

Putting these values of c_1 and c_2 in (3.13), we get the formulas (3.11), (3.12). ■

Remark 3.3. It can be verified without difficulty that if $\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) \neq 0$, then for the solution $y(n)$ of the nonhomogeneous equation (3.1) with the impulse conditions (3.2) and nonhomogeneous boundary conditions

$$\alpha y(a-1) - \beta y^{[\Delta]}(a-1) = \mu, \quad \gamma y(b) + \delta y^{[\Delta]}(b) = \nu,$$

with μ and ν constants, the formula

$$y(n) = w(n) + \sum_{s=a}^{b} G(n,s)h(s), \quad n \in [a-1, b+1]$$

holds, where the function $G(n,s)$ is defined by (3.12), and

$$w(n) = -\frac{\nu}{W_b(\varphi, \psi)}\varphi(n) - \frac{\mu}{W_a(\varphi, \psi)}\psi(n), \quad n \in [a-1, b+1].$$

4. Sign Properties of the Green Function

Consider the BVPI (3.1)–(3.3) and assume that

$$\begin{aligned} p(n) &> 0 \quad \text{for} \quad n \in [a-1, c-1] \cup [c+1, b] \\ \text{and} \quad q(n) &\geq 0 \quad \text{for } n \in [a, c-1] \cup [c+2, b], \end{aligned} \tag{4.1}$$

$$d_1 > 0,\ d_2 > 0; \quad \alpha, \beta, \gamma, \delta \geq 0,\ \alpha + \beta > 0,\ \gamma + \delta > 0. \tag{4.2}$$

Let $\varphi(n)$ and $\psi(n)$ be the solutions of the homogeneous problem (3.4), (3.5) satisfying the initial conditions (3.6) and (3.7), respectively.

Lemma 4.1. For the solutions $\varphi(n)$ and $\psi(n)$, the equations

$$\varphi^{[\Delta]}(n) = \alpha + \sum_{s=a}^{n} q(s)\varphi(s), \quad n \in [a, c-1], \tag{4.3}$$

$$\varphi(n) = \beta + \alpha \sum_{k=a-1}^{n-1} \frac{1}{p(k)} + \sum_{s=a}^{n-1} \left[\sum_{k=s}^{n-1} \frac{1}{p(k)} \right] q(s)\varphi(s), \quad n \in [a, c], \tag{4.4}$$

$$\varphi^{[\Delta]}(n) = \varphi^{[\Delta]}(c+1) + \sum_{s=c+2}^{n} q(s)\varphi(s), \quad n \in [c+2, b], \tag{4.5}$$

$$\begin{aligned}\varphi(n) &= \varphi(c+1) + \varphi^{[\Delta]}(c+1)\sum_{k=c+1}^{n-1}\frac{1}{p(k)} + \sum_{s=c+2}^{n-1}\left[\sum_{k=s}^{n-1}\frac{1}{p(k)}\right]q(s)\varphi(s),\\ n &\in [c+2, b+1],\end{aligned} \tag{4.6}$$

and

$$\psi^{[\Delta]}(n) = \psi^{[\Delta]}(c-1) - \sum_{s=n+1}^{c-1} q(s)\psi(s), \quad n \in [a-1, c-1], \tag{4.7}$$

$$\begin{aligned}\psi(n) &= \psi(c-1) - \psi^{[\Delta]}(c-1)\sum_{k=n}^{c-2}\frac{1}{p(k)} + \sum_{s=n+1}^{c-1}\left[\sum_{k=n}^{s-1}\frac{1}{p(k)}\right]q(s)\psi(s),\\ n &\in [a-1, c-1],\end{aligned} \tag{4.8}$$

$$\psi^{[\Delta]}(n) = -\gamma - \sum_{s=n+1}^{b} q(s)\psi(s), \quad n \in [c+1, b], \tag{4.9}$$

$$\psi(n) = \delta + \gamma\sum_{k=n}^{b-1}\frac{1}{p(k)} + \sum_{s=n+1}^{b}\left[\sum_{k=n}^{s-1}\frac{1}{p(k)}\right]q(s)\psi(s), \quad n \in [c+1, b], \tag{4.10}$$

respectively, hold.

Proof. To prove equations (4.3)–(4.6) for $\varphi(n)$, consider the equation

$$\Delta[p(s-1)\Delta\varphi(s-1)] = q(s)\varphi(s), \quad s \in [a, c-1] \cup [c+2, b]. \tag{4.11}$$

Summing this equation from a to k with $k \in [a, c-1]$, and taking into account the initial condition $\varphi^{[\Delta]}(a-1) = \alpha$, we get

$$\varphi^{[\Delta]}(k) = \alpha + \sum_{s=a}^{k} q(s)\varphi(s), \quad k \in [a, c-1],$$

that is the equation (4.3). Rewrite this equation as

$$\Delta\varphi(k) = \frac{\alpha}{p(k)} + \frac{1}{p(k)}\sum_{s=a}^{k} q(s)\varphi(s), \quad k \in [a, c-1],$$

and sum from a to $n-1$ with $n \in [a, c]$, to get

$$\varphi(n) - \varphi(a) = \alpha\sum_{k=a}^{n-1}\frac{1}{p(k)} + \sum_{k=a}^{n-1}\frac{1}{p(k)}\sum_{s=a}^{k} q(s)\varphi(s), \quad n \in [a, c].$$

Next, from the initial conditions in (3.6) we have

$$\varphi(a) = \frac{\alpha}{p(a-1)} + \beta.$$

Substituting this in the last equation and using the formula (changing the order of summation)

$$\sum_{k=a}^{m} u(k) \sum_{s=a}^{k} v(s) = \sum_{s=a}^{m} \left[\sum_{k=s}^{m} u(k) \right] v(s),$$

we find the equation (4.4).

Now we sum equation (4.11) from $c+2$ to k with $k \in [c+2, b]$, to get

$$\varphi^{[\Delta]}(k) = \varphi^{[\Delta]}(c+1) + \sum_{s=c+2}^{k} q(s)\varphi(s), \quad k \in [c+2, b],$$

that is (4.5). Rewrite this equation as

$$\Delta\varphi(k) = \frac{1}{p(k)} \varphi^{[\Delta]}(c+1) + \frac{1}{p(k)} \sum_{s=c+2}^{k} q(s)\varphi(s), \quad k \in [c+2, b],$$

and sum from $c+2$ to $n-1$ with $n \in [c+2, b+1]$, to get

$$\begin{aligned} \varphi(n) - \varphi(c+2) &= \varphi^{[\Delta]}(c+1) \sum_{k=c+2}^{n-1} \frac{1}{p(k)} + \sum_{k=c+2}^{n-1} \frac{1}{p(k)} \sum_{s=c+2}^{k} q(s)\varphi(s), \\ n &\in [c+2, b+1]. \end{aligned}$$

Substituting here

$$\varphi(c+2) = \varphi(c+1) + \frac{1}{p(c+1)} \varphi^{[\Delta]}(c+1)$$

and changing the order of summation in the double sum, we get (4.6).

Let us consider the solution $\psi(n)$. Summing the equation

$$\Delta[p(s-1)\Delta\psi(s-1)] = q(s)\psi(s), \quad s \in [a, c-1] \cup [c+2, b], \tag{4.12}$$

from $k+1$ to b with $k \in [c+1, b]$, and taking into account the initial condition $\psi^{[\Delta]}(b) = -\gamma$, we get

$$\psi^{[\Delta]}(k) = -\gamma - \sum_{s=k+1}^{b} q(s)\psi(s), \quad k \in [c+1, b],$$

that is (4.9). Rewrite this equation as

$$\Delta\psi(k) = -\frac{\gamma}{p(k)} - \frac{1}{p(k)} \sum_{s=k+1}^{b} q(s)\psi(s), \quad k \in [c+1, b],$$

and sum from n to $b-1$ with $n \in [c+1, b-1]$, to get

$$\psi(n) = \delta + \gamma \sum_{k=n}^{b-1} \frac{1}{p(k)} + \sum_{k=n}^{b-1} \frac{1}{p(k)} \sum_{s=k+1}^{b} q(s)\psi(s), \quad n \in [c+1, b],$$

where we have used the initial condition $\psi(b) = \delta$. Changing here the order of summation in the double sum, we get (4.10). Note that for $n = b$ equation (4.10) is understood to be $\psi(b) = \delta$.

Now we sum equation (4.12) from $k+1$ to $c-1$ with $k \in [a-1, c-1]$, to get equation (4.7):

$$\psi^{[\Delta]}(k) = \psi^{[\Delta]}(c-1) - \sum_{s=k+1}^{c-1} q(s)\psi(s), \quad k \in [a-1, c-1],$$

where the sum is understood to be zero for $k = c-1$. Rewrite this equation as

$$\Delta\psi(k) = \frac{1}{p(k)}\psi^{[\Delta]}(c-1) - \frac{1}{p(k)} \sum_{s=k+1}^{c-1} q(s)\psi(s), \quad k \in [a-1, c-1],$$

and sum from n to $c-2$ with $n \in [a-1, c-2]$, to get

$$\psi(n) = \psi(c-1) - \psi^{[\Delta]}(c-1) \sum_{k=n}^{c-2} \frac{1}{p(k)} + \sum_{k=n}^{c-2} \frac{1}{p(k)} \sum_{s=k+1}^{c-1} q(s)\psi(s), \quad n \in [a-1, c-2].$$

Changing here the order of summation in the double sum, we get (4.8), where the sums are understood to be zero for $n = c-1$. ■

Lemma 4.2. Under the conditions (4.1) and (4.2), the solutions $\varphi(n)$ and $\psi(n)$ have the following properties:

$$\varphi(a-1) = \beta \geq 0; \quad \varphi(n) > 0 \quad \text{for} \quad n \in [a, b+1];$$

$$\varphi^{[\Delta]}(n) \geq 0, \quad \text{for} \quad n \in [a-1, c-1] \cup [c+1, b];$$

$$\psi(n) > 0 \quad \text{for} \quad n \in [a-1, c-1] \cup [c+1, b-1];$$

$$\psi(b) = \delta \geq 0; \quad \psi(b+1) = \delta - \frac{\gamma}{p(b)};$$

$$\psi(c) > 0 \quad \text{if} \quad d_1 p(c-1) > d_2 p(c+1);$$

$$\psi^{[\Delta]}(n) \le 0 \quad \text{for} \quad n \in [a-1, c-1] \cup [c+1, b].$$

Proof. Since $\varphi(a-1) = \beta \ge 0$, we have from (4.4) that

$$\varphi(a) = \beta + \frac{\alpha}{p(a-1)} > 0,$$

$$\varphi(a+1) = \beta + \alpha \left[\frac{1}{p(a-1)} + \frac{1}{p(a)} \right] + \frac{1}{p(a)} q(a)\varphi(a) > 0.$$

Proceeding in this way we get from (4.4), step by step, that

$$\varphi(n) > 0 \quad \text{for} \quad n = a+2, a+3, \ldots, c.$$

Thus, $\varphi(n) > 0$ for $n \in [a, c]$. Consequently, from (4.3) we get also

$$\varphi^{[\Delta]}(n) \ge 0, \quad n \in [a, c-1].$$

Now from the impulse conditions

$$\varphi(c-1) = d_1\varphi(c+1), \quad \varphi^{[\Delta]}(c-1) = d_2\varphi^{[\Delta]}(c+1),$$

by the conditions $d_1 > 0$, $d_2 > 0$ we find that

$$\varphi(c+1) > 0, \quad \varphi^{[\Delta]}(c+1) \ge 0.$$

Passing then to equations (4.6) and (4.5) we get

$$\varphi(n) > 0 \quad \text{for} \quad n \in [c+2, b] \quad \text{and} \quad \varphi^{[\Delta]}(n) \ge 0 \quad \text{for} \quad n \in [c+2, b].$$

So, the statements of the lemma for $\varphi(n)$ are proved.

Now we consider the solution $\psi(n)$. We have $\psi(b) = \delta \ge 0$ and from (4.10) we find that

$$\psi(b-1) = \delta + \frac{\gamma}{p(b-1)} + \frac{1}{p(b-1)} q(b)\psi(b) > 0.$$

Proceeding in this way we get from (4.10) that

$$\psi(n) > 0 \quad \text{for} \quad n \in [c+1, b-1].$$

Consequently, from (4.9) we get also

$$\psi^{[\Delta]}(n) \le 0 \quad \text{for} \quad n \in [c+1, b].$$

Now from the impulse conditions

$$\psi(c-1) = d_1\psi(c+1), \quad \psi^{[\Delta]}(c-1) = d_2\psi^{[\Delta]}(c+1),$$

by the conditions $d_1 > 0$, $d_2 > 0$ we find that

$$\psi(c-1) > 0, \quad \psi^{[\Delta]}(c-1) \le 0.$$

Next, passing to equation (4.8) we get that

$$\psi(n) > 0 \quad \text{for} \quad n \in [a-1, c-1]$$

and then (4.7) yields

$$\psi^{[\Delta]}(n) \leq 0 \quad \text{for} \quad n \in [a-1, c-1].$$

From the initial conditions (3.7) we find that $\psi(b+1) = \delta - \gamma \diagup p(b)$. Finally, let us determine the sign of $\psi(c)$. To this end we use the formula (2.9) which implies that $\psi(c) > 0$ if $d_1 p(c-1) > d_2 p(c+1)$. ■

Lemma 4.3. Let conditions (4.1) and (4.2) hold. Besides, in the case $q(n) \equiv 0$ for $n \in [a, c-1] \cup [c+2, b]$, let $\alpha + \gamma > 0$. Then the Wronskian of the solutions $\varphi(n)$ and $\psi(n)$ satisfies the inequalities

$$W_s(\varphi, \psi) < 0 \quad \text{for} \quad s \in [a-1, c-1] \cup [c+1, b], \tag{4.13}$$

$$W_c(\varphi, \psi) > 0. \tag{4.14}$$

Proof. For $s \in [a-1, c-1] \cup [c+1, b]$ we have, by (3.8),

$$W_s(\varphi, \psi) = \begin{cases} -d_1 d_2[\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b)], & s \in [a-1, c-1], \\ -\gamma\varphi(b) - \delta\varphi^{[\Delta]}(b), & s \in [c+1, b]. \end{cases}$$

Therefore for proof of (4.13) it is enough to show that

$$\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) > 0.$$

Using (4.5) and (4.6), we find

$$\begin{aligned} &\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) \\ = \ &\gamma \left\{ \varphi(c+1) + \varphi^{[\Delta]}(c+1) \sum_{k=a+1}^{b-1} \frac{1}{p(k)} + \sum_{s=c+2}^{b-1} \left[\sum_{k=s}^{b-1} \frac{1}{p(k)} \right] q(s)\varphi(s) \right\} \\ &+ \delta \left\{ \varphi^{[\Delta]}(c+1) + \sum_{s=c+2}^{b} q(s)\varphi(s) \right\}. \end{aligned} \tag{4.15}$$

Further, from the impulse conditions, we have

$$\varphi(c+1) = \frac{1}{d_1}\varphi(c-1), \quad \varphi^{[\Delta]}(c+1) = \frac{1}{d_2}\varphi^{[\Delta]}(c-1).$$

Hence, by (4.4), (4.3),

$$\varphi(c+1) = \frac{1}{d_1} \left\{ \beta + \alpha \sum_{k=a-1}^{c-2} \frac{1}{p(k)} + \sum_{s=a}^{c-2} \left[\sum_{k=s}^{c-2} \frac{1}{p(k)} \right] q(s)\varphi(s) \right\},$$

$$\varphi^{[\Delta]}(c+1) = \frac{1}{d_2}\left\{\alpha + \sum_{s=a}^{c-1} q(s)\varphi(s)\right\}.$$

Substituting these into (4.15) we get

$$\begin{aligned}\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) &= \frac{\gamma}{d_1}\left\{\beta + \alpha \sum_{k=a-1}^{c-2} \frac{1}{p(k)} + \sum_{s=a}^{c-2}\left[\sum_{k=s}^{c-2} \frac{1}{p(k)}\right] q(s)\varphi(s)\right\} \\ &\quad + \frac{1}{d_2}\left\{\delta + \gamma \sum_{k=c+1}^{b-1} \frac{1}{p(k)}\right\}\left\{\alpha + \sum_{s=a}^{c-1} q(s)\varphi(s)\right\} \\ &\quad + \gamma \sum_{s=c+2}^{b-1}\left[\sum_{k=s}^{b-1} \frac{1}{p(k)}\right] q(s)\varphi(s) + \delta \sum_{s=c+2}^{b} q(s)\varphi(s).\end{aligned} \tag{4.16}$$

It follows from (4.16) that if $q(n)$ is not identically zero on $[a, c-1] \cup [c+2, b]$, then $\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) > 0$. If $q(n)$ is identically zero on $[a, c-1] \cup [c+2, b]$, then (4.16) becomes

$$\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) = \frac{\gamma}{d_1}\left\{\beta + \alpha \sum_{k=a-1}^{c-2} \frac{1}{p(k)}\right\} + \frac{\alpha}{d_2}\left\{\delta + \gamma \sum_{k=c+1}^{b-1} \frac{1}{p(k)}\right\}$$

which gives $\gamma\varphi(b) + \delta\varphi^{[\Delta]}(b) > 0$ if $\alpha + \gamma > 0$.

Finally, (4.14) follows from (2.8) by (4.13). ■

From formula (3.12) for the Green function $G(n, s)$, by Lemma 4.2 and Lemma 4.3 the following theorem follows.

Theorem 4.4. Assume that the following conditions hold:

$$\begin{aligned} p(n) &> 0 \quad \text{for} \quad n \in [a-1, c-1] \cup [c+1, b] \\ \text{and} \quad q(n) &\geq 0 \quad \text{for} \quad n \in [a, c-1] \cup [c+2, b]; \end{aligned}$$

$$d_1 > 0, \; d_2 > 0, \quad d_1 p(c-1) > d_2 p(c+1);$$

$$\alpha, \beta, \gamma, \delta \geq 0, \quad \alpha + \beta > 0, \; \gamma + \delta > 0;$$

$$\alpha + \gamma > 0 \quad \text{if } q(n) \equiv 0 \text{ on } [a, c-1] \cup [c+2, b].$$

Then

(i) $G(n, s) \geq 0$ for $n \in [a-1, b]$ and $s \in [a-1, c-1] \cup [c+1, b]$.

(ii) $G(n, s) > 0$ for $n \in [a, b-1]$ and $s \in [a, c-1] \cup [c+1, b-1]$.

(iii) If $\delta > 0$, then $G(n, s) > 0$ for $n \in [a, b]$ and $s \in [a, c-1] \cup [c+1, b]$.

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