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Oscillation Theorems for Certain Fourth Order Nonlinear Difference Equations

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Abstract

We establish some new sufficient conditions for the oscillation of solutions of fourth order nonlinear difference equations

$$\Delta\left(a(n)(\Delta^3 x(n))^\alpha\right) + q(n)f(x[g(n)]) = 0$$

and

$$\Delta\left(a(n)(\Delta^3 x(n))^\alpha\right) = q(n)f(x[g(n)]) + p(n)h(x[\sigma(n)])$$

when $\sum^\infty a^{-1/\alpha}(n) < \infty$.

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1. Introduction

This paper is concerned with the oscillatory behavior of nonlinear fourth order difference equations

$$\Delta\left(a(n)(\Delta^3 x(n))^\alpha\right) + q(n)f(x[g(n)]) = 0 \tag{1.1}$$

and

$$\Delta\left(a(n)(\Delta^3 x(n))^\alpha\right) = q(n)f(x[g(n)]) + p(n)h(x[\sigma(n)]), \tag{1.2}$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, \cdots\}$, n_0 is a nonnegative integer and Δ is the forward difference operator $\Delta x(n) = x(n+1) - x(n)$, and $\{a(n)\}$, $\{p(n)\}$, $\{q(n)\}$, $\{g(n)\}$ and $\{\sigma(n)\}$ are sequences of real numbers.

The following conditions are always assumed to hold:

(i) α is the ratio of two positive odd integers;

(ii) $a(n) > 0$ for $n \in \mathbb{N}(n_0)$ and

$$\sum_{n=n_0}^{\infty} a^{-1/\alpha}(n) < \infty; \tag{1.3}$$

(iii) $p(n)$ and $q(n) \geq 0$ for $n \in \mathbb{N}(n_0)$;

(iv) $g, \sigma : \mathbb{N}(n_0) \to \mathbb{Z}$ is such that $g(n) < n$, $\sigma(n) > n$, $\Delta g(n) \geq 0$ and $\Delta\sigma(n) \geq 0$ for $n \in \mathbb{N}(n_0)$ and $\lim_{n\to\infty} g(n) = \infty$;

(v) $f, h \in C(\mathbb{R}, \mathbb{R})$, $xf(x) \geq 0$, $xh(x) \geq 0$, $h'(x) \geq 0$ and $f'(x) \geq 0$ for $x \neq 0$, and

$$-f(-xy) \geq f(xy) \geq f(x)f(y) \quad \text{for} \quad xy > 0 \tag{1.4}$$

and

$$-h(-xy) \geq h(xy) \geq h(x)h(y) \quad \text{for} \quad xy > 0. \tag{1.5}$$

By a solution of equation $(1, i), i = 1, 2$ we mean a real sequence $\{x(n)\}$ defined on $\mathbb{N}(n_0)$, which satisfies equation $(1, i), i = 1, 2$. A nontrivial solution of equation $(1, i), i = 1, 2$ is said to be nonoscillatory if it is either eventually positive or eventually negative, and it is oscillatory otherwise. Equation $(1, i), i = 1, 2$ is said to be oscillatory if all its solutions are oscillatory.

Determining oscillation criteria for difference equations has received a great deal of attention in the last few years. Here, we refer to the monographs by Agarwal et. al. [1–4] and the references cited therein. This interest is motivated by the importance of difference equations in the numerical solutions of differential equations. Compared to second order difference equations, the study of higher order equations, and in particular fourth order equations, has received considerably less attention, see [7–10], where the special cases of (1.1) ($\alpha = 1$ and $a(n) \equiv 1$) have been considered. It seems that very little is known regarding the oscillation of equation (1.1) and (1.2) particularly when condition (1.3) holds. Therefore, our main goal here is to establish some new criteria for the oscillation of equations (1.1) and (1.2). In Section 2, we present some sufficient conditions for the oscillation of equation (1.1). Section 3 is devoted to the study of the oscillation of equation (1.2). Some general remarks are given in Section 4. The results of this paper are presented in a form which is essentially new and of high degree of generality. It contains, in particular, for the equations (1.1) and (1.2) when $\alpha = 1$ and $\sum^{\infty} 1/a(k)$ finite or infinite, many existing results.

2. Oscillation Theorems for Equation (1.1)

In this section we shall establish some sufficient conditions for the oscillation of equation (1.1). For $n \in \mathbb{N}(n_0)$, we let

$$A[n, n_0] = \sum_{k=n_0}^{n-1} \sum_{s=n_0}^{k-1} s a^{-1/\alpha}(s)$$

and

$$c(n) = \sum_{k=n}^{\infty} a^{-1/\alpha}(k).$$

Our first main result is the following theorem.

Theorem 2.1. Let conditions (i)–(v), (1.3) and (1.4) hold and assume that there exist nondecreasing sequences $\{\xi(n)\}$ and $\{\eta(n)\}$, $\xi, \eta : \mathbb{N}(n_0) \to \mathbb{Z}$ such that $g(n) < \xi(n) < \eta(n) < n - 1$ for $n \in \mathbb{N}(n_0)$. If the first order difference equations

$$\Delta y(n) + \overline{c} q(n) f(A[g(n), n_1]) f\left(y^{1/\alpha}[g(n)]\right) = 0 \tag{2.1}$$

for every $n_1 \in \mathbb{N}(n_0)$ and any constant $\overline{c}$, $0 < \overline{c} < 1$ and

$$\Delta w(n) + c^* q(n) f(g(n)) f(\xi(n) - \eta(n)) f\left(\sum_{k=\xi(n)}^{\eta(n)-1} a^{-1/\alpha}(k)\right) f\left(w^{1/\alpha}[\eta(n)]\right) = 0, \tag{2.2}$$

for any constant c^*, $0 < c^* < 1$, are oscillatory, and for $n_1 \in \mathbb{N}(n_0)$, $n_1 \geq n_0$

$$\sum_{s=n_1}^{\infty} \left(\frac{1}{a(s)} \sum_{k=n_1}^{s-1} q(k) f\left(g^2(k)\right) f(c[g(k)]) \right)^{1/\alpha} = \infty, \tag{2.3}$$

and

$$\sum_{s=n_1}^{\infty}\left(\frac{1}{a(s)}\sum_{k=n_1}^{s-1} q(k)f(\xi(k)-g(k))f(\eta(k)-\xi(k))f(c[g(k)])\right)^{1/\alpha}=\infty, \qquad (2.4)$$

then equation (1.1) is oscillatory.

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1), say, $x(n)>0$ and $x[g(n)]>0$ for $n\in\mathbb{N}(n_0)$. From equation (1.1), we see that $\Delta\left(a(n)(\Delta^3x(n))^{\alpha}\right)\le 0$ for $n\in\mathbb{N}(n_0)$. There exists an $n_1\in\mathbb{N}(n_0)$ such that $\Delta^i x(n)$, $i=1,2,3$ are of one sign for $n_1\ge n_0$. There are eight possibilities to consider. The following four cases hold:

(I) $\Delta^i x(n)>0$, $i=1,2,3$ for $n\ge n_1$;

(II) $\Delta^3x(n)>0$, $\Delta^2x(n)<0$ and $\Delta x(n)>0$ for $n\ge n_1$;

(III) $\Delta^3x(n)<0$, $\Delta^2x(n)>0$ and $\Delta x(n)>0$ for $n\ge n_1$; and

(IV) $\Delta^3x(n)<0$, $\Delta^2x(n)>0$ and $\Delta x(n)<0$ for $n\ge n_1$,

while the following other four cases:

$$\begin{aligned}
&\Delta^3x(n)>0,\ \Delta^2x(n)>0 \text{ and } \Delta x(n)<0 \text{ for } n\ge n_1;\\
&\Delta^3x(n)>0,\ \Delta^2x(n)<0 \text{ and } \Delta x(n)<0 \text{ for } n\ge n_1;\\
&\Delta^3x(n)<0,\ \Delta^2x(n)<0 \text{ and } \Delta x(n)>0 \text{ for } n\ge n_1;\\
&\Delta^3x(n)<0,\ \Delta^2x(n)<0 \text{ and } \Delta x(n)<0 \text{ for } n\ge n_1,
\end{aligned}$$

are obviously disregarded. Next, we consider:

Case (I). There exist a constant b, $0<b<1$ and an $n_2\in\mathbb{N}(n_0)$, $n_2\ge n_1$ such that

$$\begin{aligned}
\Delta^2x(n) &\ge bn\Delta^3x(n) \ \text{ for } \ n\ge n_2\\
&= b\frac{n}{a^{1/\alpha}(n)}y^{1/\alpha}(n) \ \text{ for } \ n\ge n_2,
\end{aligned} \qquad (2.5)$$

where $y(n)=a(n)(\Delta^3x(n))^{\alpha}$ for $n\ge n_2$.

Summing (2.5) twice from n_2 to $n-1$, we get

$$x(n)\ge bA[n,n_2]y^{1/\alpha}(n) \text{ for } n\ge n_2.$$

Thus, there exists an $n_3\in\mathbb{N}(n_0)$, $n_3\ge n_2$ such that

$$x[g(n)]\ge bA[g(n),n_2]y^{1/\alpha}[g(n)] \text{ for } n\ge n_3. \qquad (2.6)$$

Using (2.6) and (1.4) in equation (1.1), we have

$$\Delta y(n)+f(b)q(n)f(A[g(n),n_2])f\left(y^{1/\alpha}[g(n)]\right)\le 0 \text{ for } n\ge n_3. \qquad (2.7)$$

Summing from $n \geq n_3$ to $u \geq n$ and letting $u \to \infty$, we have

$$y(n) \geq f(b) \sum_{s=n}^{\infty} q(s) f(A[g(s), n_3]) f\left(y^{1/\alpha}[g(n)]\right).$$

The sequence $\{y(n)\}$ is obviously strictly decreasing for $n \geq n_3$, hence by the analog of [6, Theorem 1] (also, see [3]), we conclude that there exists a positive solution $\{y(n)\}$ of equation (2.1) with $\lim_{n\to\infty} y(n) = 0$, which is a contradiction.

Case (II). There exist an $n_2 \geq n_1$ and a constant d, $0 < d < 1$ such that

$$x(n) \geq dn\Delta x(n) \text{ for } n \geq n_2.$$

There exists an $n_3 \in \mathbb{N}(n_0)$, $n_3 \geq n_2$ such that

$$x[g(n)] \geq dg(n)y[g(n)] \quad \text{for} \quad n \geq n_3, \tag{2.8}$$

where $y(n) = \Delta x(n)$, $n \geq n_3$.

Using (2.8) and (1.4) in equation (1.1), we have

$$\Delta\left(a(n)(\Delta^2 y(n))^\alpha\right) + f(d)(q(n) f(g(n)) f(y[g(n)]) \leq 0 \quad \text{for} \quad n \geq n_3. \tag{2.9}$$

Clearly, $\Delta^2 y(n) > 0$, $\Delta y(n) < 0$ and $y(n) > 0$ for $n \geq n_3$. For $t \geq s \geq n_3$, we have

$$y(t) - y(s) = \sum_{k=s}^{t-1} \Delta y(k),$$

or

$$y(s) \geq (t-s)(-\Delta y(t)).$$

Replacing s and t by $g(n)$ and $\xi(n)$ respectively, we find

$$y[g(n)] \geq (\xi(n) - g(n))Z[\xi(n)] \quad \text{for} \quad n \geq n_4 \geq n_3, \tag{2.10}$$

where $Z(n) = \Delta y(n)$, $n \geq n_4$. Using (2.10) and (1.4) in (2.9), we get

$$\Delta\left(a(n)(\Delta Z(n))^\alpha\right) + f(d) q(n) f(g(n)) f(\xi(n) - g(n)) f(Z[\xi(n)]) \leq 0 \quad \text{for} \quad n \geq n_4. \tag{2.11}$$

Clearly, $\Delta Z(n) < 0$ and $Z(n) > 0$ for $n \geq n_4$. Next, for $t \geq s \geq n_4$, we get

$$\begin{aligned} Z(s) \geq \sum_{k=s}^{t-1} -\Delta Z(k) &= \sum_{k=s}^{t-1} a^{-1/\alpha}(k)\left(-a(k)(\Delta Z(k))^\alpha\right)^{1/\alpha} \\ &\geq \left(\sum_{k=s}^{t-1} a^{-1/\alpha}(k)\right) w^{1/\alpha}(t), \end{aligned} \tag{2.12}$$

where $w(n) = -a(n)(\Delta Z(n))^\alpha$ for $n \geq n_4$. Replacing s and t in (2.12) by $\xi(n)$ and $\eta(n)$ respectively, we have

$$Z[\xi(n)] \geq \left(\sum_{k=\xi(n)}^{\eta(n)-1} a^{-1/\alpha}(k)\right) w^{1/\alpha}[\eta(n)] \text{ for } n \geq n_5 \geq n_4. \tag{2.13}$$

Using (2.13) and (1.4) in (2.11), we get

$$\Delta w(n) + f(d)q(n)f(g(n))f(\xi(n) - g(n))f\left(\sum_{k=\xi(n)}^{\eta(n)-1} a^{-1/\alpha}(k)\right) f\left(w^{1/\alpha}[\eta(n)]\right) \leq 0 \text{ for } n \geq n_5.$$

The rest of the proof is similar to the proof of Case (I) above, and hence omitted.

Case (III). There exist a constant $\overline{b}$, $0 < \overline{b} < 1$ and an $n_2 \geq n_1$ such that

$$\Delta x(n) \geq \overline{b} n \Delta^2 x(n) \text{ for } n \geq n_2.$$

Summing the last inequality from n_2 to $n-1$ and then assume that there exist a constant b^*, $0 < b^* < 1$ and an $n_3 \geq n_2$ such that

$$x[g(n)] \geq b^* g^2(n) y[g(n)] \text{ for } n \geq n_3, \tag{2.14}$$

where $y(n) = \Delta^2 x(n)$, $n \geq n_3$.

Using (2.14) and (1.4) in equation (1.1), we get

$$\Delta\left(a(n)(\Delta y(n))^\alpha\right) + f(b^*)f(g^2(n))f(y[g(n)]) \leq 0 \text{ for } n \geq n_3. \tag{2.15}$$

Clearly, $\Delta y(n) < 0$ and $y(n) > 0$ for $n \geq n_3$. For $s \geq t \geq n_3$, we get

$$a(s)(-\Delta y(s))^\alpha \geq a(t)(-\Delta y(t))^\alpha,$$

or

$$-\Delta y(s) \geq a^{-1/\alpha}(s)\left(a^{1/\alpha}(t)(-\Delta y(t)\right).$$

Summing the above inequality from $t = n \geq n_3$ to $u - 1 \geq n$ and letting $u \to \infty$, we have

$$y(n) \geq \left(-a^{1/\alpha}(n)\Delta y(n)\right)\left(\sum_{s=n}^{\infty} a^{-1/\alpha}(s)\right). \tag{2.16}$$

Combining (2.16) with the inequality

$$-a^{1/\alpha}(n)\Delta y(n) \geq -a^{1/\alpha}(n_3)\Delta y(n_3), n \geq n_3$$

we get

$$y(n) \geq \left(-a^{1/\alpha}(n_3)\Delta y(n_3)\right) c(n) \text{ for } n \geq n_3.$$

Thus, there exist a constant $\ell > 0$ and an $n_4 \in \mathbb{N}(n_0)$, $n_4 \geq n_3$ such that

$$y[g(n)] \geq \ell c[g(n)] \text{ for } n \geq n_4. \tag{2.17}$$

Using (2.14) and (1.4) in (2.15), we get

$$f(b^*)f(\ell)f(g^2(n))q(n)f(c[g(n)]) \leq -\Delta\left(a(n)(\Delta y(n))^\alpha\right), n \geq n_4.$$

Summing the above inequality from n_4 to $n-1 \geq n_4$, we obtain

$$f(b^*)f(\ell)\sum_{k=n_3}^{n-1} q(k)f(g^2(k))f(c[g(k)]) \leq a(n_4)(\Delta y(n_4))^\alpha - a(n)(\Delta y(n))^\alpha,$$

or

$$(f(b^*)f(\ell))^{1/\alpha}\left(\frac{1}{a(n)}\sum_{k=n_4}^{n-1} q(k)f(g^2(k))f(c[g(k)])\right)^{1/\alpha} = -\Delta y(n) \text{ for } n \geq n_4. \tag{2.18}$$

Summing (2.18) from n_4 to $n-1 \geq n_4$, we have

$$(f(b^*)f(\ell))^{1/\alpha}\sum_{s=n_4}^{n-1}\left(\frac{1}{a(s)}\sum_{k=n_4}^{s-1} q(k)f(g^2(k))f(c[g(k)])\right)^{1/\alpha}$$
$$\leq y(n_4) - y(n) \leq y(n_4) < \infty.$$

Letting $n \to \infty$ in the above inequality, we obtain a contradiction to condition (2.3).

Case (IV). For $t \geq s \geq n_1$, we see that

$$x(s) \geq (t-s)y(t),$$

where $y(n) = -\Delta x(n)$, $n \geq n_1$. Replacing t and s by $g(n)$ and $\xi(n)$, we get

$$x[g(n)] \geq (\xi(n) - g(n))y[\xi(n)] \text{ for } n \geq n_2 \geq n_1. \tag{2.19}$$

Using (2.19) and (1.4) in equation (1.1), we get

$$\Delta\left(a(n)(\Delta^2 y(n))^\alpha\right) + q(n)f(\xi(n) - g(n))f(y[\xi(n)]) \leq 0. \tag{2.20}$$

Repeating the above procedure, we see that there exists an $n_3 \geq n_2$ such that

$$y[\xi(n)] \geq (\eta(n) - \xi(n))Z[\eta(n)] \text{ for } n \geq n_3, \tag{2.21}$$

where $Z(n) = \Delta y(n)$, $n \geq n_3$. Using (2.21) and (1.4), we get

$$\Delta\left(a(n)(\Delta Z(n))^\alpha\right) + q(n) f(\xi(n) - g(n)) f(\eta(n) - \xi(n)) f(Z[\eta(n)]) \leq 0 \quad \text{for} \quad n \geq n_3.$$

The rest of the proof is similar to that of Case (III) and hence omitted. This completes the proof. ■

Next, we combine equations (2.1) and (2.2) in one by letting

$$Q(n) = \min\left\{\overline{c} q(n) f(A[g(n), n_1]), c^* q(n) f(g(n)) f(\xi(n) - g(n)) f\left(\sum_{k=\xi(n)}^{\eta(n)-1} a^{-1/\alpha}(k)\right)\right\}, \quad n \geq n_1 \in \mathbb{N}(n_0) \tag{2.22}$$

for constants $\overline{c}$, c^*, $0 < \overline{c}, c^* < 1$.

In this case one can easily see that equations (2.1) and (2.2) can be replaced by

$$\Delta v(n) + Q(n) f\left(v^{1/\alpha}[\eta(n)]\right) = 0. \tag{2.23}$$

We also let

$$\overline{Q}(n) = \min\{q(n) f(g^2(n)) f(c[g(n)]), q(n) f(\xi(n) - g(n)) f(\eta(n) - \xi(n)) f(c[g(n)])\}. \tag{2.24}$$

Here, conditions (2.3) and (2.4) may be replaced by

$$\sum_{s=n_1}^{\infty} \left(\frac{1}{a(s)} \sum_{k=n_1}^{s-1} \overline{Q}(k)\right)^{1/\alpha} = \infty, \quad n_1 \in \mathbb{N}(n_0). \tag{2.25}$$

Remark 2.2. We note that the results of this paper are presented in a form that it allows us to extract from it results for equation (1.1) which are valid when

$$\sum^{\infty} a^{-1/\alpha}(n) = \infty. \tag{2.26}$$

In fact, we have the following result.

Theorem 2.3. Let conditions (i)–(v), (1.4) and (2.26) hold and assume that there exist nondecreasing sequences $\{\xi(n)\}$ and $\{\eta(n)\}$, $\xi, \eta : \mathbb{N}(n_0) \to \mathbb{Z}$ such that $g(n) < \xi(n) < \eta(n) < n - 1$ for $n \in \mathbb{N}(n_0)$. If equation (2.23) is oscillatory, then equation (1.1) is oscillatory.

Proof. The proof of Theorem 2.3 is exactly the same as that of Theorem 2.1 – Cases (I) and (II) and employ $Q(n)$ in (2.22). ■

The following corollary is immediate.

Corollary 2.4. Let conditions (i)–(v), (1.3), (1.4) and (2.25) hold and assume that there exist nondecreasing sequences $\{\xi(n)\}$ and $\{\eta(n)\}$, $\xi, \eta : \mathbb{N}(n_0) \to \mathbb{Z}$ such that $g(n) < \xi(n) < \eta(n) < n-1$ for $n \in \mathbb{N}(n_0)$. Equation (1.1) is oscillatory if one of the following conditions holds:

(I_1) $\dfrac{f(u^{1/\alpha})}{u} \geq 1$ for $u \neq 0$ and

$$\limsup_{n\to\infty} \sum_{k=\eta(n)}^{n-1} Q(k) > 1,$$

where $Q(n)$ is defined as in (2.22);

(I_2) $\displaystyle\int_{\pm 0} \frac{du}{f(u^{1/\alpha})} < \infty$ and

$$\sum^{\infty} Q(n) = \infty;$$

(I_3) $\dfrac{u}{f(u^{1/\alpha})} \to 0$ as $u \to \infty$ and

$$\limsup_{n\to\infty} \sum_{k=\eta(n)}^{n-1} Q(k) > 0.$$

3. Oscillation Theorems for Equation (1.2)

In this section we are interested in obtaining criteria for the oscillation of all solutions of equation (1.2) of mixed arguments.

Theorem 3.1. Let conditions (i)–(v) and (1.3)–(1.5) hold and assume that there exist nondecreasing sequences $\{\xi(n)\}$ and $\{\rho(n)\}$, ξ and $\rho : \mathbb{N}(n_0) \to \mathbb{Z}$ such that

$$g(n) < \xi(n) < n \text{ and } \sigma(n) > \rho(n) > n \quad \text{for} \quad n \in \mathbb{N}(n_0). \tag{3.1}$$

If the first order difference equations

$$\Delta y(n) - p(n)h\left(\frac{(\sigma(n)-\rho(n))^{(3)}}{3!} a^{-1/\alpha}[\rho(n)]\right) h\left(y^{1/\alpha}[\rho(n)]\right) = 0 \tag{3.2}$$

$$\Delta z(n) + cq(n)f(g^2(n))f\left(\frac{\xi(n)-g(n)}{a^{1/\alpha}[\xi(n)]}\right) f\left(z^{1/\alpha}[\xi(n)]\right) = 0, \tag{3.3}$$

for every constant c, $0 < c < 1$

$$\Delta w(n) + q(n) f\left(\frac{(\xi(n) - g(n))^{(3)}}{3!} a^{-1/\alpha}[\xi(n)]\right) f\left(w^{1/\alpha}[\xi(n)]\right) = 0, \tag{3.4}$$

where $t^{(m)} = \prod_{i=0}^{m-1}(t - i)$, are oscillatory, and

$$\sum_{s=n_1 \geq n_0}^{\infty} \left(\frac{1}{a(s)} \sum_{k=n_1 \geq n_0}^{s-1} q(k) f(g(k)) f(\xi(k) - g(k)) f(c[\xi(k)])\right)^{1/\alpha} = \infty, \tag{3.5}$$

then equation (1.2) is oscillatory.

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.2), say, $x(n) > 0$, $x[g(n)] > 0$ and $x[\sigma(n)] > 0$ for $n \geq n_0 \geq 0$. Since $\Delta\left(a(n)(\Delta^3 x(n))^{\alpha}\right) \geq 0$ for $n \geq n_0$, there exists an $n_1 \geq n_0$ such that $\Delta^i x(n)$, $i = 1, 2, 3$ are of one sign for $n \geq n_1$. As earlier there are eight possibilities to consider. The following four cases hold:

(I) $\Delta^i x(n) > 0$, $i = 1, 2, 3$ for $n \geq n_1$;

(II) $\Delta^3 x(n) < 0$ and $\Delta^i x(n) > 0$, $i = 1, 2$ for $n \geq n_1$;

(III) $(-1)^i \Delta^i x(n) > 0$, $i = 1, 2, 3$ for $n \geq n_1$; and

(IV) $\Delta^3 x(n) > 0$, $\Delta^2 x(n) < 0$ and $\Delta x(n) > 0$ for $n \geq n_1$.

The remaining other four cases

$$\begin{array}{l}
\Delta^3 x(n) > 0,\ \Delta^2 x(n) > 0 \ \text{ and } \ \Delta x(n) < 0 \ \text{ for } \ n \geq n_1; \\
\Delta^3 x(n) > 0,\ \Delta^2 x(n) < 0 \ \text{ and } \ \Delta x(n) < 0 \ \text{ for } \ n \geq n_1; \\
\Delta^3 x(n) < 0,\ \Delta^2 x(n) < 0 \ \text{ and } \ \Delta x(n) < 0 \ \text{ for } \ n \geq n_1; \\
\Delta^3 x(n) < 0,\ \Delta^2 x(n) < 0 \ \text{ and } \ \Delta x(n) > 0 \ \text{ for } \ n \geq n_1,
\end{array}$$

have to be disregarded. Next, we consider:

Case (I). By Taylor series (see [1, p. 26]) for $t \geq s \geq n_1$, we have

$$x(t) \geq \frac{(t-s)^{(3)}}{3!} \Delta^3 x(s) := \frac{(t-s)^{(3)}}{3!} a^{-1/\alpha}(s) \left(a(s)(\Delta^3 x(s))^{\alpha}\right)^{1/\alpha},$$

or

$$x(t) \geq \frac{(t-s)^{(3)}}{3!} a^{-1/\alpha}(s)(y(s))^{1/\alpha},$$

where $y(n) = a(n)(\Delta^3 x(n))^\alpha$, $n \geq n_1$. Replacing t and s by $\sigma(n)$ and $\rho(n)$ respectively, we find

$$x[\sigma(n)] \geq \frac{(\sigma(n) - \rho(n))^{(3)}}{3!} a^{-1/\alpha}[\rho(n)] y^{1/\alpha}[\rho(n)] \text{ for } n \geq n_2 \geq n_1. \tag{3.6}$$

Using (3.6) and (1.5) in equation (1.2), we have

$$\Delta y(n) \geq p(n) h \left(\frac{(\sigma(n) - \rho(n))^{(3)}}{3!} a^{-1/\alpha}[\rho(n)] \right) f \left(y^{1/\alpha}[\rho(n)] \right) \text{ for } n \geq n_2.$$

Now by known results, see [3, 5], we arrive at the desired contradiction.

Case (II). There exist a constant b, $0 < b < 1$ and an integer $n_2 \geq n_1$ such that

$$\Delta x(n) \geq bn \Delta^2 x(n) \text{ for } n \geq n_2.$$

Summing this inequality from n_2 to $n - 1$, we see that there exist a constant $\overline{b}$, $0 < \overline{b} < 1$ and an $n_3 \geq n_2$ such that

$$x[g(n)] \geq bg^2(n) z[g(n)] \text{ for } n \geq n_3, \tag{3.7}$$

where $z(n) = \Delta^2 x(n)$, $n \geq n_3$. Using (3.7) and (1.4) in equation (1.2), we have

$$\Delta \left(a(n)(\Delta z(n))^\alpha \right) \geq f(\overline{b}) q(n) f(g^2(n)) f(z[g(n)]) \text{ for } n \geq n_3. \tag{3.8}$$

Clearly, $z(n) > 0$ and $\Delta z(n) < 0$ for $n \geq n_3$. As in the proof of Theorem 2.1 – Case (II), there exists an $n_4 \geq n_3$ such that

$$z[g(n)] \geq (\xi(n) - g(n))(-\Delta z[\xi(n)]) \text{ for } n \geq n_4,$$

or

$$z[g(n)] \geq (\xi(n) - g(n)) a^{-1/\alpha}[\xi(n)] w^{1/\alpha}[\xi(n)] \text{ for } n \geq n_4, \tag{3.9}$$

where $w(n) = -a(n)(\Delta z(n))^\alpha$ for $n \geq n_4$. Using (3.9) and (1.4) in (3.8), we get

$$\Delta w(n) + f(\overline{b}) q(n) f(g^2(n)) f \left(\frac{\xi(n) - g(n)}{a^{1/\alpha}[\xi(n)]} \right) f \left(w^{1/\alpha}[\xi(n)] \right) \leq 0 \text{ for } n \geq n_4.$$

The rest of the proof is similar to that of Theorem 2.1 – Case (I) and hence omitted.

Case (III). By Taylor's series one can easily see that there exists an $n_2 \geq n_1$ such that

$$x[g(n)] \geq \frac{(\xi(n) - g(n))^{(3)}}{3!} a^{-1/\alpha}[\xi(n)] y^{1/\alpha}[\xi(n)] \text{ for } n \geq n_2, \tag{3.10}$$

where $y(n) = -a(n)(\Delta^3 x(n))^\alpha, n \geq n_2$. Using (3.10) and (1.4) in equation (1.1), we have

$$\Delta y(n) + q(n) f\left(\frac{(\xi(n) - g(n))^{(3)}}{3! a^{1/\alpha}[\xi(n)]}\right) f\left(y^{1/\alpha}[\xi(n)]\right) \leq 0 \text{ for } n \geq n_2.$$

The rest of the proof is similar to that of Theorem 2.1 – Case (I) and hence omitted.

Case (IV). There exist a constant b, $0 < b < 1$ and an $n_2 \geq n_1$ such that

$$x[g(n)] \geq b g(n) y[g(n)] \text{ for } n \geq n_2, \tag{3.11}$$

where $y(n) = \Delta x(n), n \geq n_2$. Using (3.11) and (1.4) in equation (1.2), we get

$$\Delta\left(a(n)(\Delta^2 y(n))^\alpha\right) \geq f(b) q(n) f(g(n)) f(y[g(n)]) \text{ for } n \geq n_2. \tag{3.12}$$

Clearly, $\Delta^2 y(n) > 0$, $\Delta y(n) < 0$ and $y(n) > 0$ for $n \geq n_2$. Thus, there exists an integer $n_3 \geq n_2$ such that

$$y[g(n)] \geq (\xi(n) - g(n)) z[\xi(n)] \text{ for } n \geq n_3, \tag{3.13}$$

where $z(n) = -\Delta y(n), n \geq n_3$. Using (3.13) and (1.4) in (3.12), we get

$$\Delta\left(a(n)(\Delta z(n))^\alpha\right) + f(b) q(n) f(g(n)) f(\xi(n) - g(n)) f\left(z^{1/\alpha}[\xi(n)]\right) \leq 0 \text{ for } n \geq n_3.$$

The rest of the proof is similar to that of Theorem 2.1 – Case (III) and hence omitted. This completes the proof. ■

We may combine equations (3.3) and (3.4) in one by letting

$$\tilde{Q}(n) = \min\left\{c q(n) f(g^2(n)) f\left(\frac{\xi(n) - g(n)}{a^{1/\alpha}[\xi(n)]}\right), \right.$$
$$\left. q(n) f\left(\frac{(\xi(n) - g(n))^{(3)}}{3! a^{1/\alpha}[\xi(n)]}\right)\right\} \tag{3.14}$$

for any constant c, $0 < c < 1$.

In this case one can easily see that equations (3.3) and (3.4) can be replaced by

$$\Delta w(n) + \tilde{Q}(n) f\left(w^{1/\alpha}[\xi(n)]\right) = 0. \tag{3.15}$$

Next, we extract from Theorem 3.1 the following result which is valid when condition (2.16) holds.

Theorem 3.2. Let conditions (i)–(iv), (1.4), (1.5) and (2.26) hold and assume that there exist nondecreasing sequences $\{\xi(n)\}$ and $\{\rho(n)\}$ such that (3.1) holds. If equations (3.2) and (3.15) are oscillatory, then equation (1.2) is oscillatory.

Proof. The proof of Theorem 3.2 is the same as that of Theorem 3.1 – Cases (I) – (III) and make use of $\tilde{Q}(n)$ in (3.14). ■

The following corollary is immediate.

Corollary 3.3. Let conditions (i)–(v) and (1.3)–(1.5) and (3.5) hold and assume that there exist nondecreasing sequences $\{\xi(n)\}$ and $\{\rho(n)\}$ such that (3.1) holds. Equation (2.1) is oscillatory if one of the following conditions holds:

(II_1) $\dfrac{h(u^{1/\alpha})}{u} \geq 1$ and $\dfrac{f(u^{1/\alpha})}{u} \geq 1$ for $u \neq 0$,

$$\limsup_{n\to\infty} \sum_{k=n}^{\rho(n)-1} p(k)h\left(\frac{(\sigma(k)-\rho(k))^{(3)}}{3!}a^{-1/\alpha}[\rho(k)]\right) > 1$$

and

$$\limsup_{n\to\infty} \sum_{k=\xi(n)}^{n-1} \tilde{Q}(k) > 1,$$

where $\tilde{Q}$ is as in (3.14);

(II_2) $\displaystyle\int^{\pm\infty} \frac{du}{h(u^{1/\alpha})} < \infty$ and $\displaystyle\int_{\pm 0} \frac{du}{f(u^{1/\alpha})} < \infty$,

$$\sum^{\infty} p(n)h\left(\frac{(\sigma(n)-\rho(n))^{(3)}}{3!}a^{-1/\alpha}[\rho(n)]\right) = \infty$$

and

$$\sum^{\infty} \tilde{Q}(n) = \infty;$$

(II_3) $\dfrac{u}{h(u^{1/\alpha})} \to 0$ as $u \to \infty$ and $\dfrac{y}{f(y^{1/\alpha})} \to 0$ as $y \to 0$,

$$\limsup_{n\to\infty} \sum_{k=n}^{\rho(n)-1} p(k)h\left(\frac{(\sigma(k)-\rho(k))^{(3)}}{3!}a^{-1/\alpha}[\rho(k)]\right) > 0$$

and

$$\limsup_{n\to\infty} \sum_{k=\xi(n)}^{n-1} \tilde{Q}(k) > 0.$$

4. General Remarks

1. Conditions (1.4) and (1.5) can be discarded if we let $f(x) = x^{\beta}$ and $h(x) = x^{\gamma}$, where β and γ are ratios of two positive odd integers.

2. By applying other known oscillation results for first order difference equations (see [5]) one can easily drive oscillation criteria similar to Corollaries 2.4 and 3.3 from Theorems 2.1 and 3.1 respectively. The details are left to the reader.

3. The conditions that involve $\sum_{k=n_0}^{n-1} a^{-1/\alpha}(k)$ or $\sum_{k=n}^{\infty} a^{-1/\alpha}(k)$ etc., may be reduced to simpler form by replacing it with its upper bound, i.e., by a constant. We also note that condition (1.3) may be replaced by

$$\sum^{\infty} \frac{k}{a^{1/\alpha}(k)} < \infty \text{ or } \sum^{\infty} \left(\frac{k}{a(k)}\right)^{1/\alpha} < \infty$$

and the results remain valid possibly with some minor changes. The details are left to the reader.

4. The results of this paper can be easily extended to neutral difference equations of the form

$$\Delta \left(a(n) \left(\Delta^3 (x(n) + c(n)x[\tau(n)]) \right)^{\alpha} \right) + q(n) f(x[g(n)]) = 0$$

and

$$\Delta \left(a(n) \left(\Delta^3 (x(n) + c(n)x[\tau(n)]) \right)^{\alpha} \right) = q(n) f(x[g(n)]) + p(n)h(x[\sigma(n)]),$$

where $\{c(n)\}$ is a sequence of real numbers with $0 < c(n) < 1$ or $c(n) > 1$, and $\tau : \mathbb{N}(n_0) \to \mathbb{Z}$ is nondecreasing and $\lim_{n\to\infty} \tau(n) = \infty$. The formulation of results for the above neutral equations are easy. The details are left to the reader.

5. It will be of interest to employ different techniques rather than used in this paper, and obtain other criteria for the oscillation of equations (1.1) and (1.2).

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