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On a Rational Recursive Sequence with Parameter near the Boundary

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Abstract

This note studies existence of positive prime periodic solutions of higher order for rational recursive equations of the form $y_n = A + y_{n-k}/y_{n-m}$, $n = 0, 1, 2, \ldots$, with $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty)$, k odd and $m \in \{1, 2, 3, 4, \ldots\}$, where $s = \max\{k, m\}$. In particular, we show that for $k \geq 5$, odd, $m \geq 1$, $\gcd(k, m) = 1$ and sufficiently small $A > 0$, there exist periodic solutions with prime period $2m^* + U_{m^*}$, for some m^*, where $U_m = \min\{i \in \mathbb{N} : i(i+1) \geq 2m\}$. A value of $m^* > (k-1)^2/2 + m$ is given explicitly.

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1. Introduction

This note studies existence of prime periodic solutions of higher order for rational recursive equations of the form

$$y_n = A + \frac{y_{n-k}}{y_{n-m}}, \quad n = 0, 1, \dots, \tag{1.1}$$

with $A > 0$, $y_{-s}, y_{-s+1}, \dots, y_{-1} \in (0, \infty)$ and $m \in \{1, 2, 3, 4, \dots\}$, where $s = \max\{k, m\}$. Equation (1.1) has been studied by many authors in the recent past. In [1], conditions for global asymptotic stability of solutions are presented for $k = 1$. In [4], some quantitative bounds for solutions are provided. Properties of solutions for $A < 0$ are considered in [17] and [18]. Further results for equations of the type in (1.1) can be found in [1–20] and the references therein.

It is known that all positive solutions to (1.1) are bounded (c.f. [1, 6]), and that a sufficient condition for global asymptotic stability of the positive equilibrium of Equation (1.1) is $A > 1$, but little is known regarding possible behavior of solutions for small $A > 0$ and large k and m. One particularly well-known conjecture regarding solutions for $A < 1$ is the following (see for instance, [1]).

Conjecture 1.1. Suppose that $(k, m) = (1, 3)$. Prove that when $A > \sqrt{2} - 1$, the unique positive equilibrium of Equation (1.1) is globally asymptotically stable.

In [3], it was shown that for $k = 1$ and almost all m, for sufficiently small A, there exists a prime period $2m + U_m$ solution to (1.1), where $U_m = \max\{i \in \mathbb{N} : i(i+1) \leq 2(m-1)\} + 1 = \min\{i \in \mathbb{N} : i(i+1) \geq 2m\}$. In particular, the following theorem was proven.

Theorem 1.2. Set $\mathcal{V} = \bigcup_{j>0} \left\{ \frac{j(j+1)}{2}, \frac{j(j+1)}{2} + 1 \right\}$. If $m > 1$ satisfies $m \notin \mathcal{V}$ and $k = 1$, then there exists an $\epsilon_m > 0$ such that for all $0 < A < \epsilon_m$, there exists a prime period $2m + U_m$ solution to (1.1).

Here we will generalize the work in [3] to cover the case of general odd k by proving the following theorem.

Theorem 1.3. For $k, m \in \mathbb{N}^+$, set

$$U_{k,m} = \max\{i \in \mathbb{N} : i(i + k - 2) \leq 2m\}. \tag{1.2}$$

If $k \geq 5$ is odd, then <u>**for all**</u> $m \geq 1$ with $\gcd(k, m) = 1$, there exists an $\epsilon_{k,m} > 0$ such that for all $0 < A < \epsilon_{k,m}$, there exists a prime period

$$P_{m^*} = 2m^* + U_{m^*} = k(k-1) + 2m + kU_{k,m}$$

solution to (1.1), where

$$m^* = \frac{(k-1)^2}{2} + m + \frac{k-1}{2} U_{k,m}. \tag{1.3}$$

Table 1: Prime periods of existing positive solutions to Equation (1.1) for sufficiently small A

k/m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1					13			20	22			29	31	33	
3		13		20	22		29	31			40		44	46	
5	22	29	31	33		42	44	46	53		57	59	61	68	
7	44	46	55	57	59	61		72	74	76	78	87	89		93
9	74	76		89	91		95	97		110	112		116	118	
11	112	114	116	118	131	133	135	137	139	141		156	158	160	162
13	158	160	162	164	166	181	183	185	187	189	191	193		210	212
15	212	214	216	218		222	239	241	243		247	249	251	253	
17	274	276	278	280	282	284	286	305	307	309	311	313	315	317	319
19	344	346	348	350	352	354	356	358	379	381	383	385	387	389	391
21	422	424		428	430			436		461	463		467		
23	508	510	512	514	516	518	520	522	524	526	551	553	555	557	559
25	602	604	606	608		612	614	616	618		622	649	651	653	

Remark 1.4. For the case $k = 3$, see Theorem 2.1, below.

Some periods implied by the results given here are provided in Table 1.4. Theorem 1.3 as well as a result covering the case $k = 3$ are proven in the next section.

2. Proof of the Main Theorem

In this section we prove Theorem 1.3. The essential idea is to show that m^* given in (1.3) (which was initially suggested through computations) satisfies

(i) $U_{m^*} = U_{k,m} + k - 1$

(ii) $m^* \notin \mathcal{V}$

(iiii) $\gcd(k, P_{m^*}) = 1$

(iv) $m^* k = m \mod P_{m^*}$.

The result will then follow upon employing Theorem 1.2 for $(k, m) = (1, m^*)$ to obtain a prime periodic solution, $\{y_i\}$, to the equation $y_n = A + y_{n-1}/y_{n-m^*}$ and verifying that $\{y_i^*\}$ defined via $y_i^* = y_j$ whenever $kj = i \mod P_{m^*}$ is a prime periodic solution, as required.

Proof of Theorem 1.3. From the definition of $U_{k,m}$, we have

$$U_{k,m}(U_{k,m} + (k-2)) \leq 2m \text{ and } (U_{k,m} + 1)(U_{k,m} + (k-1)) \geq 2m + 1. \tag{2.1}$$

Now, note that via (2.1) and (1.3)

$$\begin{aligned}(U_{k,m}+(k-1))(U_{k,m}+k) &= (U_{k,m}+(k-1))(U_{k,m}+1)+U_{k,m}(k-1)+(k-1)^2\\ &\geq 2m+1+U_{k,m}(k-1)+(k-1)^2 > 2m^* \qquad (2.2)\end{aligned}$$

and

$$\begin{aligned}&(U_{k,m}+(k-2))(U_{k,m}+(k-1))\\ &\quad=(U_{k,m}+(k-2))U_{k,m}+U_{k,m}(k-1)+(k-1)(k-2)\\ &\quad=(k-1)^2-(k-1)+U_{k,m}(k-1)+(U_{k,m}+(k-2))U_{k,m}\\ &\quad\leq (k-1)^2+(k-1)U_{k,m}+2m-(k-1)\\ &\quad=2m^*-(k-1)<2m^*. \qquad (2.3)\end{aligned}$$

The inequalities in (2.2) and (2.3) and the definition of U_{m^*} give that

$$U_{m^*}=U_{k,m}+(k-1). \qquad (2.4)$$

We then have via (1.3) and (2.4) that

$$\begin{aligned}P_{m^*} &= 2m^*+U_{m^*}=(k-1)^2+2m+(k-1)U_{k,m}+U_{m^*}\\ &=k(k-1)+2m+kU_{k,m}+(U_{m^*}-U_{k,m}-(k-1))=k(k-1)+2m+kU_{k,m}. \qquad (2.5)\end{aligned}$$

If $(k-1)/2>1$ (i.e., $k>3$), the inequalities in (2.2) and (2.3) also guarantee that $m^*\notin\mathcal{V}$. Hence, suppose that $\{a_i\}$ is a solution to the equation

$$y_n=A+\frac{y_{n-1}}{y_{n-m^*}},\quad n=0,1,\ldots. \qquad (2.6)$$

Note that Equation (2.5) gives that $\gcd(k,P_{m^*})=1$ (since k is odd and $\gcd(k,m)=1$) and define the sequence $\{a_i^*\}$ via $a_i^*=a_j$, whenever $kj=i \mod P_{m^*}$.

Now, for $n>s^*\stackrel{\text{def}}{=}\max\{k,m,m^*\}$, consider a_n^*, a_{n-k}^* and a_{n-m}^*. We have $a_n^*=a_{nk^{-1}}$ where k^{-1} is taken so that $k^{-1}k=1 \mod P_{m^*}$ and $k^{-1}>0$. Similarly $a_{n-k}^*=a_{(n-k)k^{-1}}=a_{nk^{-1}-1}$ and $a_{n-m}^*=a_{(n-m)k^{-1}}=a_{nk^{-1}-mk^{-1}}$. Employing (1.3) and (2.5) gives

$$\begin{aligned}m^*k-m &= \frac{k(k-1)^2}{2}+m(k-1)+\frac{k(k-1)}{2}U_{k,m}\\ &=\frac{k-1}{2}(k(k-1)+2m+kU_{k,m})=\frac{k-1}{2}P_m^*=0 \mod P_m^*. \qquad (2.7)\end{aligned}$$

Thus, by the definition of $\{a_i^*\}$ and the P_{m^*}-periodicity of $\{a_i\}$, we have

$$\begin{aligned} a_n^* &= a_{nk^{-1}} = A + \frac{a_{nk^{-1}-1}}{a_{nk^{-1}-m^*}} = A + \frac{a_{(n-k)k^{-1}}}{a_{nk^{-1}-m^*}} \\ &= A + \frac{a_{(n-k)k^{-1}}}{a_{nk^{-1}-mk^{-1}}} = A + \frac{a_{(n-k)k^{-1}}}{a_{(n-m)k^{-1}}} \\ &= A + \frac{a_{n-k}^*}{a_{n-m}^*} \end{aligned} \tag{2.8}$$

and $\{a_i^*\}_{i>s^*}$ is a periodic solution of (1.1) with period P_{m^*}. To verify that the constructed solution, $\{a_i^*\}$, has *prime* period P_{m^*}, note that for sufficiently small A, the solution $\{a_i\}$ of period P_{m^*} constructed in [3] has only one value in the interval $[1 - A, 1 + 3A]$ or one in the interval $[1/A - 3, 1/A + 4]$ per each P_{m^*}-cycle. Any P_{m^*} consecutive terms of the solution $\{a_i^*\}$ comprise a simple reordering of the values in the cycle, and hence $\{a_i^*\}$ has prime period P_{m^*}. ■

For the case $k = 3$, we have the following.

Theorem 2.1. Suppose $k = 3$ and set $\mathcal{W} = \bigcup_{j>0} \left\{ \frac{j(j+1)}{2} \right\}$ (the set of positive triangular numbers). If $m > 1$ satisfies $m \notin \mathcal{W}$ and $\gcd(m, 3) = 1$, then there exists an $\epsilon_{3,m} > 0$ such that for all $0 < A < \epsilon_{3,m}$, there exists a prime period $Q = 6+2m+3U_{3,m}$ solution to (1.1).

Proof. Note that for $k = 3$ and m satisfying $\gcd(3, m) = 1$, all parts of the proof of Theorem 1.3 hold except perhaps that m^* could be an element of $\mathcal{V}$. Considering Equation (2.3), this can happen only if $(U_{k,m} + (k - 2))U_{k,m} = (U_{3,m} + 1)U_{3,m} = 2m$, or equivalently if $m \in \mathcal{W}$. The proof then follows as in the case of $k \geq 5$. ■

We close with the following conjecture.

Conjecture 2.2. For k, m satisfying the requirements of either Theorem 1.2, 1.3 or 2.1, there exists an $\epsilon_{k,m} > 0$ such that all nontrivial solutions to Equation (1.1) with $0 < A < \epsilon_{k,m}$ are asymptotically periodic with prime period as indicated in the theorem.

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