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Global Attractivity for Nonlinear Delay Dynamic Equations

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Abstract

Conditions under which solutions of a first-order nonlinear variable-delay dynamic equation go to zero at infinity are given, for arbitrary time scales that are unbounded above. In two examples, we apply our techniques to dynamic equations on isolated, unbounded time scales.

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1. Introduction to the Delay Dynamic Equation

Utilizing Hilger's landmark paper [11], in which he unifies, extends, and generalizes ideas from continuous calculus, discrete calculus, and quantum calculus to arbitrary time-scale calculus, we extend to dynamic equations on time scales some discrete results on delay difference equations. In particular, we consider the nonlinear variable delay dynamic equation

$$x^{\Delta}(t) = F(t, x(\tau(t))), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale unbounded above, and the function $F : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to its two variables with $F(t, 0) \equiv 0$ for each $t \in \mathbb{T}$. Moreover, the variable delay $\tau : \mathbb{T} \to \mathbb{T}$ is increasing with $\tau(t) \leq t$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ such that $\lim_{t\to\infty} \tau(t) = \infty$. The initial function associated with (1.1) takes the form $x(t) = \psi(t)$ for $t \in [\tau(t_0), t_0]$, where ψ is rd-continuous on $[\tau(t_0), t_0]$. Equation (1.1) is studied

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extensively by Zhang [16] and Zhang and Yan [17] in the case when $\mathbb{T} = \mathbb{Z}$; indeed much of the organization of this paper is motivated by these two papers. See also related discussions by Erbe, Xia, and Yu in [9]; Matsunaga, Miyazaki, and Hara [13]; Matsunaga, Hara, and Sakata [12]; and the recent work by Graef and Qian [10]. Related papers on delay dynamic equations include [1–4] and work by Bohner [5], Čermák and M. Urbánek [8], Pötzsche [14], and Wu and Zhou [15]. For more on dynamic equations on time scales consult the recent texts by Bohner and Peterson [6, 7]. To clarify some notation, take $\tau^{-1}(t) := \sup\{s : \tau(s) \leq t\}$, $\tau^{-(n+1)}(t) = \tau^{-1}(\tau^{-n}(t))$ for $t \in [\tau(t_0), \infty)_{\mathbb{T}}$, and $\tau^{n+1}(t) = \tau(\tau^n(t))$ for $t \in [\tau^{-3}(t_0), \infty)_{\mathbb{T}}$. By our choice of the delay τ, there exists large $T \in \mathbb{T}$ such that $\tau(t) \geq t_0$ and $\tau^2(t) \leq \tau(t) \leq t \leq \tau^{-1}(\sigma(t))$ for all $t \geq T$. In addition, we always suppose

(H1) there exists continuous $f : \mathbb{R} \to \mathbb{R}$ such that $xf(x) > 0$ and $|f(x)| < |x|$ for $x \neq 0$, with

$$f^{\dagger}(x) := \max\left\{ \sup_{0 \leq u \leq |x|} f(u), \quad \sup_{0 \leq u \leq |x|} (-f(-u)) \right\}, \qquad x \in \mathbb{R};$$

(H2) there exist rd-continuous functions $a, b : \mathbb{T} \to [0, \infty)$ such that for large $t \in \mathbb{T}$, $x \in \mathbb{R}$, and f from (H1),

$$a(t) \min\{0, f(-x)\} \leq F(t, x) \leq b(t) \max\{0, f(-x)\};$$

(H3) there exist $A, B > 0$ such that for large $t \in \mathbb{T}$ and a, b from (H2),

$$\int_{\tau(t)}^{\sigma(t)} a(s)\Delta s \leq \lambda A \quad \text{and} \quad \int_{\tau(t)}^{\sigma(t)} b(s)\Delta s \leq \lambda B,$$

where

$$\lambda := \frac{3}{2} + \frac{1}{2} \frac{\inf\{\mu(t) : t \in \mathbb{T}\}}{\sup\{\tau^{-1}(\sigma(t)) - t : t \in \mathbb{T}\}}. \tag{1.2}$$

It is understood that $\lambda = 3/2$ if either $\inf\{\mu(t)\} = 0$ or $\sup\{\tau^{-1}(\sigma(t)) - t\} = \infty$. Equation (1.1) has the related special form

$$x^{\Delta}(t) + p(t)g(x(\tau(t))) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.3}$$

where $p : \mathbb{T} \to \mathbb{R}$ is rd-continuous and $g : \mathbb{R} \to \mathbb{R}$ is continuous with $xg(x) > 0$ for $x \neq 0$. The following results were established in the discrete case by Zhang and Yan [17] and extended to arbitrary time scales [3].

Theorem 1.1. [3, Theorem 3.1] Suppose that (H1)–(H3) hold, and that

$$\max\{A, A^2\} \cdot \max\{B, B^2\} \leq 1. \tag{1.4}$$

If there exists $t^* \in \mathbb{T}$ and a collection of continuous functions $\{H(t, \cdot) : (0, \infty) \to [0, \infty)\}_{t \in \mathbb{T}}$ such that, for any $\epsilon > 0$ and $t \in [t^*, \infty)_{\mathbb{T}}$,

$$\sup_{x \geq \epsilon} F(t, x) \leq -H(t, \epsilon), \quad \inf_{x \leq -\epsilon} F(t, x) \geq H(t, \epsilon), \quad \int_{t^*}^{\infty} H(t, \epsilon) \Delta t = \infty, \tag{1.5}$$

then every solution of (1.1) goes to zero in the limit.

Corollary 1.2. [3, Corollary 3.2] Let $\alpha, \beta > 0$ such that

$$-\alpha|x| \leq g(x) \leq \beta|x|, \qquad x \neq 0.$$

Assume for all large $t \in \mathbb{T}$ that

$$(\alpha\beta^2)^{1/3} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s \leq \lambda \text{ if } \alpha \leq \beta, \; (\alpha^2\beta)^{1/3} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s \leq \lambda \text{ if } \alpha > \beta. \tag{1.6}$$

If $\int_{t_0}^{\infty} p(t)\Delta t = \infty$, then every solution of (1.3) goes to zero in the limit.

Note that in condition (1.4), the larger of A and B may be greater than one, but with its square very large, the smaller of the two must be correspondingly small. It is the purpose of this note to replace (1.4) with the weaker condition

$$AB \leq 1, \tag{1.7}$$

following the discrete case by Zhang [16].

2. Foundational Lemmas

We will need Lemma 2.1 in the proof of Lemma 2.3.

Lemma 2.1. [1, Lemma 2.1] For right-dense continuous functions $w : \mathbb{T} \to \mathbb{R}$ and points $c, t \in \mathbb{T}$,

$$\int_c^t \left(w(s) \int_c^{\sigma(s)} w(z)\Delta z \right) \Delta s = \frac{1}{2} \left(\int_c^t w(s)\Delta s \right)^2 + \frac{1}{2} \int_c^t \mu(s) w^2(s)\Delta s.$$

Lemma 2.2. For any $K > 0$ and λ given in (1.2), set

$$\Lambda_K := \max_{0 \leq z \leq 1} \left\{ \lambda K z - (\lambda - 1) z^2 \right\}.$$

If $A, B \in (0, \infty)$ with $AB \leq 1$, then $0 < \Lambda_A \Lambda_B \leq 1$.

Proof. For $K > 0$, consider the function

$$\phi_K(z) = \lambda K z - (\lambda - 1) z^2,$$

which has a positive root $z_r = \dfrac{\lambda K}{\lambda - 1}$. If $\hat{z}_r = \min\{1, z_r\}$, then $\Lambda_K = \max\limits_{0 \le z \le \hat{z}_r} \phi_K(z) > 0$. For $z_1 \in \left[0, \hat{z}_B\right]$ and $z_2 \in \left[0, \hat{z}_A\right]$, we have $\sqrt{z_1 z_2} \in \left[0, \hat{z}_{\sqrt{AB}}\right]$, and

$$\begin{aligned} \phi_B(z_1)\phi_A(z_2) &= \left[\lambda B z_1 - (\lambda - 1)z_1^2\right]\left[\lambda A z_2 - (\lambda - 1)z_2^2\right] \\ &= \lambda^2 AB z_1 z_2 - \lambda(\lambda - 1)\left[A z_1^2 z_2 + B z_1 z_2^2\right] + [(\lambda - 1)z_1 z_2]^2 \\ &\le \lambda^2 AB z_1 z_2 - 2\lambda(\lambda - 1)\sqrt{AB}(z_1 z_2)^{3/2} + [(\lambda - 1)z_1 z_2]^2 \\ &= \left[\phi_{\sqrt{AB}}\left(\sqrt{z_1 z_2}\right)\right]^2. \end{aligned}$$

Since $0 < \phi_{\sqrt{AB}}\left(\sqrt{z_1 z_2}\right) \le \phi_{\sqrt{AB}}(1)$ and $\phi_K(1)$ is increasing for $K \in [0, 1]$, we see that

$$\phi_B(z_1)\phi_A(z_2) \le \phi_1^2(1) = 1$$

for $z_1 \in \left[0, \hat{z}_B\right]$ and $z_2 \in \left[0, \hat{z}_A\right]$. Thus, $\Lambda_A \Lambda_B \le 1$. ■

Lemma 2.3. Assume (H1)–(H3) hold, and let x be a solution of (1.1). Suppose there exists $t_1 \in (\tau^{-2}(T), \infty)_{\mathbb{T}}$ such that $\tau^2(t_1) \ge t_0$, $(H3)$ holds for $t \ge \tau^2(t_1)$, and $x(t_1)x^\sigma(t_1) \le 0$. Let $M > 0$ be given.

(i) If $x(t) \ge -M$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$, then

$$x(t) \le \Lambda_B f^\dagger(M) \quad \text{for} \quad t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}.$$

(ii) If $x(t) \le M$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$, then

$$x(t) \ge -\Lambda_A f^\dagger(M) \quad \text{for} \quad t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}.$$

Proof. We concentrate solely on (i), since the proof of (ii) is similar. Following from $x(t_1)x^\sigma(t_1) \le 0$, there exists a real number $\xi \in [t_1 - 1, t_1]$ such that

$$x(t_1) + [x^\sigma(t_1) - x(t_1)](\xi - t_1 + 1) = 0. \tag{2.1}$$

Because $f^\dagger$ is nonnegative and nondecreasing, $f(x(t)) \ge -f^\dagger(x(t)) \ge -f^\dagger(M)$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$; from (1.1) and (H2) we have

$$x^\Delta(t) \le b(t) f^\dagger(M), \qquad t \in [\tau(t_1), \tau^{-1}(t_1)]_{\mathbb{T}}, \tag{2.2}$$

so that integration and the fundamental theorem of time scale calculus yield

$$x(t_1) - x(\tau(t)) \le f^\dagger(M) \int_{\tau(t)}^{t_1} b(s)\Delta s, \quad t \in [t_1, \tau^{-1}(t_1)]_{\mathbb{T}}.$$

Using the characterization of ξ in (2.1), we obtain that for $t \in [t_1, \tau^{-1}(t_1)]_{\mathbb{T}}$,

$$\begin{aligned} x(\tau(t)) &\geq x(t_1) - f^{\dagger}(M)\int_{\tau(t)}^{t_1} b(s)\Delta s \\ &= -[x^{\sigma}(t_1) - x(t_1)](\xi - t_1 + 1) - f^{\dagger}(M)\int_{\tau(t)}^{t_1} b(s)\Delta s \\ &\geq -f^{\dagger}(M)\left[(\xi - t_1)\int_{t_1}^{\sigma(t_1)} b(s)\Delta s + \int_{\tau(t)}^{\sigma(t_1)} b(s)\Delta s\right], \end{aligned}$$

where we used (2.2) to arrive at the last line. Continuing in this manner, from (H2) and the fact that $f^{\dagger}(x) < x$ for positive x, we see that

$$\begin{aligned} x^{\Delta}(t) &\leq b(t)f^{\dagger}\left(f^{\dagger}(M)\left[(\xi - t_1)\int_{t_1}^{\sigma(t_1)} b(s)\Delta s + \int_{\tau(t)}^{\sigma(t_1)} b(s)\Delta s\right]\right) \\ &\leq b(t)f^{\dagger}(M)\left[(\xi - t_1)\int_{t_1}^{\sigma(t_1)} b(s)\Delta s + \int_{\tau(t)}^{\sigma(t_1)} b(s)\Delta s\right] \end{aligned} \tag{2.3}$$

for $t \in [t_1, \tau^{-1}(t_1)]_{\mathbb{T}}$. Now by (H3) we know that

$$0 \leq \zeta := (t_1 - \xi)\int_{t_1}^{\sigma(t_1)} b(s)\Delta s + \int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b(s)\Delta s \leq \lambda B, \tag{2.4}$$

which we consider in the following two cases.

Case 1:

Suppose ζ defined in (2.4) satisfies $\zeta \in (0, 1)$. For $t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$, we have

$$\begin{aligned} x(t) &= x^{\sigma}(t_1) + \int_{\sigma(t_1)}^{t} x^{\Delta}(s)\Delta s \\ &\overset{(2.1)}{=} [x^{\sigma}(t_1) - x(t_1)](t_1 - \xi) + \int_{\sigma(t_1)}^{t} x^{\Delta}(s)\Delta s \\ &= (t_1 - \xi)\mu(t_1)x^{\Delta}(t_1) + \int_{\sigma(t_1)}^{t} x^{\Delta}(s)\Delta s \\ &\overset{(2.3)}{\leq} f^{\dagger}(M)(t_1 - \xi)\mu(t_1)b(t_1)\left[(\xi - t_1)\int_{t_1}^{\sigma(t_1)} b(s)\Delta s + \int_{\tau(t_1)}^{\sigma(t_1)} b(s)\Delta s\right] \\ &\quad + f^{\dagger}(M)\int_{\sigma(t_1)}^{t} b(s)\left[(\xi - t_1)\int_{t_1}^{\sigma(t_1)} b(u)\Delta u + \int_{\tau(s)}^{\sigma(t_1)} b(u)\Delta u\right]\Delta s \\ &= f^{\dagger}(M)\left\{(t_1 - \xi)\mu(t_1)b(t_1)\left[\int_{\tau(t_1)}^{\sigma(t_1)} b(s)\Delta s - (t_1 - \xi)\mu(t_1)b(t_1)\right]\right. \\ &\quad \left. + \int_{\sigma(t_1)}^{t} b(s)\left[\int_{\tau(s)}^{\sigma(t_1)} b(u)\Delta u - (t_1 - \xi)\mu(t_1)b(t_1)\right]\Delta s\right\}, \end{aligned}$$

where the last equality follows from simple factoring. Continuing,

$$\begin{aligned}
x(t) &\overset{\text{(H3)}}{\leq} f^{\dagger}(M)\left\{(t_1-\xi)\mu(t_1)b(t_1)\left[\lambda B-(t_1-\xi)\mu(t_1)b(t_1)\right]\right.\\
&\quad\left.+\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b(s)\left[\lambda B-\int_{\sigma(t_1)}^{\sigma(s)} b(u)\Delta u-(t_1-\xi)\mu(t_1)b(t_1)\right]\Delta s\right\}\\
&= f^{\dagger}(M)\left\{-[(t_1-\xi)\mu(t_1)b(t_1)]^2-(t_1-\xi)\mu(t_1)b(t_1)\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b(s)\Delta s\right.\\
&\quad\left.+\lambda B\zeta-\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b(s)\left[\int_{\sigma(t_1)}^{\sigma(s)} b(u)\Delta u\right]\Delta s\right\}.
\end{aligned}$$

Using Lemma 2.1 on the last double integral involving b, we obtain

$$\begin{aligned}
x(t) &\leq f^{\dagger}(M)\left\{-[(t_1-\xi)\mu(t_1)b(t_1)]^2-(t_1-\xi)\mu(t_1)b(t_1)\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b(s)\Delta s\right.\\
&\quad\left.+\lambda B\zeta-\frac{1}{2}\left(\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b(s)\Delta s\right)^2-\frac{1}{2}\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))}\mu(s)b^2(s)\Delta s\right\}\\
&= f^{\dagger}(M)\left(\lambda B\zeta-\left[\frac{\zeta^2}{2}+\frac{((t_1-\xi)\mu(t_1)b(t_1))^2}{2}+\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))}\frac{\mu(s)}{2}b^2(s)\Delta s\right]\right).
\end{aligned}$$

Define

$$m(s) := \begin{cases} (t_1-\xi)\sqrt{\mu(s)}b(s) & : s \leq t_1 \\ \sqrt{\mu(s)}b(s) & : s > t_1, \end{cases}$$

so that m is right-dense continuous and

$$x(t) \leq f^{\dagger}(M)\left(\lambda B\zeta-\frac{\zeta^2}{2}-\frac{1}{2}\int_{t_1}^{\tau^{-1}(\sigma(t_1))} m^2(s)\Delta s\right).$$

By the Cauchy–Schwarz inequality [6, Theorem 6.15],

$$\begin{aligned}
\int_{t_1}^{\tau^{-1}(\sigma(t_1))} m^2(s)\Delta s &\geq \frac{1}{\tau^{-1}(\sigma(t_1))-t_1}\left(\int_{t_1}^{\tau^{-1}(\sigma(t_1))} m(s)\Delta s\right)^2\\
&= \frac{1}{\tau^{-1}(\sigma(t_1))-t_1}\left((t_1-\xi)(\mu(t_1))^{3/2}b(t_1)\right.\\
&\quad\left.+\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b(s)\sqrt{\mu(s)}\Delta s\right)^2\\
&\overset{(1.2)}{\geq} 2(\lambda-3/2)\zeta^2.
\end{aligned}$$

Thus, for $t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$,

$$x(t) \leq f^{\dagger}(M)\left(\lambda B\zeta - (\lambda - 1)\zeta^2\right) \leq \Lambda_B f^{\dagger}(M) \tag{2.5}$$

by Lemma 2.2.

Case 2:
Suppose $1 \leq \zeta \leq \lambda B$ for ζ as in (2.4). Actually, from (H3) we have in this case that $\int_{t_1}^{\tau^{-1}(\sigma(t_1))} b(s)\Delta s \in [1, \lambda B]$. Note that

$$g(t) := \int_{t}^{\tau^{-1}(\sigma(t_1))} b(s)\Delta s - 1, \quad t \in [t_1, \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$$

is a delta-differentiable and decreasing function, so that by [6, Theorem 1.16 (i)], g is continuous on $t \in [t_1, \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$. Since $g(t_1) \geq 0$ and $g(\tau^{-1}(\sigma(t_1))) = -1 < 0$, by the intermediate value theorem [6, Theorem 1.115], there exists $t_2 \in [t_1, \tau^{-1}(\sigma(t_1)))_{\mathbb{T}}$ such that either $g(t_2) = 0$ or $g(t_2) > 0 > g^{\sigma}(t_2)$. Either way,

$$\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b(s)\Delta s < 1 \leq \int_{t_2}^{\tau^{-1}(\sigma(t_1))} b(s)\Delta s = \mu(t_2)b(t_2) + \int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b(s)\Delta s,$$

ergo there exists a real number $\phi \in [t_2 - 1, t_2)$ such that

$$\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b(s)\Delta s + (t_2 - \phi)\mu(t_2)b(t_2) = 1. \tag{2.6}$$

Using (2.1) and (2.2), we have for $t \in [t_1, t_2]_{\mathbb{T}}$ that

$$\begin{aligned}
x(t) &= (t_1 - \xi)\mu(t_1)x^{\Delta}(t_1) + \int_{\sigma(t_1)}^{t} x^{\Delta}(s)\Delta s \\
&\leq (t_1 - \xi)\mu(t_1)b(t_1)f^{\dagger}(M) + \int_{\sigma(t_1)}^{t} b(s)f^{\dagger}(M)\Delta s \\
&\leq f^{\dagger}(M)\left[(t_1 - \xi)\mu(t_1)b(t_1) + \int_{\sigma(t_1)}^{t_2} b(s)\Delta s\right] \\
&\leq f^{\dagger}(M)\int_{t_1}^{t_2} b(s)\Delta s \leq (\lambda B - 1)f^{\dagger}(M) \leq \Lambda_B f^{\dagger}(M),
\end{aligned}$$

where the last inequality follows from our choice of t_2. For $t \in [t_2, \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$, with (2.1) we see that

$$\begin{aligned}
x(t) &= (t_1 - \xi)\mu(t_1)x^{\Delta}(t_1) + \int_{\sigma(t_1)}^{t} x^{\Delta}(s)\Delta s \\
&= \left[(t_1 - \xi)\mu(t_1)x^{\Delta}(t_1) + (\phi - t_2 + 1)\mu(t_2)x^{\Delta}(t_2) + \int_{\sigma(t_1)}^{t_2} x^{\Delta}(s)\Delta s\right] \\
&\quad + \left[(t_2 - \phi)\mu(t_2)x^{\Delta}(t_2) + \int_{\sigma(t_2)}^{t} x^{\Delta}(s)\Delta s\right] = S_1 + S_2,
\end{aligned}$$

where S_1 is the first grouping and S_2 the second. Using (2.2) for S_1 and (2.3) for S_2,

$$S_1 \leq f^\dagger(M)\left[(t_1-\xi)\mu(t_1)b(t_1)+(\phi-t_2)\mu(t_2)b(t_2)+\int_{\sigma(t_1)}^{\sigma(t_2)} b(s)\Delta s\right]$$

and

$$\begin{aligned} S_2 \leq\ & f^\dagger(M)(t_2-\phi)\mu(t_2)b(t_2)\left[\int_{\tau(t_2)}^{\sigma(t_1)} b(s)\Delta s-(t_1-\xi)\mu(t_1)b(t_1)\right] \\ &+ f^\dagger(M)\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b(s)\left[\int_{\tau(s)}^{\sigma(t_1)} b(u)\Delta u-(t_1-\xi)\mu(t_1)b(t_1)\right]\Delta s. \end{aligned}$$

Then continuing for $t \in [t_2, \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$ while recalling (2.6),

$$\begin{aligned} x(t) \leq f^\dagger(M)\Bigg(&\left[(t_1-\xi)\mu(t_1)b(t_1)+(\phi-t_2)\mu(t_2)b(t_2)+\int_{\sigma(t_1)}^{\sigma(t_2)} b(s)\Delta s\right] \\ &\times\left[\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b(s)\Delta s+(t_2-\phi)\mu(t_2)b(t_2)\right] \\ &+(t_2-\phi)\mu(t_2)b(t_2)\left[\int_{\tau(t_2)}^{\sigma(t_1)} b(s)\Delta s-(t_1-\xi)\mu(t_1)b(t_1)\right] \\ &+\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b(s)\left[\int_{\tau(s)}^{\sigma(t_1)} b(u)\Delta u-(t_1-\xi)\mu(t_1)b(t_1)\right]\Delta s\Bigg). \end{aligned}$$

Proceeding by rearranging,

$$\begin{aligned} x(t) &\leq f^\dagger(M)\Bigg(\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b(s)\left[(\phi-t_2)\mu(t_2)b(t_2)+\int_{\tau(s)}^{\sigma(t_2)} b(u)\Delta u\right]\Delta s \\ &\quad+(t_2-\phi)\mu(t_2)b(t_2)\left[(\phi-t_2)\mu(t_2)b(t_2)+\int_{\tau(t_2)}^{\sigma(t_2)} b(s)\Delta s\right]\Bigg) \\ &\overset{\text{(H3)}}{\leq} f^\dagger(M)\Bigg(\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b(s)\left[(\phi-t_2)\mu(t_2)b(t_2)+\lambda B-\int_{\sigma(t_2)}^{\sigma(s)} b(u)\Delta u\right]\Delta s \\ &\quad+(t_2-\phi)\mu(t_2)b(t_2)\left[(\phi-t_2)\mu(t_2)b(t_2)+\lambda B\right]\Bigg) \\ &\overset{(2.6)}{=} f^\dagger(M)\Bigg(\lambda B-\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b(s)\int_{\sigma(t_2)}^{\sigma(s)} b(u)\Delta u\Delta s-[(t_2-\phi)\mu(t_2)b(t_2)]^2 \\ &\quad-(t_2-\phi)\mu(t_2)b(t_2)\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b(s)\Delta s\Bigg) \\ &= f^\dagger(M)\left(\lambda B-\frac{1}{2}-\frac{1}{2}\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))}\mu(s)b^2(s)\Delta s-\frac{1}{2}[(t_2-\phi)\mu(t_2)b(t_2)]^2\right) \end{aligned}$$

using Lemma 2.1 and (2.6) again. Thus, as in (2.5),

$$x(t) \le f^\dagger(M)\left(\lambda B - \frac{1}{2} - (\lambda - 3/2)\right) = \Lambda_B f^\dagger(M), \quad t \in [t_2, \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}.$$

This completes the proof. ■

Lemma 2.4. Suppose that (H1)–(H3) hold, and that $AB \le 1$. Let x be a solution of (1.1) and $t_1 \in \mathbb{T}$ be as in Lemma 2.3. Then for any $M > 0$ the following hold.

(i) If $B \le 1$ and $-\Lambda_A M \le x(t) \le M$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$, then

$$-\Lambda_A f^\dagger(M) \le x(t) \le f^\dagger(M), \quad t \in [\sigma(t_1), \infty)_{\mathbb{T}}.$$

(ii) If $A \le 1$ and $-M \le x(t) \le \Lambda_B M$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$, then

$$-f^\dagger(M) \le x(t) \le \Lambda_B f^\dagger(M), \quad t \in [\sigma(t_1), \infty)_{\mathbb{T}}.$$

Proof. The proof of (ii) is similar to that for (i) and is omitted. Thus, suppose that $B \le 1$ and $-\Lambda_A M \le x(t) \le M$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$. By Lemma 2.3, for $t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$ we have

$$-\Lambda_A f^\dagger(M) \le x(t) \le f^\dagger(M). \tag{2.7}$$

If (2.7) is false for some $t > \tau^{-1}(\sigma(t_1))$, let

$$T_1 := \inf\left\{t > \tau^{-1}(\sigma(t_1)) : x(t) < -\Lambda_A f^\dagger(M) \text{ or } x(t) > f^\dagger(M)\right\}.$$

Clearly

$$-\Lambda_A f^\dagger(M) \le x(t) \le f^\dagger(M), \quad t \in [\sigma(t_1), T_1)_{\mathbb{T}}, \tag{2.8}$$

and we have the following cases.

Case 1.1:
Suppose $x(T_1) > f^\dagger(M)$. By continuity and the choice of T_1, T_1 is a left-scattered point with $x(\rho(T_1)) \in [-\Lambda_A f^\dagger(M), f^\dagger(M)]$ and $x^\Delta(\rho(T_1)) > 0$. By (1.1) and (H2), $x(\tau(\rho(T_1))) < 0$. Set

$$T_0 := \max\left\{t \in [\tau(\rho(T_1)), \rho(T_1))_{\mathbb{T}} : x(t)x^\sigma(t) \le 0\right\}.$$

Then $x(T_0)x^\sigma(T_0) \le 0$ and $\tau^2(t_1) \le \tau^3(T_1) \le \tau^2(T_0) \le T_0 < T_1$. By (i) above and (2.8), $x(t) \ge -\Lambda_A M$ on $[\tau^2(T_0), T_0]_{\mathbb{T}}$. As a result, from Lemma 2.3 (i) and the fact that $\Lambda_A\Lambda_B \le 1$ by Lemma 2.2, $x(t) \le \Lambda_A\Lambda_B f^\dagger(M)$ on $[\sigma(T_0), \tau^{-1}(\sigma(T_0))]_{\mathbb{T}}$. Since $\tau(\rho(T_1)) \le T_0 < \rho(T_1)$ and τ is increasing, $\sigma(T_0) \le T_1$ and $f^\dagger(M) < x(T_1) \le f^\dagger(M)$, a contradiction.

Case 1.2:
Suppose $x(T_1) = f^\dagger(M)$. Then T_1 is a right-dense point, $x^\Delta(T_1) \ge 0$, and there

exists $T_2 \in (T_1, \tau^{-1}(T_1)]_{\mathbb{T}}$ such that $x(t) > f^{\dagger}(M)$ on $(T_1, T_2]_{\mathbb{T}}$. By (1.1) and (H2), $x(\tau(T_1)) \leq 0$. Set

$$T_0 := \max\left\{t \in [\tau(T_1), T_1)_{\mathbb{T}} : x(t)x^{\sigma}(t) \leq 0\right\}.$$

Then $x(T_0)x^{\sigma}(T_0) \leq 0$ and $\tau^2(t_1) \leq \tau^3(T_2) \leq \tau^2(T_0) \leq T_0 < T_2$. By (i) above and (2.8), $x(t) \geq -\Lambda_A M$ on $[\tau^2(T_0), T_0]_{\mathbb{T}}$. Consequently, from Lemma 2.3 (i) and the fact that $\Lambda_A \Lambda_B \leq 1$ by Lemma 2.2 again, $x(t) \leq \Lambda_A \Lambda_B f^{\dagger}(M)$ on $[\sigma(T_0), \tau^{-1}(\sigma(T_0))]_{\mathbb{T}}$. Since $\tau(T_1) \leq T_0 < T_2$ and τ is increasing, $\sigma(T_0) \leq T_2 \leq \tau^{-1}(\sigma(T_0))$ and $f^{\dagger}(M) < x(T_2) \leq f^{\dagger}(M)$, a contradiction.

Case 2:
If $x(T_1) \leq -\Lambda_A f^{\dagger}(M)$, then (2.8) implies either $x^{\Delta}(T_1) \leq 0$ or $x^{\Delta}(\rho(T_1)) < 0$. Again by (1.1) and (H2), either $x(\tau(T_1)) \geq 0$ or $x(\tau(\rho(T_1))) > 0$. Pick T_0 as above for either case. Then $x(t) \leq M$ on $[\tau^2(T_0), T_0]_{\mathbb{T}}$. This time, from Lemma 2.3 (ii), $x(t) \geq -\Lambda_A f^{\dagger}(M)$ on $[\sigma(T_0), \tau^{-1}(\sigma(T_0))]_{\mathbb{T}}$. Similar to above, either $-\Lambda_A f^{\dagger}(M) > x(T_2) \geq -\Lambda_A f^{\dagger}(M)$ or $-\Lambda_A f^{\dagger}(M) > x(T_1) \geq -\Lambda_A f^{\dagger}(M)$, both contradictions. ■

3. Solutions of (1.1) go to Zero

We now state our main result on the asymptotic behavior of solutions of (1.1), namely that solutions of (1.1) go to zero at infinity. This improves Theorem 1.1, and extends [16, Theorem 3.1] to arbitrary time scales.

Theorem 3.1. Suppose that (H1)–(H3) hold, and that $AB \leq 1$. If there exists $t^* \in \mathbb{T}$ and a collection of continuous functions $\{H(t, \cdot) : (0, \infty) \to [0, \infty)\}_{t \in \mathbb{T}}$ such that, for any $\epsilon > 0$ and $t \in [t^*, \infty)_{\mathbb{T}}$,

$$\sup_{x \geq \epsilon} F(t, x) \leq -H(t, \epsilon), \quad \inf_{x \leq -\epsilon} F(t, x) \geq H(t, \epsilon), \quad \int_{t^*}^{\infty} H(t, \epsilon)\Delta t = \infty, \tag{3.1}$$

then every solution of (1.1) goes to zero in the limit.

Proof. Let x be a solution of (1.1). If x is eventually negative or eventually positive, then x must be eventually nondecreasing or eventually nonincreasing by (H2). Without loss of generality suppose x is eventually positive, and thus eventually nonincreasing. If $x(t) \to \varepsilon > 0$, then $\sup_{x \geq \varepsilon} F(t, x) = 0$ for all $t \geq t^*$ by (1.1), so $0 \leq -H(t, \varepsilon) \leq 0$ for all $t \geq t^*$ by (3.1); in other words, $H \equiv 0$ on $[t^*, \infty)_{\mathbb{T}}$, a contradiction of $\int_{t^*}^{\infty} H(t, \epsilon)\Delta t = \infty$. Therefore any solution x that is eventually of one sign goes to zero in the limit. Next assume that x is an oscillatory solution of (1.1). Then x has a generalized zero in any neighborhood of infinity, so there exists an increasing sequence of points $\{t_n \in \mathbb{T}\}_{n \in \mathbb{N}}$ such that $t_n \to \infty$ as $n \to \infty$ and $x(t_n)x^{\sigma}(t_n) \leq 0$. As $AB \leq 1$, either $A \leq 1$ or $B \leq 1$. If the latter, let $M > 0$ be given such that $-\Lambda_A M \leq x(t) \leq M$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$. By

Lemma 2.4 (i), $-\Lambda_A f^\dagger(M) \le x(t) \le f^\dagger(M)$ for $t \in [\sigma(t_1), \infty)_{\mathbb{T}}$. Let $f_1^\dagger := f^\dagger$ and $f_{n+1}^\dagger := f^\dagger \circ f_n^\dagger$ for $n \in \mathbb{N}$. By successive use of the above argument, we deduce that

$$-\Lambda_A f_n^\dagger(M) \le x(t) \le f_n^\dagger(M), \quad t \in [\sigma(t_n), \infty)_{\mathbb{T}}. \tag{3.2}$$

Assumption (H1) implies that the sequence $\{f_n^\dagger(M)\}_{n\in\mathbb{N}}$ is nonincreasing. If $M_0 := \lim_{n\to\infty} f_n^\dagger(M)$, then $M_0 = f^\dagger(M_0)$. If $M_0 > 0$, then since $f(M_0) < M_0$ by (H1), set $f(M_0) = M_0 - \varepsilon$. By the continuity of f, there exists $\delta > 0$ such that $|f(x) - f(M_0)| < \varepsilon/2$ for all $x \in (M_0 - \delta, M_0]$. Therefore $|f(x) - M_0 + \varepsilon| < \varepsilon/2$, which means $M_0 - 3\varepsilon/2 < f(x) < M_0 - \varepsilon/2$ for all $x \in (M_0 - \delta, M_0]$. But $f(x) < x$ on $[0, m_0 - \delta]$ implies that $f(x) < M_0 - \delta < M_0$ on $[0, M_0 - \delta]$. Consequently f is bounded away from M_0 on $[0, M_0]$, a contradiction of $M_0 = f^\dagger(M_0)$, so that $M_0 = 0$. From (3.2) we have that $\lim_{t\to\infty} x(t) = 0$. In a similar way we arrive at the identical conclusion if $A \le 1$. ■

Recall the special case of (1.1) given in (1.3), namely

$$x^\Delta(t) + p(t)g(x(\tau(t))) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

Then we have the following corollary to Theorem 3.1.

Corollary 3.2. Assume $p : \mathbb{T} \to \mathbb{R}$ is rd-continuous and $g : \mathbb{R} \to \mathbb{R}$ is continuous with $xg(x) > 0$ for $x \ne 0$. Let $\alpha, \beta > 0$ such that

$$-\alpha|x| \le g(x) \le \beta|x|, \qquad x \ne 0. \tag{3.3}$$

Assume for all large $t \in \mathbb{T}$ that

$$(\alpha\beta)^{1/2} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s \le \lambda. \tag{3.4}$$

If $\int_{t_0}^{\infty} p(t)\Delta t = \infty$, then every solution of (1.3) goes to zero in the limit.

Proof. Take $F(t, x) := -p(t)g(x)$, $a(t) := \alpha p(t)$, $b(t) := \beta p(t)$,

$$f(x) := \begin{cases} g(x)/\beta & : \ x \ge 0 \\ g(x)/\alpha & : \ x < 0, \end{cases}$$

and $H(t, \epsilon) := p(t) \min\left\{\inf_{x\ge\epsilon} g(x), -\sup_{x\le-\epsilon} g(x)\right\}$. Then (H1) and (H2) are satisfied, as are the conditions in (3.1). Set $A := (\alpha/\beta)^{1/2}$ and $B := (\beta/\alpha)^{1/2}$, then by (3.4) we have that (H3) holds and $AB = 1$. By Theorem 3.1, every solution of (1.3) goes to zero in the limit. ■

Remark 3.3. The condition in (3.3) improves those in Corollary 1.2.

4. Examples on Isolated Time Scales

Let $\mathbb{T}$ be a time scale unbounded above, with every point both left and right scattered. Consider the logistic dynamic equation [7, (2.28)],

$$y^\Delta = \frac{py(1-y/N)}{1+\frac{\mu p}{N}y} \quad \text{or} \quad y^\sigma = \frac{y(1+\mu p)}{1+\frac{\mu p}{N}y}, \quad \frac{py}{N} \in \mathcal{R},$$

with $\mathcal{R}$ given in [6, Definition 2.25]. Because of the isolated nature of each point in $\mathbb{T}$ in this section, these two forms are equivalent. We introduce a delay $\tau : \mathbb{T} \to \mathbb{T}$ and concentrate on

$$y^\sigma(t) = \frac{y(t)(1+\mu(t)p(t))}{1+\frac{\mu(t)p(t)}{N}y(\tau(t))}, \quad t \in \mathbb{T}, \tag{4.1}$$

assuming $N > 0$ is fixed, $p > 0$ on $\mathbb{T}$, and $p(y \circ \tau)/N \in \mathcal{R}$. We would like to show conditions under which positive solutions of (4.1) go to the "carrying capacity" N. In (4.1), let $y = Ne^x$, $y^\sigma = Ne^{x\circ\sigma}$, and $y \circ \tau = Ne^{x\circ\tau}$ to obtain

$$x^\Delta(t) = \frac{1}{\mu(t)} \ln\left\{\frac{1+\mu(t)p(t)}{1+\mu(t)p(t)e^{x(\tau(t))}}\right\}. \tag{4.2}$$

If

$$F(t,x) := \frac{1}{\mu(t)} \ln\left\{\frac{1+\mu(t)p(t)}{1+\mu(t)p(t)e^{x}}\right\},$$

then $F(t,\cdot)$ is continuous with $F(t,0) \equiv 0$. For fixed $t \in \mathbb{T}$ and $x > 0$, $F(t,x) \geq -x/\mu(t)$, so we take $a(t) := 1/\mu(t)$; for $x < 0$, $F(t,x) \leq -xp(t)/(1+\mu(t)p(t))$ implies we should have $b(t) := p(t)/(1+\mu(t)p(t))$. A direct consequence of these choices is that conditions (H1) and (H2) are met. Finally, notice that $-F(t,\epsilon) \geq F(t,-\epsilon)$ for $t \in \mathbb{T}$ and $\epsilon > 0$, so that

$$\sup_{x\geq\epsilon} F(t,x) = F(t,\epsilon) \leq -F(t,-\epsilon), \qquad \inf_{x\leq-\epsilon} F(t,x) = F(t,-\epsilon)$$

leads us to define $H(t,\epsilon) := F(t,-\epsilon)$.

Example 4.1. Let $\mathbb{T} = h\mathbb{Z}$ for some $h \in (0,1)$, $\tau(t) := t - hk$ for $t \in \mathbb{T}$ and $k \in \mathbb{N}$, and $p(t) \in (0,\bar{p}]$ for all $t \in \mathbb{T}$ for some $\bar{p} > 0$, such that $\lim_{t\to\infty} p(t) \neq 0$. If

$$\bar{p} \in \left(0, \frac{\lambda^2}{h[(k+1)^2-\lambda^2]}\right] \quad \text{for} \quad \lambda = \frac{3k+4}{2(k+1)},$$

then every positive solution of (4.1) goes to N in the limit.

Proof. Note that $\mu(t) \equiv h$. Take $A := \dfrac{k+1}{\lambda}$ and $B := \dfrac{h\bar{p}(k+1)}{\lambda(1+h\bar{p})}$. Then

$$\int_{\tau(t)}^{\sigma(t)} a(s)\Delta s = \int_{t-hk}^{t+h} \frac{1}{h}\Delta s = k+1 = \lambda A,$$

$$\int_{\tau(t)}^{\sigma(t)} b(s)\Delta s = \int_{t-hk}^{t+h} \frac{p(s)}{1+hp(s)}\Delta s \leq \int_{t-hk}^{t+h} \frac{\bar{p}}{1+h\bar{p}}\Delta s = \frac{(k+1)h\bar{p}}{1+h\bar{p}} = \lambda B,$$

$$\text{and} \quad \int_h^\infty F(t,-\epsilon)\Delta t = \infty.$$

Moreover, if $\bar{p} \in \left(0, \dfrac{\lambda^2}{h[(k+1)^2-\lambda^2]}\right]$, then

$$AB = \frac{h\bar{p}(k+1)^2}{\lambda^2(1+h\bar{p})} \leq 1.$$

Hence by Theorem 3.1, every solution x of (4.2) goes to zero in the limit. But then every positive solution $y = Ne^x$ of (4.1) goes to N. In addition, since

$$\frac{\lambda^2}{h[(k+1)^2-\lambda^2]} > \frac{\lambda^3}{h[(k+1)^3-\lambda^3]}$$

for all $k \in \mathbb{N}$ and all $h > 0$, this improves [3, Example 4.1]. ■

Again let $\mathbb{T}$ be a time scale unbounded above, with every point both left and right scattered, and consider the food-limited population model given by the delay dynamic equation

$$\frac{y^\sigma(t)}{y(t)} = \exp\left(\mu(t)p(t)\frac{1-y(\tau(t))}{1+cy(\tau(t))}\right) \tag{4.3}$$

for some delay $\tau : \mathbb{T} \to \mathbb{T}$ and constant $c > 0$.

Example 4.2. Assume for large $t \in \mathbb{T}$ that we have

$$\int_{\tau(t)}^{\sigma(t)} p(s)\Delta s \leq 2\lambda\sqrt{c}. \tag{4.4}$$

If $\int_0^\infty p(t)\Delta t = \infty$, then every positive solution of (4.3) goes to 1 in the limit.

Proof. In (4.3), let $y = e^x$, $y^\sigma = e^{x\circ\sigma}$, and $y \circ \tau = e^{x\circ\tau}$ to obtain

$$x^\Delta(t) + p(t)\frac{e^{x(\tau(t))}-1}{1+ce^{x(\tau(t))}} = 0, \tag{4.5}$$

which is of form (1.3). If

$$g(x) := \frac{e^x-1}{1+ce^x},$$

then g is continuous with $xg(x) > 0$ for $x \neq 0$ and $g(0) = 0$. As in [16],

$$\max_{x>0} g'(x) = \begin{cases} \dfrac{1+c}{4c} & : c \leq 1, \\[2ex] \dfrac{1}{1+c} & : c > 1, \end{cases} \qquad \max_{x<0} g'(x) = \begin{cases} \dfrac{1+c}{4c} & : c > 1, \\[2ex] \dfrac{1}{1+c} & : c \leq 1. \end{cases}$$

Take $\alpha = \max_{x<0} g'(x)$ and $\beta = \max_{x>0} g'(x)$, so that (3.3) holds, and (4.4) implies (3.4). By Corollary 3.2, every solution of (4.5) tends to zero. Thus every solution of (4.3) tends to one. ■

References

[1] Douglas R. Anderson, Asymptotic behavior of solutions for neutral dynamic equations on time scales, *Adv. Difference Equ.*, pages Art. ID 80850, 11, 2006.

[2] Douglas R. Anderson, Boundedness and vanishing of solutions for a forced delay dynamic equation, *Adv. Difference Equ.*, pages Art. ID 35063, 19, 2006.

[3] Douglas R. Anderson and Zackary R. Kenz, Global asymptotic behavior for delay dynamic equations, *Nonlinear Anal.*, 66(7):1633–1644, 2007.

[4] Douglas R. Anderson, Robert J. Krueger, and Allan C. Peterson, Delay dynamic equations with stability, *Adv. Difference Equ.*, pages Art. 94051, 1–19, 2006.

[5] Martin Bohner, Some oscillation criteria for first order delay dynamic equations, *Far East J. Appl. Math.*, 18(3):289–304, 2005.

[6] Martin Bohner and Allan Peterson, *Dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA, 2001. An introduction with applications.

[7] Martin Bohner and Allan Peterson, *Advances in dynamic equations on time scales*, Birkhäuser, Boston, 2003.

[8] Jan Čermák and Miroslav Urbánek, On the asymptotics of solutions of delay dynamic equations on time scales, *Math. Comput. Modelling*, 46(3-4):445–458, 2007.

[9] L.H. Erbe, H. Xia, and J.S. Yu, Global stability of a linear nonautonomous delay difference equation, *J. Differ. Equations Appl.*, 1(2):151–161, 1995.

[10] John R. Graef and Chuanxi Qian, Global attractivity in a nonlinear difference equation and its applications, *Dynam. Systems Appl.*, 15(1):89–96, 2006.

[11] Stefan Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, *Results Math.*, 18(1-2):18–56, 1990.

[12] H. Matsunaga, T. Hara, and S. Sakata, Global attractivity for a nonlinear difference equation with variable delay, *Comput. Math. Appl.*, 41(5-6):543–551, 2001.

[13] Hideaki Matsunaga, Rinko Miyazaki, and Tadayuki Hara, Global attractivity results for nonlinear delay differential equations, *J. Math. Anal. Appl.*, 234(1):77–90, 1999.

[14] Christian Pötzsche, Delay equations on measure chains: basics and linearized stability, In *Proceedings of the Eighth International Conference on Difference Equations and Applications*, pages 227–234. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[15] Haihua Wu and Zhan Zhou, Stability for first order delay dynamic equations on time scales, *Comput. Math. Appl.*, 53(12):1820–1831, 2007.

[16] Xiaosheng Zhang, Global attractivity for nonlinear difference equations with delay, *J. Math. Anal. Appl.*, 336(2):975–986, 2007.

[17] Xiaosheng Zhang and Jurang Yan, Global asymptotic behavior of nonlinear difference equations, *Comput. Math. Appl.*, 49(9-10):1335–1345, 2005.