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Eigenvalue Problems and Oscillation of Linear Hamiltonian Systems

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Abstract

This paper deals with the oscillation behavior of linear Hamiltonian systems and related eigenvalue problems with general, linearly independent self-adjoint boundary conditions. The main new aspect of this paper is the fact that we do not require controllability, strong observability or strong normality of the system. In view of this generalization it is necessary to introduce a new notion of "proper" eigenvalues and their multiplicities of the related eigenvalue problem. We show that the "proper" eigenvalues of the related eigenvalue problems are always isolated. Furthermore, we introduce a new notion of the multiplicity of a "proper" focal point of so-called conjoined bases of the differential system. We derive oscillation theorems which give a formula for the number of all "proper" eigenvalues (including multiplicities) smaller or equal than a certain constant with respect to the number of all "proper" focal points (including multiplicities) of a certain conjoined basis of the Hamiltonian system. Due to this generalization we are able to treat more general Sturm–Liouville eigenvalue problems as in the existing literature.

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1. Introduction and Main Results

Second order *Sturm–Liouville eigenvalue problems* became very famous after their birth in 1836 by J.C.F. STURM and J. LIOUVILLE [9,12] because of their importance in the field of physics. Today many lecture books about ordinary differential equations dedicate a short section to those eigenvalue problems, often in connection with *Dirichlet boundary conditions*. A well-known theorem states that the *principal solution* of such an eigenvalue problem

$$(py')' - (q - \lambda y) = 0, \quad y(0) = y(1) = 0,$$

with continuous functions p, q, $p(t) > 0$ for every $t \in [0, 1]$ and $\lambda \in \mathbb{R}$, i.e., the solution of the differential equation above with the initial conditions $y(0) = 0$, $p(0)y'(0) = 1$, has exactly as many roots as the number of eigenvalues of the given problem smaller or equal to λ. Here we call $\lambda \in \mathbb{R}$ an eigenvalue if there is a nontrivial solution y which satisfies the related Sturm–Liouville eigenvalue problem and the boundary conditions. Many authors (cf., e.g., [1,4,7,8,10,11]) generalized this problem and embedded it in a *linear Hamiltonian system*

$$\dot{x} = Ax + Bu, \ \dot{u} = (C - \lambda C_0)x - A^T u \text{ on } \mathcal{I} := [a, b], a < b. \tag{H}$$

Recently the focus on the analysis of this kind of problems lies on discrete eigenvalue problems [2,3,6]. The tricky part on these kind of problems is to define so-called "focal points" of special solutions of the related difference equation.

In this paper we deal with self-adjoint eigenvalue problems consisting of such linear Hamiltonian systems and certain separated boundary conditions, i.e.,

$$\begin{aligned} &\dot{x} = Ax + Bu, \ \dot{u} = (C - \lambda C_0)x - A^T u, \ t \in \mathcal{I} \\ &R_1^a x(a) + R_2^a u(a) = R_1^b x(b) R_2^b u(b) = 0. \end{aligned} \tag{E}$$

Problems of this kind cover a wide range of eigenvalue problems, among them the higher order Sturm–Liouville eigenvalue problems. The main new aspect of this paper is the fact that we do not require controllability, strong observability or strong normality of the system, terms which we will define below. Due to this generalization we can treat more general Sturm–Liouville eigenvalue problems as in existing literature and we can reduce problems with general boundary conditions to the case of separated boundary conditions. For applications of the theory presented in this paper we refer to [13].

Throughout this paper we use the following **setting** and **assumptions**. There are given piecewise continuous matrix-valued functions $A(t)$, $B(t)$, $C(t)$, $C_0(t)$ defined on the interval $\mathcal{I} := [a, b]$ which fulfill

$$\begin{cases} A(t), B(t), C(t), C_0(t) \text{ are real } n \times n\text{-matrices for all } t \in \mathcal{I}, \\ B(t), C(t), C_0(t) \text{ are symmetric for all } t \in \mathcal{I}, \\ B(t), C(t) \text{ are nonnegative definite for all } t \in \mathcal{I}. \end{cases}$$

Furthermore, there are given real $n \times n$-matrices R_1^a, R_2^a, R_1^b, R_2^b with

$$\operatorname{rank}(R_1^a, R_2^a) = \operatorname{rank}(R_1^b, R_2^b) = n, \quad R_1^a R_2^{aT} = R_2^a R_1^{a\mathrm{T}}, R_1^b R_2^{bT} = R_2^b R_1^{b\mathrm{T}}.$$

We will use the following terminology. For a (real) matrix M we denote by Im M, ker M, rank$M$, def M, det M, M^T, M^{-1} and $M^\dagger$ the image of M, the kernel of M, the rank of M, the defect of M (i.e., the dimension of the kernel of M), the determinant of M, the transpose of M, the inverse of M and the Moore–Penrose inverse of M, respectively. We write $M \geq 0$, $M > 0$ if the (real and) symmetric matrix M is nonnegative definite, positive definite, respectively. Moreover, I and 0 denote the identity matrix and the zero matrix, respectively. If the context does not clarify their dimension we will add the dimension in their index. For any linear subspace V of $\mathbb{R}^n$ we denote by dim V the dimension of V. Of course, limits, differentiation, and integration of matrix-valued functions are always performed elementwise. Furthermore, given a function f we will denote the usual left-hand limit of f at the point t by $f(t-)$ and the right-hand limit by $f(t+)$.

For any two (matrix-valued) solutions (X_1, U_1) and (X_2, U_2) of (H) the *Wronskian* $X_1^T(t)U_2(t) - U_1^T(t)X_2(t)$ is a constant matrix on $\mathbb{R}$ (cf., [4, Prop. 1.1.1]). Of particular importance are so-called *conjoined bases* of (H), i.e., matrix-valued solutions (X, U) of (H) with $\operatorname{rank}(X^T(t), U^T(t)) = n$ and $X^T(t)U(t) = U^T(t)X(t)$. Hence, conjoined bases are special half fundamental systems of (H). From now on (X_a, U_a) will always denote a special conjoined basis of (H) with the initial values $X_a(a) = X_a(a; \lambda) = -R_2^{aT}$, $U_a(a) = U_a(a; \lambda) = R_1^{aT}$.

Definition 1.1. A point $t_0 \in (a, b]$ is called *proper focal point* of a conjoined basis (X, U) of (H), if $\ker X(t_0) \supsetneqq \ker X(t_0-)$. The number def$X(t_0)$ − def$X(t_0-)$ is called *multiplicity of the focal point* t_0 of X.

Definition 1.2. We call $\lambda \in \mathbb{C}$ a *proper eigenvalue of* (E), if there exists a solution (x, u) of (H) with $C_0 x \not\equiv 0$ on $\mathcal{I}$ which satisfies the separated boundary conditions. Such a solution (x, u) is called *eigenfunction to the proper eigenvalue* λ. The number

$$\dim\{C_0(t)x(t) \mid (x, u) \text{ is a solution of (E)}\}$$

is called the *multiplicity of the proper eigenvalue* λ.

Remark 1.3.

(i) By [4, Definition 2.3.1] the Hamiltonian system (H) (or the pair (A, B)) is called *(completely) controllable* or *identically normal* on $\mathcal{I}$, if $\dot{v} = -A^T v$, $Bv \equiv 0$ on some nondegenerate interval $\mathcal{J} \subset \mathcal{I}$ always implies that $v \equiv 0$ on $\mathcal{J}$. Note that $B(t)$ is symmetric by our assumption. If (X, U) is a conjoined basis of (H), then by [4, Definition 1.1.1(i)] $t_0 \in \mathcal{I}$ is called a focal point of X if the matrix $X(t_0)$ is invertible. [4, Theorem 4.1.3] states under our assumptions that (A, B) is controllable on $\mathcal{I}$ if and only if the focal points of every conjoined basis of (H) are isolated in $\mathcal{I}$. Observe that every focal point of a conjoined basis of (H) is also a proper focal point of that conjoined basis but not vice versa. [5, Theorem 3] implies however that the proper focal points of every conjoined basis of (H) are isolated as well. Note that the term “generalized focal point” is used in [5] instead of “proper focal point”.

(ii) According to [4, Definition 4.1.2] a Hamiltonian system is called *strictly normal* if it is controllable and if a pair (x, u) solves the Hamiltonian system for different parameters λ on some nondegenerate interval $\mathcal{J} \subset \mathcal{I}$, then necessarily $x = u \equiv 0$. Note that we do not postulate strict normality in this paper.

(iii) The self-adjointness of the given eigenvalue problem (E) implies that every proper eigenvalue of (E) is real. Furthermore, let (x, u) be an eigenfunction of a proper eigenvalue of (E). Then the real parts of x and of u form an eigenfunction as well, so that eigenfunctions of (E) can be assumed to be real. Therefore we will always assume that every given eigenfunction is real in this paper.

(iv) Two eigenfunctions (x_1, u_1) and (x_2, u_2) to different proper eigenvalues λ_1 and λ_2 of (E) are orthogonal, i.e.,

$$< x_1, x_2 >:= \int_{\mathcal{I}} \left\{\overline{x}_1^T C_0 x_2\right\}(t)dt = 0.$$

This fact follows directly from [4, Proposition 2.2.2].

We will now formulate the oscillation theorems which give a formula that connects the number of the proper focal points of the special conjoined basis (X_a, U_a) and the number of proper eigenvalues of (E) $\leq \lambda$ except of a certain discrete set

$$\mathcal{M} := \{\lambda \in \mathbb{R} \,|\, \text{def } X_a(b; \lambda) = \text{def } X_a(b; \lambda+)\}.$$

In Section 2 we will prove that def $X_a(b; \lambda+)$ exists for every $\lambda \in \mathbb{R}$ and that the set $\mathcal{M}$ is indeed discrete. The oscillation theorems below can be extended to hold also on this set $\mathcal{M}$ as discussed in [13] but this is rather technical. Furthermore, we will use the following notation.

$$V := \bigcap_{t\in\mathcal{I}} \ker \left\{X_a^T(t;\lambda)C_0(t)X_a(t;\lambda)\right\} \cap \ker X_a(b;\lambda), \text{ and}$$

$\mathcal{V} \in \mathbb{R}^{n\times q}$ is a matrix with $\mathbb{R}^n = V \oplus \text{Im } \mathcal{V}$, where $q := n - \dim V$;

$$W := \bigcap_{t\in\mathcal{I}} \ker \left\{X_a^T(t;\lambda)C_0(t)X_a(t;\lambda)\right\} \cap \ker \left\{R_1^b X_a(b;\lambda) + R_2^b U_a(b;\lambda)\right\} \text{ and}$$

$\mathcal{W} \in \mathbb{R}^{n\times r}$ is a matrix with $\mathbb{R}^n = W \oplus \text{Im } \mathcal{W}$, where $r := n - \dim W$.

Theorem 1.4. (Local Oscillation Theorem) Let $\Lambda(\lambda) := R_1^b X_a(b;\lambda) + R_2^b U_a(b;\lambda)$. Furthermore, let

$n_1(\lambda)$ denote the number of proper focal points of $X_a(t;\lambda)$ in $(a,b]$ (including multiplicities),
$n_2(\lambda)$ $:= \operatorname{ind}\{R^T(S_1 + (U_a X_a^\dagger)(b;\lambda))R\} + \operatorname{def}\{\Lambda(\lambda)\mathcal{W}\}$, where $R_1^b = R_2^b S_1 + S_2$ with a symmetric matrix S_1 and a matrix S_2 with $\ker R_2^b = \operatorname{Im} S_2^T$, and $R \in \mathbb{R}^{n\times n}$ with $\operatorname{Im} R = \operatorname{Im} R_2^{bT} \cap \operatorname{Im} X_a(b;\lambda+)$,
$n_3(\lambda)$ denote the number of proper focal points of (E) $\le \lambda$ (including multiplicities).

Then

$$n_1 := \lim_{\lambda\to-\infty} n_1(\lambda), \quad n_2 := \lim_{\lambda\to-\infty} n_2(\lambda)$$

exist and the equation

$$n_1(\lambda) + n_2(\lambda) = n_3(\lambda) + n_1 + n_2$$

holds for all $\lambda \in \mathbb{R} \setminus \mathcal{M}$.

Theorem 1.5. (Global Oscillation Theorem) Assume the setting of Theorem 1.4 and let $S_1^a, S_2^a \in \mathbb{R}^{n\times n}$ be such that S_1^a is symmetric and the equations $R_1^a = R_2^a S_1^a + S_2^a$, $\ker R_2^a = \operatorname{Im} S_2^{aT}$ hold. Furthermore, let the quadratic functional

$$\mathcal{F}(x;\lambda_0) := \int_a^b \{x^T(C - \lambda_0 C_0)x + u^T B u\}(t)\,dt + x^T(b)S_1^b x(b) - x^T(a)S_1^a x(a)$$

be positive definite for a $\lambda_0 \in \mathbb{R}$, i.e., $\mathcal{F}(x;\lambda_0) > 0$ for every $x \not\equiv 0$ satisfying $\dot{x} = Ax + Bu$ on $[a,b]$ for some ("control") $u \in C_s(\mathcal{I})$ and $x(a) \in \operatorname{Im} R_2^a$, $x(b) \in \operatorname{Im} R_2^b$. Then

$$n_1(\lambda) + n_2(\lambda) = n_3(\lambda)$$

holds for all $\lambda \in \mathbb{R} \setminus \mathcal{M}$.

Proof. According to [5, Theorem 1], the definiteness of the quadratic functional $\mathcal{F}$ implies that there is no proper focal point of $X_a(t;\lambda)$ for all $\lambda \le \lambda_0$ and hence $n_1 = \lim_{\lambda\to-\infty} n_1(\lambda) = 0$. Furthermore, this theorem implies that $n_2(\lambda) = 0$ for all $\lambda \le \lambda_0$ such that $n_2 = \lim_{\lambda\to-\infty} n_2(\lambda) = 0$. Thus, the assertion of the global oscillation theorem follows directly from the local oscillation theorem, Theorem 1.4. ■

Remark 1.6.

(i) [4, Corollary 3.1.3(i)] implies that there are always matrices $S_1, S_2 \in \mathbb{R}^{n\times n}$ which satisfy the conditions stated in Theorem 1.5 above.

(ii) Postulating the positive semi-definiteness of the quadratic functional $\mathcal{F}(x;\lambda_0)$ is not enough. If we consider e.g., the eigenvalue problem consisting of the linear Hamiltonian system

$$\dot{x} = \begin{cases} u, & t \in [0,\pi] \\ 0, & t \in [\pi, 2\pi], \end{cases} \qquad \dot{u} = x$$

with Dirichlet boundary conditions, then the solution $x_0(t) = \sin t, t \in [0,\pi]$, $x_0(t) = 0, t \in [\pi, 2\pi]$, possesses a proper focal point in the open interval (a,b) independent of λ.

Corollary 1.7. (Dirichlet Boundary Conditions) Let (X_0, U_0) denote the *principal solution* of (H) at a, i.e., a conjoined basis of (H) with $X_0(a) = 0, U_0(a) = I$. As in Theorem 1.4 let $n_1(\lambda)$ denote the number of proper focal points of X_0, $n_1 := \lim\limits_{\lambda\to-\infty} n_1(\lambda)$ and $n_3(\lambda)$ denote the number of proper eigenvalues of (E) $\leq \lambda$. Then

$$n_1(\lambda) = n_3(\lambda) + n_1$$

holds for all $\lambda \in \mathbb{R}$.

Proof. This is a direct consequence of Theorem 1.4 and Lemma 2.4 below as in this case the matrix R vanishes using the notation of Theorem 1.4. Hence $n_2(\lambda)$ reduces to def $\{\Lambda(\lambda)\mathcal{W}\}$, which is in this special case equal to def $\{X_0(b;\lambda)\mathcal{V}\}$. Now Lemma 2.4 finishes the proof. ■

2. Proper Focal Points

In this section we will examine proper focal points of conjoined bases of (H). In particular, we are interested in the local influence of the parameter λ on the number of proper focal points of a conjoined basis of (H) in $\mathcal{I}$. The matrices $\mathcal{V}$ and $\mathcal{W}$ are well defined as the following proposition shows.

Proposition 2.1. The spaces V, W and $\bigcap\limits_{t\in\mathcal{I}} \ker X_a(t;\lambda)$ are independent of the parameter λ.

Proof. Let $c \in \widetilde{V} := \bigcap\limits_{t\in\mathcal{I}} \ker\left\{X_a^T(t;\lambda_0)C_0(t)X_a(t;\lambda_0)\right\}$ and $(x,u) := (X_a, U_a)(t;\lambda_0)c$ for a fixed $\lambda_0 \in \mathbb{R}$. Then $x(a)$ and $u(a)$ are independent of λ and $C_0(\cdot)x(\cdot;\lambda_0) \equiv 0$ holds on $\mathcal{I}$. Hence

$$\dot{x} = Ax + Bu, \quad \dot{u} = Cx - A^T u,$$

i.e., (x,u) solves (H) independent of λ. The uniqueness of solutions of (H) and the fact that $x(b) = X_a(b)c$ and $u(b) = U_a(b)c$ are independent of λ finish the proof. ■

Proposition 2.2. Let (X_1, U_1), (X_2, U_2) be normalized conjoined bases of (H) with constant initial conditions at the point a according to λ, i.e., let (X_1, U_1) and (X_2, U_2) be conjoined bases of (H) with $X_1^T(t)U_2(t) - U_1^T(t)X_2(t) \equiv I$ on $\mathcal{I}$ and $X_\nu(a;\lambda) \equiv X_\nu(a)$, $U_\nu(a;\lambda) \equiv U_\nu(a)$, $\nu = 1, 2$. Furthermore, let $t_0 \in \mathcal{I}$, $\lambda_0 \in \mathbb{R}$ and $X_1(t_0;\lambda)$ be invertible on $[\lambda_0 - \delta, \lambda_0 + \delta]$ for a $\delta > 0$. Then for all $c \in \ker X_2(t_0;\lambda)$ and for all $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ the equation

$$\frac{\partial}{\partial\lambda} c^T \left\{X_1^{-1} X_2\right\}(t_0;\lambda)c = c^{\mathrm{T}} \int_a^{t_0} X_2^T(t;\lambda) C_0(t) X_2(t;\lambda)\, dt\, c$$

holds.

Proof. We refer to the results of [4, Lemma 4.1.4] and use the same notation. With

$$X_*(t;\lambda) := \begin{pmatrix} 0 & I \\ X_1(t;\lambda) & X_2(t;\lambda) \end{pmatrix}$$

we have

$$X_*^{-1}(t;\lambda) := \begin{pmatrix} -X_1^{-1}(t;\lambda)X_2(t;\lambda) & X_1^{-1}(t;\lambda) \\ I & 0 \end{pmatrix}$$

for all $t \in \mathcal{I}$ and $\lambda \in \mathbb{R}$, for which $X_1(t;\lambda)$ is invertible. Furthermore, let

$$\Phi(t;\lambda) = \begin{pmatrix} X_1 & X_2 \\ U_1 & U_2 \end{pmatrix}(t;\lambda).$$

The lemma cited above implies the following equation for all $t \in \mathcal{I}$ and for all $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$:

$$\begin{aligned}
\frac{\partial}{\partial\lambda} c^T \left\{X_1^{-1} X_2\right\}(t_0;\lambda)c &= -\begin{pmatrix} c \\ 0 \end{pmatrix}^T \frac{\partial}{\partial\lambda} \begin{pmatrix} -X_1^{-1}X_2 & X_1^{-1} \\ (X_1^{-1})^T & U_1 X_1^{-1} \end{pmatrix}(t_0;\lambda) \begin{pmatrix} c \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} c \\ 0 \end{pmatrix}^T (X_*^{-1})^T(t_0;\lambda) \int_a^{t_0} \Phi^T(t;\lambda) \begin{pmatrix} C_0(t) & 0 \\ 0 & 0 \end{pmatrix} \Phi(t;\lambda)\,dt\, X_*^{-1}(t_0;\lambda) \begin{pmatrix} c \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ c \end{pmatrix}^T \int_a^{t_0} \Phi^T(t;\lambda) \begin{pmatrix} C_0(t) & 0 \\ 0 & 0 \end{pmatrix} \Phi(t;\lambda)\,dt \begin{pmatrix} 0 \\ c \end{pmatrix} \\
&= c^T \int_a^{t_0} X_2^T(t;\lambda) C_0(t) X_2(t;\lambda)\,dt\, c.
\end{aligned}$$

This concludes the proof. ■

Proposition 2.3. The equation

$$\ker X_a(b;\lambda) = V \text{ holds for all } \lambda \in \mathbb{R} \setminus \mathcal{M},$$

and the set $\mathcal{M}$ is discrete. Thus, for all $\lambda \in \mathbb{R}$, $\ker X_a(b;\lambda+)$, $\ker X_a(b;\lambda-)$ exist and the following equations hold:

$$\operatorname{def} X_a(b;\lambda) - \operatorname{def} X_a(b;\lambda+) = \operatorname{def} X_a(b;\lambda) - \operatorname{def} X_a(b;\lambda-) = \operatorname{def}\{X_a(b;\lambda)\mathcal{V}\}.$$

Proof. Using the fact that $V \subset \ker X_a(b;\lambda)$ for all $\lambda \in \mathbb{R}$, we derive (by definition of the matrix $\mathcal{V}$)

$$\operatorname{def} X_a(b;\lambda) = \dim V + \operatorname{def}\{X_a(b;\lambda)\mathcal{V}\},\ \lambda \in \mathbb{R}.$$

Now let $\lambda_0 \in \mathbb{R}$ be fixed. To prove the assertion we show that there are $\delta_1 > 0$ and $\delta_2 > 0$ such that $\operatorname{def}\{X_a(b;\lambda)\mathcal{V}\} = 0$ for all $\lambda \in (\lambda_0, \lambda_0 + \delta_1]$ and for all $[\lambda_0 - \delta_2, \lambda_0)$.

We first show that there is a $\delta_1 > 0$ such that $\operatorname{def}\{X_a(b;\lambda)\mathcal{V}\} = 0$ for all $\lambda \in (\lambda_0, \lambda_0 + \delta_1]$. By [4, Proposition 4.1.1] there exists a conjoined basis $(\widetilde{X}, \widetilde{U})$ of (H) such that $(\widetilde{X}, \widetilde{U})$ and (X_a, U_a) are normalized conjoined bases of (H). Then $(X_1, U_1) := (\widetilde{X} + cX_a, \widetilde{U} + cU_a)$ and (X_a, U_a) are normalized conjoined bases for all $c \in \mathbb{R}$ as well and by [4, Theorem 3.1.2] we may choose c sufficiently large such that $X_1(b;\lambda_0)$ is invertible and $\{X_1^{-1}X_a\}(b;\lambda_0) \geq 0$. The continuity of solutions of (H) implies that there is a $\delta_1 > 0$ such that $X_1(b;\lambda)$ is invertible on $[\lambda_0, \lambda_0 + \delta_1]$. Proposition 2.2 implies $c^T\{X_1^{-1}X_a\}c > 0$ for all $c \in \operatorname{Im}\mathcal{V}\setminus\{0\}$ and for all $\lambda \in (\lambda_0, \lambda_0 + \delta_1]$. We especially have $X_a(b;\lambda)c \neq 0$ for all $c \in \operatorname{Im}\mathcal{V}\setminus\{0\}$ and for all $\lambda \in (\lambda_0, \lambda_0 + \delta_1]$. Since $\ker\mathcal{V} = \{0\}$ due to the construction of the matrix $\mathcal{V}$, we can follow $X_a(b;\lambda)\mathcal{V}d \neq 0$ for all $d \neq 0$ and all $\lambda \in (\lambda_0, \lambda_0 + \delta_1]$. This implies the first assertion in this proof.

The second part of this proof follows analogously if we choose $\delta_2 > 0$ and $\tilde{c} \in \mathbb{R}$ sufficiently small such that for the conjoined basis $(X_2, U_2) := (\widetilde{X} + \tilde{c}X_a, \widetilde{U} + \tilde{c}U_a)$ the matrix-valued function $X_2(b;\lambda)$ is invertible on $[\lambda_0 - \delta_2, \lambda_0)$ and we have $\{X_2^{-1}X_a\}(b;\lambda) \leq 0$ for all $\lambda \in [\lambda_0 - \delta_2, \lambda_0)$. ∎

The forthcoming lemma gives a formula for the local change of the number of proper focal points of X_a regarding to λ. Basically this lemma would lead to a local oscillation theorem if we had to deal with Dirichlet boundary conditions, i.e., $x(a) = x(b) = 0$, only. Hence it is the most important part of this paper.

Lemma 2.4. Let $n_1(\lambda)$ denote the number of proper focal points of $X_a(t;\lambda)$ in $(a, b]$ (including multiplicities). Then $n_1(\lambda+)$, $n_1(\lambda-)$ exist and

$$n_1(\lambda+) = n_1(\lambda), \quad n_1(\lambda-) = n_1(\lambda) - \operatorname{def}\{X_a(b;\lambda)\mathcal{V}\}$$

hold for all $\lambda \in \mathbb{R}$.

Proof. First we show that, for any fixed $\lambda_0 \in \mathbb{R}$, $n_1(\lambda_0+)$ exists and the equation $n_1(\lambda_0+) = n_1(\lambda_0)$ holds. As in the proof of Proposition 2.3 let $(\widetilde{X}, \widetilde{U})$ be a conjoined basis of (H) such that $(\widetilde{X}_{t_0}, \widetilde{U}_{t_0})$, (X_a, U_a) are normalized conjoined bases of (H), $(\widetilde{X}_{t_0}, \widetilde{U}_{t_0})$ is independent of λ in the left endpoint a, $\widetilde{X}_{t_0}(t_0, \lambda_0)$ is invertible for a fixed $t_0 \in \mathcal{I}$ and such that $\{\widetilde{X}_{t_0}^{-1}X_a\}(t_0;\lambda_0) \geq 0$. By [5, Theorem 3] and by the continuity of conjoined bases of (H) there is an $\varepsilon_{t_0} > 0$ such that

(i) $\ker X_a(t;\lambda_0) = \ker X_a(t-;\lambda_0)$ for all $t \in [t_0-\varepsilon_{t_0}, t_0+\varepsilon_{t_0}] \setminus \{0\}$,
$\ker X_a(t;\lambda_0) = \ker X_a(t+;\lambda_0)$ for all $t \in [t_0-\varepsilon_{t_0}, t_0+\varepsilon_{t_0}] \setminus \{0\}$.

(ii) $\widetilde{X}_{t_0}(t;\lambda_0)$ is invertible on the interval $[t_0-\varepsilon_{t_0}, t_0+\varepsilon_{t_0}]$.

Since the point $t_0 \in \mathcal{I}$ is arbitrary, the Heine–Borel theorem implies that there are finitely pairwise different points $t_1,\dots,t_N \in \mathcal{I}$ and $\varepsilon_{t_\nu} > 0$ such that (i) and (ii) hold with $t_\nu, \varepsilon_{t_\nu}, (\widetilde{X}_{t_\nu}, \widetilde{U}_{t_\nu})$ instead of $t_0, \varepsilon_{t_0}, (\widetilde{X}_{t_0}, \widetilde{U}_{t_0})$, but with analogous properties and with $\mathcal{I} \subset \bigcup_{\nu=1}^{N} (t_\nu-\varepsilon_{t_\nu}, t_\nu+\varepsilon_{t_\nu})$. Because of (i) all proper focal points of $X_a(\cdot;\lambda_0)$ are contained in $\{t_1,\dots,t_N\}$. Let

$$V_\xi := \bigcap_{t\in[a,\xi]} \ker\left\{X_a^T C_0 X_a\right\}(t;\lambda) \cap \ker X_a(\xi;\lambda),\ \xi \in \mathcal{I}.$$

Proposition 2.1 implies that the space V_ξ is independent of λ for all $\xi \in \mathcal{I}$. Furthermore, let $\mathcal{V}_1 \in \mathbb{R}^{n\times m_1}, \mathcal{V}_2 \in \mathbb{R}^{n\times m_2}, \mathcal{V}_3 \in \mathbb{R}^{n\times m_3}$ with $m_1 := n - \dim V_{t_\nu-\varepsilon_{t_\nu}}$, $m_2 := n - \dim V_{t_\nu}$, $m_3 := n - \dim V_{t_\nu+\varepsilon_{t_\nu}}$, such that

$$\mathbb{R}^n = \operatorname{Im} \mathcal{V}_1 \oplus V_{t_\nu-\varepsilon_{t_\nu}} = \operatorname{Im} \mathcal{V}_2 \oplus V_{t_\nu} = \operatorname{Im} \mathcal{V}_3 \oplus V_{t_\nu+\varepsilon_{t_\nu}}.$$

Because of the continuity of conjoined bases of (H), there exists a $\delta_{t_\nu} > 0$ for each $\nu = 1,\dots,N$, such that

(iii) $\widetilde{X}_{t_\nu}(t;\lambda)$ is invertible on $[t_\nu-\varepsilon_{t_\nu}, t_\nu+\varepsilon_{t_\nu}] \times [\lambda_0-\delta_{t_\nu}, \lambda_0+\delta_{t_\nu}]$ and

(iv) $\mathcal{V}_1^T \left\{\widetilde{X}_{t_\nu}^{-1} X_a\right\}(t_\nu-\varepsilon_{t_\nu};\lambda)\mathcal{V}_1$ is invertible for all $\lambda \in [\lambda_0-\delta_{t_\nu}, \lambda_0+\delta_{t_\nu}] \setminus \{\lambda_0\}$,
$\mathcal{V}_2^T \left\{\widetilde{X}_{t_\nu}^{-1} X_a\right\}(t_\nu;\lambda)\mathcal{V}_2$ is invertible for all $\lambda \in [\lambda_0-\delta_{t_\nu}, \lambda_0+\delta_{t_\nu}] \setminus \{\lambda_0\}$ and
$\mathcal{V}_3^T \left\{\widetilde{X}_{t_\nu}^{-1} X_a\right\}(t_\nu+\varepsilon_{t_\nu};\lambda)\mathcal{V}_3$ is invertible for all $\lambda \in [\lambda_0-\delta_{t_\nu}, \lambda_0+\delta_{t_\nu}] \setminus \{\lambda_0\}$.

Let $\delta := \min_{\nu=1,\dots,N} \delta_{t_\nu} > 0$. According to (iii), the matrix-valued function $H(t;\lambda) := \{\widetilde{X}_{t_\nu}^{-1} X_a\}(t;\lambda)$ is well defined on $[t_\nu-\varepsilon_{t_\nu}, t_\nu+\varepsilon_{t_\nu}] \times [\lambda_0-\delta, \lambda_0+\delta]$ for all $\nu = 1,\dots,N$. Let $\mu_1(t;\lambda),\dots,\mu_n(t;\lambda)$ denote the eigenvalues of $H(t;\lambda)$. Since H is symmetric, continuous in both variables and monotone increasing in t as well as in λ in $[t_\nu-\varepsilon_{t_\nu}, t_\nu+\varepsilon_{t_\nu}] \times [\lambda_0-\delta, \lambda_0+\delta]$ (compare [4, Proposition 4.1.2] and Proposition 2.2), also the eigenvalues $\mu_1(t;\lambda),\dots,\mu_n(t;\lambda)$ are continuous and monotone increasing in t as well as in λ if properly ordered by [4, Proposition 3.2.3].

Furthermore, (iv) implies: If $\mu_i(t_\nu-\varepsilon_{t_\nu};\lambda) = 0$ for a $\lambda \in [\lambda_0-\delta, \lambda_0+\delta] \setminus \{\lambda_0\}$, then we have $\mu_i(t_\nu-\varepsilon_{t_\nu};\lambda) \equiv 0$ on $[\lambda_0-\delta, \lambda_0+\delta]$; if $\mu_i(t_\nu+\varepsilon_{t_\nu};\lambda) = 0$ for a $\lambda \in [\lambda_0-\delta, \lambda_0+\delta] \setminus \{\lambda_0\}$, then we have $\mu_i(t_\nu+\varepsilon_{t_\nu};\lambda) \equiv 0$ on $[\lambda_0-\delta, \lambda_0+\delta]$; if $\mu_i(t_\nu;\lambda) = 0$ for a $\lambda \in [\lambda_0-\delta, \lambda_0+\delta] \setminus \{\lambda_0\}$, then $\mu_i(t_\nu;\lambda) \equiv 0$ on $[\lambda_0-\delta, \lambda_0+\delta]$. We always impose proper notation.

Now let t_ν be a proper focal point of $X_a(\cdot;\lambda_0)$ with multiplicity $k \geq 0$. We show that for all $\lambda \in [\lambda_0, \lambda_0+\delta]$ there are exactly k proper focal points of $X_a(\cdot;\lambda)$ (including multiplicities) in $(t_\nu - \varepsilon_{t_\nu}, t_\nu + \varepsilon_{t_\nu})$. Since the union of these intervals overlaps $\mathcal{I}$, the existence of $n_1(\lambda_0+)$ and the equation $n_1(\lambda_0+) = n_1(\lambda_0)$ follows.

Because of $\{\widetilde{X}_{t_\nu}^{-1} X_a\}(t_\nu; \lambda_0) \geq 0$ and since k is the multiplicity of the proper focal point t_ν of $X_a(\cdot; \lambda_0)$, there is a $c \in \{0, \dots, n-k\}$ with

$$\begin{array}{rcccccl} \mu_1(t_\nu;\lambda_0) & = & \dots & = & \mu_k(t_\nu;\lambda_0) & = & 0, \\ \mu_{k+1}(t_\nu;\lambda_0) & = & \dots & = & \mu_{k+c}(t_\nu;\lambda_0) & = & 0, \\ \mu_{k+c+1}(t_\nu;\lambda_0) & , & \dots & , & \mu_n(t_\nu;\lambda_0) & > & 0. \end{array}$$

Since $\ker X_a(t; \lambda_0)$ is constant on $[t_\nu - \varepsilon_{t_\nu}, t_\nu)$ and on $(t_\nu, t_\nu + \varepsilon_{t_\nu}]$, according to (i) and because of the monotonicity of the eigenvalues μ_i in t, it holds that for all $t \in [t_\nu - \varepsilon_{t_\nu}, t_\nu)$

$$\begin{array}{rcccccl} \mu_1(t;\lambda_0) & , & \dots & , & \mu_k(t;\lambda_0) & < & 0, \\ \mu_{k+1}(t;\lambda_0) & = & \dots & = & \mu_{k+c}(t;\lambda_0) & = & 0, \\ \mu_{k+c+1}(t;\lambda_0) & , & \dots & , & \mu_n(t;\lambda_0) & > & 0, \end{array}$$

and for all $t \in (t_\nu, t_\nu + \varepsilon_{t_\nu}]$

$$\begin{array}{rcccccl} \mu_1(t;\lambda_0) & , & \dots & , & \mu_k(t;\lambda_0) & \geq & 0, \\ \mu_{k+1}(t;\lambda_0) & , & \dots & , & \mu_{k+c}(t;\lambda_0) & \geq & 0, \\ \mu_{k+c+1}(t;\lambda_0) & , & \dots & , & \mu_n(t;\lambda_0) & > & 0. \end{array}$$

Because of (iv) and the monotonicity of the eigenvalues μ_i in λ, we have for all $\lambda \in [\lambda_0, \lambda_0 + \delta]$

$$\begin{array}{rcccccl} \mu_1(t_\nu - \varepsilon_{t_\nu};\lambda) & , & \dots & , & \mu_k(t_\nu - \varepsilon_{t_\nu};\lambda) & < & 0, \\ \mu_{k+1}(t_\nu - \varepsilon_{t_\nu};\lambda) & , & \dots & , & \mu_{k+c}(t_\nu - \varepsilon_{t_\nu};\lambda) & \geq & 0, \\ \mu_{k+c+1}(t_\nu - \varepsilon_{t_\nu};\lambda) & , & \dots & , & \mu_n(t_\nu - \varepsilon_{t_\nu};\lambda) & > & 0. \end{array}$$

Furthermore, (iv) and the monotonicity of the eigenvalues μ_i in λ imply that for all $\lambda \in [\lambda_0, \lambda_0 + \delta]$

$$\begin{array}{rcccccl} \mu_1(t_\nu;\lambda) & , & \dots & , & \mu_k(t_\nu;\lambda) & \geq & 0, \\ \mu_{k+1}(t_\nu;\lambda) & , & \dots & , & \mu_{k+c}(t_\nu;\lambda) & \geq & 0, \\ \mu_{k+c+1}(t_\nu;\lambda) & , & \dots & , & \mu_n(t_\nu;\lambda) & > & 0 \end{array}$$

holds. Finally, the monotonicity of the eigenvalues $\mu_i(\cdot; \lambda)$ in t for all $\lambda \in [\lambda_0, \lambda_0 + \delta]$ implies

$$\begin{array}{rcccccl} \mu_1(t_\nu + \varepsilon_{t_\nu};\lambda) & , & \dots & , & \mu_k(t_\nu + \varepsilon_{t_\nu};\lambda) & \geq & 0, \\ \mu_{k+1}(t_\nu + \varepsilon_{t_\nu};\lambda) & , & \dots & , & \mu_{k+c}(t_\nu + \varepsilon_{t_\nu};\lambda) & \geq & 0, \\ \mu_{k+c+1}(t_\nu + \varepsilon_{t_\nu};\lambda) & , & \dots & , & \mu_n(t_\nu + \varepsilon_{t_\nu};\lambda) & > & 0. \end{array}$$

Because of the continuity and the monotonicity of the eigenvalues μ_i in t, there are uniquely determined $\tau_{i,\lambda} \in (t_\nu - \varepsilon_{t_\nu}, t_\nu]$ such that $\mu_i(\tau_{i,\lambda}; \lambda) = 0$ and $\mu_i(\tau; \lambda) < 0$ for all $\tau \in [t_\nu - \varepsilon_{t_\nu}, \tau_{i,\lambda})$, $i = 1, \dots, k$, and all $\lambda \in [\lambda_0, \lambda_0 + \delta]$. These points are exactly the proper focal points of $X_a(\cdot; \lambda)$, $\lambda \in [\lambda_0, \lambda_0 + \delta)$, since the defect of $X_a(t; \lambda)$ equals to the number of those eigenvalues of $H(t;\lambda)$ that are equal to zero, and these are the only points with this property in $(t_\nu - \varepsilon_{t_\nu}, t_\nu + \varepsilon_{t_\nu})$, $\nu = 1, \dots, N$, i.e., for all $\lambda \in [\lambda_0, \lambda_0 + \delta]$ there are exactly k proper focal points of $X_a(\cdot;\lambda)$ in $(t_\nu - \varepsilon_{t_\nu}, t_\nu + \varepsilon_{t_\nu})$. This implies the existence of $n_1(\lambda_0+)$ and proves the equation $n_1(\lambda_0+) = n_1(\lambda_0)$.

It remains to prove that $n_1(\lambda_0-)$ exists and $n_1(\lambda_0) = n_1(\lambda_0) - \operatorname{def}\{X_a(b;\lambda_0)\mathcal{V}\}$ holds. Only for this part, we introduce the term "right proper focal point" of a conjoined basis of (H): for $t_0 \in (a,b)$ let $r \in \{0,\dots,n\}$ be the number of the linearly independent vectors $d_1,\dots,d_r \in \mathbb{R}^n$ with

$$d_i \in \ker X_a(t_0),\;\; d_i \notin \ker X_a(t_0+),\;\; d_i \notin \bigcap_{t\in[a,t_0]} \ker X_a(t).$$

Given this setup, we call t_0 a *right proper focal point of X_a with multiplicity r* if $r \geq 1$. For clarity, we will give a short interpretation of right proper focal points of X_a and point out the connection to the (left) proper focal points of X_a. The defect of $X_a(t;\lambda)$ equals to the number of eigenvalues of $H(t;\lambda)$ which are zero. A proper focal point $(t;\lambda)$ of X_a with multiplicity k is characterized as follows: There are exactly k eigenvalues μ_i of H with $\mu_i(t;\lambda) = 0$, $\mu_i(t-\varepsilon;\lambda) < 0$ for every sufficiently small $\varepsilon > 0$ if properly numbered. All of these eigenvalues of H are (if properly numbered) identically zero on some compact (maybe degenerated) interval of $\mathcal{I}$. The right endpoint of such an interval is just a right proper focal point of X_a if this point is different from b. Furthermore, we want to guarantee that this point is connected to a proper focal point of X_a, i.e., that the eigenvalue of H is not identically zero on $[a, t_0]$.

Let $\widetilde{n}_1(\lambda)$ denote the number of the right proper focal points of $X_a(\cdot;\lambda)$ in (a,b) (including multiplicities). Then the following relation holds between the number of the proper focal points of X_a and the number of right focal points of X_a:

$$\widetilde{n}_1(\lambda) = n_1(\lambda) - \left(\operatorname{def} X_a(b;\lambda) - \dim \bigcap_{t\in\mathcal{I}} \ker X_a(t;\lambda)\right) \text{ for all } \lambda \in \mathbb{R}.$$

We now show that $\widetilde{n}_1(\lambda-)$ exists and $\widetilde{n}_1(\lambda-) = \widetilde{n}_1(\lambda)$ holds for all $\lambda \in \mathbb{R}$. Since $\operatorname{def} X_a(b;\lambda-)$ exists due to Proposition 2.3 and since $\bigcap_{t\in\mathcal{I}} \ker X_a(t;\lambda)$ is independent of λ, according to Proposition 2.1, the existence of $\widetilde{n}_1(\lambda-)$ for all $\lambda \in \mathbb{R}$ implies the existence of $n_1(\lambda-)$ for all $\lambda \in \mathbb{R}$, and with Proposition 2.3 we can determine the following identity for every $\lambda \in \mathbb{R}$, if we show the identity above. We have

$$\begin{aligned} n_1(\lambda) - n_1(\lambda-) &= \widetilde{n}_1(\lambda) + \widetilde{n}_1(\lambda-) + \operatorname{def} X_a(b;\lambda) - \operatorname{def} X_a(b;\lambda-) \\ &= \operatorname{def} X_a(b;\lambda) - \operatorname{def} X_a(b;\lambda-) = \operatorname{def}\{X_a(b;\lambda)\mathcal{V}\}, \end{aligned}$$

i.e., it suffices to prove that $\widetilde{n}_1(\lambda-) = \widetilde{n}_1(\lambda)$ holds for all $\lambda \in \mathbb{R}$.

Analogously to the first part of this proof, we show that $\widetilde{n}_1(\lambda_0-)$ exists for a fixed $\lambda_0 \in \mathbb{R}$ and $\widetilde{n}_1(\lambda_0-) = \widetilde{n}_1(\lambda_0)$ holds. Therefore, let $t_1,\dots,t_N, \varepsilon_{t_1},\dots,\varepsilon_{t_N}$ be defined as above and let (i)–(iv) hold. Because of (i), every right focal point of $X_a(\cdot;\lambda_0)$ is included in the set $\{t_1, \dots, t_N\}$. Let $t_\nu \in (a, b)$ be a right focal point of $X_a(\cdot; \lambda_0)$ with multiplicity $r \geq 0$. We show that for every $\lambda \in [\lambda_0 - \delta, \lambda_0]$ there are exactly r right focal points of $X_a(\cdot; \lambda)$ (including multiplicities) in $(t_\nu - \varepsilon_{t_\nu}, t_\nu + \varepsilon_{t_\nu})$. Since the union of those intervals overlaps $\mathcal{I}$, the existence of $\widetilde{n}_1(\lambda_0-)$ and the equation $\widetilde{n}_1(\lambda_0-) = \widetilde{n}_1(\lambda_0)$ follow.

Because of $\{\widetilde{X}_{t_\nu}^{-1} X_a\}(t_\nu; \lambda_0) \geq 0$ and since r is the multiplicity of the right focal point of $X_a(\cdot; \lambda_0)$, there is a $d \in \{0, \dots, n - r\}$ with

$$\begin{array}{rcccll} \mu_1(t_\nu; \lambda_0) & = & \dots & = & \mu_r(t_\nu; \lambda_0) & = 0, \\ \mu_{r+1}(t_\nu; \lambda_0) & = & \dots & = & \mu_{r+d}(t_\nu; \lambda_0) & = 0, \\ \mu_{r+d+1}(t_\nu; \lambda_0) & , & \dots & , & \mu_n(t_\nu; \lambda_0)(t_\nu; \lambda_0) & > 0, \end{array}$$

if the eigenvalues of H are properly numbered. Since $\ker X_a(t; \lambda_0)$ is constant on $[t_\nu - \varepsilon_{t_\nu}, t_\nu)$ and $(t_\nu, t_\nu + \varepsilon_{t_\nu}]$, respectively according to (i) and because of the monotonicity of the eigenvalues μ_i in t, we have for all $t \in [t_\nu - \varepsilon_{t_\nu}, t_\nu)$

$$\begin{array}{rcccll} \mu_1(t; \lambda_0) & , & \dots & , & \mu_r(t; \lambda_0) & \leq 0, \\ \mu_{r+1}(t; \lambda_0) & , & \dots & , & \mu_{r+d}(t; \lambda_0) & \leq 0, \\ \mu_{r+d+1}(t; \lambda_0) & , & \dots & , & \mu_n(t; \lambda_0) & > 0, \end{array}$$

and for all $t \in (t_\nu, t_\nu + \varepsilon_{t_\nu}]$

$$\begin{array}{rcccll} \mu_1(t; \lambda_0) & , & \dots & , & \mu_r(t; \lambda_0) & > 0, \\ \mu_{r+1}(t; \lambda_0) & = & \dots & = & \mu_{r+d}(t; \lambda_0) & = 0, \\ \mu_{r+d+1}(t; \lambda_0) & , & \dots & , & \mu_n(t; \lambda_0) & > 0. \end{array}$$

Furthermore, (iv) and the monotonicity of the eigenvalues μ_i in λ for all $\lambda \in [\lambda_0 - \delta, \lambda_0)$ imply

$$\begin{array}{rcccll} \mu_1(t_\nu; \lambda) & , & \dots & , & \mu_r(t_\nu; \lambda) & \leq 0, \\ \mu_{r+1}(t_\nu; \lambda) & , & \dots & , & \mu_{r+d}(t_\nu; \lambda) & \leq 0, \\ \mu_{r+d+1}(t_\nu; \lambda) & , & \dots & , & \mu_n(t_\nu; \lambda) & > 0. \end{array}$$

Finally, (iv) and the monotonicity of the eigenvalues μ_i in λ imply for every $\lambda \in [\lambda_0 - \delta, \lambda_0)$:

$$\begin{array}{rcccll} \mu_1(t_\nu - \varepsilon_{t_\nu}; \lambda) & , & \dots & , & \mu_r(t_\nu - \varepsilon_{t_\nu}; \lambda) & \leq 0, \\ \mu_{r+1}(t_\nu - \varepsilon_{t_\nu}; \lambda) & , & \dots & , & \mu_{r+d}(t_\nu - \varepsilon_{t_\nu}; \lambda) & \leq 0, \\ \mu_{r+d+1}(t_\nu - \varepsilon_{t_\nu}; \lambda) & , & \dots & , & \mu_n(t_\nu - \varepsilon_{t_\nu}; \lambda) & > 0 \end{array}$$

and

$$\begin{array}{rcccll} \mu_1(t_\nu + \varepsilon_{t_\nu}; \lambda) & , & \dots & , & \mu_r(t_\nu + \varepsilon_{t_\nu}; \lambda) & > 0, \\ \mu_{r+1}(t_\nu + \varepsilon_{t_\nu}; \lambda) & , & \dots & , & \mu_{r+d}(t_\nu + \varepsilon_{t_\nu}; \lambda) & \leq 0, \\ \mu_{r+d+1}(t_\nu + \varepsilon_{t_\nu}; \lambda) & , & \dots & , & \mu_n(t_\nu + \varepsilon_{t_\nu}; \lambda) & > 0. \end{array}$$

The continuity and monotonicity of the eigenvalues μ_i in t imply for all $i \in \{0, \dots, r\}$ and for all $\lambda \in [\lambda_0 - \delta, \lambda_0]$ that there are uniquely determined $\tau_{i,\lambda} \in [t_\nu, t_\nu + \varepsilon_{t_\nu})$ with $\mu_i(\tau_{i,\lambda}; \lambda) = 0$; $\mu_i(\tau; \lambda) > 0$ for all $\tau \in (\tau_{i,\lambda}, t_\nu + \varepsilon_{t_\nu}]$. Consequently, for these points there is a $c \in \mathbb{R}^n$ with $c \in \ker X_a(\tau_{i,\lambda}; \lambda)$, $c \notin \ker X_a(\tau_{i;\lambda}+; \lambda)$ and $c \notin \bigcap_{t \in \mathcal{I}} \ker X_a(t)$. If there was a $c \in \bigcap_{t \in [a, \tau_{i,\lambda}]} \ker X_a(t)$, the independence of $\bigcap_{t \in [a, \tau_{i,\lambda}]} \ker X_a(t)$ of λ would imply that $c \in \ker X_a(\tau_{i,\lambda}; \lambda_0)$, $c \notin \ker X_a(\tau_{i,\lambda}+; \lambda_0)$, and this contradicts the fact that the kernel of $X_a(\cdot; \lambda_0)$ is constant on $(t_\nu, t_\nu + \varepsilon_{t_\nu}]$ according to (i). Hence these points are the right focal points of $X_a(\cdot; \lambda)$ for $\lambda \in [\lambda_0 - \delta, \lambda_0]$. These are the only points in $(t_\nu - \varepsilon_{t_\nu}, t_\nu + \varepsilon_{t_\nu})$ with this property, i.e., for all $\lambda \in [\lambda_0 - \delta, \lambda_0]$ there are exactly r right focal points of $X_a(\cdot; \lambda)$ in $(t_\nu - \varepsilon_{t_\nu}, t_\nu + \varepsilon_{t_\nu})$. ■

3. Proper Eigenvalues

Proposition 3.1. Let $M \in \mathbb{R}^{m \times n}$, $m \le n$, and $r \in \{0, \dots, m\}$. Then there exists an invertible matrix $Z \in \mathbb{R}^{n \times n}$ and a matrix $C \in \mathbb{R}^{r \times (m-r)}$ such that

$$MZ = \begin{pmatrix} M_{11} & CM_{22} \\ 0 & M_{22} \end{pmatrix}$$

with certain matrices $M_{11} \in \mathbb{R}^{r \times r}$ and $M_{22} \in \mathbb{R}^{(m-r) \times (n-r)}$.

Proof. Let k_1 be the maximal number of linearly independent vectors $c_1, \dots, c_{k_1} \in \mathbb{R}^r$ for which there are $z_1, \dots, z_{k_1} \in \mathbb{R}^n$ with $Mz_\nu = (c_\nu^T, 0)^T$, $\nu = 1, \dots, k_1$. Suppose $\{z_{k_1+1}, \dots, z_{k_2}\}$ is a basis of $\ker M$. Then the vectors $z_1, \dots, z_{k_2}$ are linearly independent. Since

$$\operatorname{rank} M \le k_1 + m - r,$$

we have

$$r \le m - n + k_1 + \operatorname{def} M \le k_1 + \operatorname{def} M = k_2.$$

Now, let $z_{k_2+1}, \dots, z_n \in \mathbb{R}^n$ be further vectors such that $\{z_1, \dots, z_n\}$ is a basis of $\mathbb{R}^n$. With $Z := (z_1, \dots, z_n)$ we have

$$MZ = (Mz_1, \dots, Mz_n) = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

with certain matrices $M_{11} \in \mathbb{R}^{r \times r}$, $M_{12} \in \mathbb{R}^{r \times (n-r)}$, $M_{22} \in \mathbb{R}^{(m-r) \times (n-r)}$. Because of the relation

$$\mathbb{R}^{n-r} = \ker M_{22} \oplus \operatorname{Im} M_{22}^T = \ker M_{12} \oplus \operatorname{Im} M_{12}^T,$$

it remains to prove that $\ker M_{22} \subset \ker M_{12}$. With $d \in \ker M_{22}$ we have

$$MZ \begin{pmatrix} 0 \\ d \end{pmatrix} = \begin{pmatrix} M_{12} d \\ 0 \end{pmatrix}.$$

Now assume that $d \notin \ker M_{12}$. Then $Z \begin{pmatrix} 0 \\ d \end{pmatrix}$ would be linearly dependent of the vectors $z_1, \ldots, z_{k_1}, k_1 \leq r$, i.e.,

$$Z \begin{pmatrix} 0 \\ d \end{pmatrix} = \sum_{\nu=1}^{k_1} \alpha_\nu z_\nu = \sum_{\nu=r+1}^{n} d_\nu z_\nu$$

with $d = (d_{r+1}, \ldots, d_n)^T$ and certain $\alpha_1, \ldots, \alpha_{k_1} \in \mathbb{R}$. But since the vectors $z_1, \ldots, z_{k_1}$, $z_{r+1}, \ldots, z_n$ are linearly independent, we would have $d = 0$ in contradiction to $d \notin \ker M_{12}$. ■

Proposition 3.2. The proper eigenvalues of (E) are isolated.

Proof. Assume that the proper eigenvalues of (E) are not isolated. Then there is a sequence of proper eigenvalues (λ_k) of (E) with related eigenfunctions (x_k, u_k) which converges to a proper eigenvalue λ of (E). Let $(\widetilde{X}, \widetilde{U})$ be a conjoined basis of (H) such that $\widetilde{X}(a)$ and $\widetilde{U}(a)$ are independent of λ and such that $(\widetilde{X}, \widetilde{U})$, (X_a, U_a) are normalized conjoined bases of (H), i.e., fulfill the equation $\widetilde{X}^T U_a - \widetilde{U}^T X_a \equiv I$ on $\mathcal{I}$. For $k \in \mathbb{N}$ consider the fundamental matrix Φ_k with

$$\dot{\Phi}_k = \begin{pmatrix} A & B \\ C - \lambda_k C_0 & -A^T \end{pmatrix} \Phi_k, \quad \Phi_k(a) = \Phi_a := \begin{pmatrix} \widetilde{X}(a) & X_a(a) \\ \widetilde{U}(a) & U_a(a) \end{pmatrix}.$$

Because of the continuity of solutions of (H) and by the compactness of the interval $\mathcal{I} = [a, b]$ it follows that Φ_k converges uniformly on $\mathcal{I}$ to Φ with

$$\dot{\Phi} = \begin{pmatrix} A & B \\ C - \lambda C_0 & -A^T \end{pmatrix} \Phi, \quad \Phi(a) := \Phi_a.$$

Let $(c_k), (d_k) \subset \mathbb{R}^n$ with

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix}(t) = \Phi_k(t) \begin{pmatrix} c_k \\ d_k \end{pmatrix}.$$

Since $\Phi(a)$ is invertible and

$$\operatorname{rank}\left(R_1^a, R_2^a\right) = n,$$
$$R_1^a x_k(a) + R_2^a u_k(a) = \left(R_1^a \widetilde{X}(a) + R_2^a \widetilde{U}(a)\right) c_k + \left(R_1^a X_a(a) + R_2^a U_a(a)\right) d_k = 0,$$

it follows that $(c_k) \equiv 0$. Hence

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix}(t) = \Phi_k(t) \begin{pmatrix} 0 \\ d_k \end{pmatrix}.$$

Without loss of generality we may assume $d_k \in \operatorname{Im} \mathcal{W}$, $\|d_k\| = 1$ for $k \in \mathbb{N}$. Hence (d_k) is bounded and there is a subsequence which we also want to denote by (d_k) with

$$d_k \to d \ (k \to \infty), \quad d \in \operatorname{Im} \mathcal{W}, \quad \|d\| = 1.$$

It follows that

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix}(t) \to \begin{pmatrix} x \\ u \end{pmatrix} := \Phi(t) \begin{pmatrix} 0 \\ d \end{pmatrix} \ (k \to \infty) \text{ uniformly on } \mathcal{I}.$$

According to Remark 1.3(iv), eigenfunctions belonging to different proper eigenfunctions are orthogonal to each other, i.e., $< x_k, x_l >= 0$ for all $k \neq l$. Consequently, $< x, x >= \lim_{k\to\infty} < x_k, x_{k+1} >= 0$, i.e., $C_0 x \equiv 0$ on $\mathcal{I}$ and $R_1^b x(b) + R_2^b u(b) = 0$. Hence $d \in W$ in contradiction to $d \in \operatorname{Im} \mathcal{W} \setminus \{0\}$. ■

Proposition 3.3. $\lambda \in \mathbb{R}$ is a proper eigenvalue of (E) if and only if $\operatorname{def}\{\Lambda(\lambda)\mathcal{W}\} > 0$, and then this number equals to the multiplicity of the proper eigenvalue λ of (E).

Proof. As in the proof of Proposition 2.3, there is a conjoined basis $(\widetilde{X}, \widetilde{U})$ of (H) such that $(\widetilde{X}, \widetilde{U})$, (X_a, U_a) are normalized conjoined bases of (H). The vector-valued solution (x, u) of (H) is of the form

$$\begin{pmatrix} x \\ u \end{pmatrix}(t) = \Phi(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

with constants $c_1, c_2 \in \mathbb{R}^n$, where $\Phi(t)$ is defined as in the previous proof. Observe that (x, u) fulfills the boundary conditions and $C_0 x \not\equiv 0$ on $\mathcal{I}$ if and only if

$$\begin{pmatrix} R_1^a \widetilde{X}(a;\lambda) + R_2^a \widetilde{U}(a;\lambda) & R_1^a X_a(a;\lambda) + R_2^a U_a(a;\lambda) \\ R_1^b \widetilde{X}(b;\lambda) + R_2^b \widetilde{U}(b;\lambda) & R_1^b X_a(b;\lambda) + R_2^b U_a(b;\lambda) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0,$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^T \int_{\mathcal{I}} \begin{pmatrix} \widetilde{X} & X_a \end{pmatrix}^T (t;\lambda) C_0(t) \begin{pmatrix} \widetilde{X} & X_a \end{pmatrix} (t;\lambda)\, dt \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} > 0.$$

The initial values of the special conjoined basis (X_a, U_a) of (H) imply $R_1^a X_a(a) + R_2^a U_a(a) = 0$. Since $\operatorname{rank}(R_1^a, R_2^a) = n$ and since $\Phi(a)$ is invertible, it follows that $c_1 = 0$. Therefore (x, u) is an eigenfunction of the proper eigenvalue λ with multiplicity k if and only if there are k linearly independent vectors c_2 with

$$\Lambda(\lambda) c_2 = 0 \text{ and } C_0(t) X_a(t;\lambda) c_2 \not\equiv 0 \text{ on } \mathcal{I}.$$

Hence $\lambda \in \mathbb{R}$ is an eigenvalue of (E) if and only if $\operatorname{def}\{\Lambda(\lambda)\mathcal{W}\} > 0$, and this number equals to the multiplicity of the proper eigenvalue λ of (E). ■

Proposition 3.4. There are orthogonal matrices $P, Q \in \mathbb{R}^{n\times n}$ such that

$$X_a(b;\lambda) = Q^T \begin{pmatrix} X_{11}(\lambda) & 0 \\ 0 & 0 \end{pmatrix} P^T, \quad U_a(b;\lambda) = Q^T \begin{pmatrix} U_{11}(\lambda) & 0 \\ U_{21}(\lambda) & U_{22} \end{pmatrix} P^T$$

with a matrix-valued function $X_{11}(\lambda) \in \mathbb{R}^{r\times r}$, $r := \operatorname{rank} X_a(b;\lambda+)$, which is invertible on the set $\mathbb{R} \setminus \mathcal{M}$, matrix-valued functions $U_{11}(\lambda) \in \mathbb{R}^{r\times r}$, $U_{21}(\lambda) \in \mathbb{R}^{(n-r)\times r}$, and a constant matrix $U_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$.

Proof. According to Proposition 2.3, the set $\mathcal{M}$ is discrete such that there is a $\lambda_0 \in \mathbb{R}$ and an orthogonal matrix $P \in \mathbb{R}^{n\times n}$ with $\ker X_a(b;\lambda_0) = \ker X_a(b;\lambda_0+)$ and

$$X_a(b;\lambda_0)P = (\underbrace{*}_{r}, 0) \text{ with } r := \operatorname{rank} X_a(b;\lambda_0).$$

By the Gram–Schmidt theorem there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$QX_a(b;\lambda_0)P = \begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

with an invertible matrix $X_{11} \in \mathbb{R}^{r\times r}$. Now, let

$$\begin{aligned} \widetilde{X}(\lambda) &= \begin{pmatrix} X_{11}(\lambda) & X_{12}(\lambda) \\ X_{21}(\lambda) & X_{22}(\lambda) \end{pmatrix} := QX_a(b;\lambda)P, \\ \widetilde{U}(\lambda) &= \begin{pmatrix} U_{11}(\lambda) & U_{12}(\lambda) \\ U_{21}(\lambda) & U_{22}(\lambda) \end{pmatrix} := QU_a(b;\lambda)P \end{aligned}$$

with certain matrices $X_{11}(\lambda), U_{11}(\lambda) \in \mathbb{R}^{r\times r}$, $X_{12}(\lambda), U_{12}(\lambda) \in \mathbb{R}^{r\times(n-r)}$, $X_{21}(\lambda)$, $U_{21}(\lambda) \in \mathbb{R}^{(n-r)\times r}$, $X_{22}(\lambda), U_{22}(\lambda) \in \mathbb{R}^{(n-r)\times(n-r)}$, $\lambda \in \mathbb{R}$. The identity

$$X_{12}(\lambda) = X_{22}(\lambda) = 0 \text{ for all } \lambda \in \mathbb{R}$$

follows directly from the definition of P and Proposition 2.3. Since $X_a(b;\lambda)c \equiv 0$ on $\mathbb{R}$ and hence $c \in V$ according to Proposition 2.3, $C_0(\cdot)X_a(\cdot;\lambda)c \equiv 0$ holds on $\mathcal{I}$ for all $\lambda \in \mathbb{R}$ and so $U_{12}(\cdot)$ and $U_{22}(\cdot)$ are constant. We set $U_{12} := U_{12}(\lambda_0)$, $U_{22} := U_{22}(\lambda_0)$. Now

$$\begin{aligned} \widetilde{X}^T(\lambda)\widetilde{U}(\lambda) &= \begin{pmatrix} X_{11}^T(\lambda)U_{11}(\lambda) + X_{21}^T(\lambda)U_{21}(\lambda) & X_{11}^T(\lambda)U_{12} + X_{21}^T(\lambda)U_{22} \\ 0 & 0 \end{pmatrix} \\ &= P^{\mathrm{T}}\{X_a^T U_a\}(b;\lambda)P \end{aligned}$$

is symmetric for all $\lambda \in \mathbb{R}$ according to the definition of a conjoined basis of (H). Furthermore, the definition of Q implies that $X_{21}(\lambda_0) = 0$ and hence $U_{21} = 0$ because of the invertibility of $X_{11}(\lambda_0)$. Next, U_{22} is invertible because of

$$n = \operatorname{rank}(\widetilde{X}^T(\lambda_0), \widetilde{U}^T(\lambda_0)) = \operatorname{rank} \begin{pmatrix} X_{11}^T(\lambda_0) & U_{11}^T(\lambda_0) & U_{21}^T(\lambda_0) \\ 0 & 0 & U_{22}^T \end{pmatrix}.$$

We get

$$\widetilde{X}^T(\lambda)\widetilde{U}(\lambda) = \begin{pmatrix} X_{11}^T(\lambda)U_{11}(\lambda) + X_{21}^T(\lambda)U_{21}(\lambda) & X_{21}^T(\lambda)U_{22} \\ 0 & 0 \end{pmatrix}$$

for all $\lambda \in \mathbb{R}$. The symmetry and the invertibility of U_{22} imply $X_{21}(\cdot) \equiv 0$. This and Proposition 2.3 finish the proof. ■

Proposition 3.5. Let (X, U) be a conjoined basis of (H) with $X(a; \lambda) \equiv X(a)$ and $U(a; \lambda) \equiv U(a)$. Furthermore, let

$$X(b; \lambda) = \begin{pmatrix} X_{11}(\lambda) & 0 \\ 0 & 0 \end{pmatrix}, \; U(b; \lambda) = \begin{pmatrix} U_{11}(\lambda) & 0 \\ U_{21}(\lambda) & U_{22} \end{pmatrix},$$

with an $r \times r$-matrix-valued function X_{11}, $r := \operatorname{rank} X(b; \lambda+)$, which is invertible up to isolated points, matrix-valued functions $U_{11}(\lambda) \in \mathbb{R}^{r\times r}$, $U_{21}(\lambda) \in \mathbb{R}^{(n-r)\times r}$ and a constant invertible matrix $U_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$. Then the matrix-valued function $\{U_{11}X_{11}^{-1}\}(\lambda)$ is monotone decreasing on every interval on which $X_{11}(\lambda)$ is invertible.

Proof. On every interval, on which $X_{11}(\lambda)$ is regular, and with $\dfrac{\partial}{\partial\lambda} ='$ and with suppressing the argument λ, the following equation holds:

$$\begin{aligned}
\{U_{11}X_{11}^{-1}\}' &= (X_{11}^{-1})^T\{X_{11}^T U_{11}' - U_{11}^T X_{11}'\}X_{11}^{-1} \\
&= \begin{pmatrix} X_{11}^{-1} \\ 0 \end{pmatrix}^T \begin{pmatrix} X_{11}^T U_{11}' - U_{11}^T X_{11}' & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{11}^{-1} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} X_{11}^{-1} \\ 0 \end{pmatrix}^T \left\{ \begin{pmatrix} X_{11}^T & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_{11}' & 0 \\ U_{21}' & U_{22}' \end{pmatrix} - \begin{pmatrix} U_{11}^T & U_{21}^T \\ 0 & U_{22}^T \end{pmatrix} \begin{pmatrix} X_{11}' & 0 \\ 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} X_{11}^{-1} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} X_{11}^{-1} \\ 0 \end{pmatrix}^T \{X^T(b;\cdot)U'(b;\cdot) - U^T(b;\cdot)X'(b;\cdot)\} \begin{pmatrix} X_{11}^{-1} \\ 0 \end{pmatrix}.
\end{aligned}$$

We now have

$$\begin{aligned}
\frac{d}{dt}\{X^T(t;\cdot)U'(t;\cdot) - U^T(t;\cdot)X'(t;\cdot)\} &= (A(t)X(t;\cdot) + B(t)U(t;\cdot))^T U'(t;\cdot) \\
&\quad + X^T(t;\cdot)((C(t) - \lambda C_0(t))X'(t;\cdot) - A^T(t)U'(t;\cdot) - C_0(t)X(t;\cdot)) \\
&\quad - ((C(t) - \lambda C_0(t))X(t;\cdot) - A^T(t)U(t;\cdot))^T X'(t;\cdot) \\
&\quad - U^T(t;\cdot)(A(t)X'(t;\cdot) + B(t)U'(t;\cdot)) \\
&= -X^T(t;\cdot)C_0(t)X(t;\cdot),
\end{aligned}$$

such that

$$\left\{U_{11}X_{11}^{-1}\right\}' = -\begin{pmatrix} X_{11}^{-1} \\ 0 \end{pmatrix}^T \int_{\mathcal{I}} X^T(\tau;\cdot)C_0(\tau)X(\tau;\cdot)d\tau \begin{pmatrix} X_{11}^{-1} \\ 0 \end{pmatrix}.$$

Since the matrix-valued function C_0 is positive semi-definite on $\mathcal{I}$ according to our assumptions, the assertion follows. ∎

Proposition 3.6. Let $r := \operatorname{rank} X_a(b;\lambda+)$. There exist certain matrices $R_{11}^{(1)}, R_{11}^{(2)} \in \mathbb{R}^{r\times r}$, $R_{12}^{(1)} \in \mathbb{R}^{(n-r)\times r}$, $R_{22}^{(1)}, R_{22}^{(2)} \in \mathbb{R}^{(n-r)\times(n-r)}$, $C_1, C_2 \in \mathbb{R}^{(n-r)\times r}$ such that with

$$\widetilde{R}_1^b := \begin{pmatrix} R_{11}^{(1)} & R_{12}^{(1)} \\ R_{22}^{(2)} C_1 & R_{22}^{(1)} \end{pmatrix}, \quad \widetilde{R}_2^b := \begin{pmatrix} R_{11}^{(2)} & 0 \\ R_{22}^{(2)} C_2 & R_{22}^{(2)} \end{pmatrix}$$

the following statements hold:

(i) $\operatorname{Im} R_2^{b^T} = \operatorname{Im} \widetilde{R}_2^{bT}$.

(ii) $\ker(\widetilde{R}_1^b X_a(b;\lambda) + \widetilde{R}_2^b U_a(b;\lambda)) = \ker(R_1^b X_a(b;\lambda) + R_2^b U_a(b;\lambda))$ for all $\lambda \in \mathbb{R}$.

(iii) Let S_1 be a real symmetric matrix and S_2 be a real matrix with $R_1^b = R_2^b S_1 + S_2$, $\ker R_2^b = \operatorname{Im} S_2^T$. Then there is a real matrix $\widetilde{S}_2$ such that $\widetilde{R}_1^b = \widetilde{R}_2^b S_1 + \widetilde{S}_2$, $\ker \widetilde{R}_2^b = \operatorname{Im} \widetilde{S}_2^T$.

(iv) $\operatorname{rank}\left(R_{11}^{(1)}, R_{11}^{(2)}\right) = r$ and $R_{11}^{(1)} R_{11}^{(2)^T}$ is symmetric.

Proof. According to [4, Corollary 3.1.3(i)] there is a symmetric real matrix S_1 and a real matrix S_2 such that $R_1^b = R_2^b S_1 + S_2$, $\ker R_2^b = \operatorname{Im} S_2^T$. Proposition 3.1 implies that there is an invertible matrix $Z \in \mathbb{R}^{n\times n}$ and a matrix $C_2 \in \mathbb{R}^{(n-r)\times r}$ such that

$$R_2^{b^T} Z = \begin{pmatrix} R_{11}^{(2)^T} & C_2^T R_{22}^{(2)^T} \\ 0 & R_{22}^{(2)^T} \end{pmatrix}$$

with certain matrices $R_{11} \in \mathbb{R}^{r\times r}$, $R_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$. We set

$$\widehat{R}_1^b := Z^T R_1^b, \quad \widetilde{R}_2^b := Z^T R_2^b = \begin{pmatrix} R_{11}^{(2)} & 0 \\ R_{22}^{(2)} C_2 & R_{22}^{(2)} \end{pmatrix}, \quad \widehat{S}_2 := Z^T S_2.$$

Then $\operatorname{Im} R_2^{b^T} = \operatorname{Im} \widetilde{R}_2^{bT}$, i.e., (i) holds, and

$$\operatorname{rank}\left(\widehat{R}_1^b, \widetilde{R}_2^b\right) = \operatorname{rank}\left(Z^T\left(R_1^b, R_2^b\right)\right) = \operatorname{rank}\left(R_1^b, R_2^b\right) = n$$

and

$$\widehat{R}_1^b \widetilde{R}_2^{bT} = Z^T R_1^b R_2^{bT} Z = Z^T R_2^b R_1^{bT} Z = \widetilde{R}_2^b \widehat{R}_1^{bT}.$$

We apply [4, Corollary 3.1.3(ii)] with $R_2 = R_{11}^{(2)}, = R_{22}^{(2)}$ and $S_1 = 0$ in each of the two cases and conclude the existence of the matrices $\widetilde{S}_{11}^{(2)} \in \mathbb{R}^{r\times r}$ and $\widetilde{S}_{22}^{(2)} \in \mathbb{R}^{(n-r)\times(n-r)}$ for which

$$\begin{aligned} \operatorname{Im} \widetilde{S}_{11}^{(2)T} &= \ker R_{11}^{(2)}, \quad \operatorname{rank}\left(\widetilde{S}_{11}^{(2)}, R_{11}^{(2)}\right) = r, \\ \operatorname{Im} \widetilde{S}_{22}^{(2)T} &= \ker R_{22}^{(2)}, \quad \operatorname{rank}\left(\widetilde{S}_{22}^{(2)}, R_{22}^{(2)}\right) = n-r \end{aligned}$$

holds. With $\widetilde{S}_{12}^{(2)} := -\widetilde{S}_{11}^{(2)} C_2^T$ we have

$$R_{22}^{(2)} C_2 \widetilde{S}_{11}^{(2)T} + R_{22}^{(2)} \widetilde{S}_{12}^{(2)T} = 0.$$

Together with

$$\widetilde{S}_2 := \begin{pmatrix} \widetilde{S}_{11}^{(2)} & \widetilde{S}_{12}^{(2)} \\ 0 & \widetilde{S}_{22}^{(2)} \end{pmatrix}$$

we get the result

$$\widetilde{S}_2 \widetilde{R}_2^{bT} = \begin{pmatrix} \widetilde{S}_{11}^{(2)} R_{11}^{(2)T} & \widetilde{S}_{11}^{(2)} C_2^T R_{22}^{(2)^T} + \widetilde{S}_{12}^{(2)} R_{22}^{(2)T} \\ 0 & \widetilde{S}_{22}^{(2)} R_{22}^{(2)T} \end{pmatrix} = 0$$

and

$$\begin{aligned} \operatorname{rank}(\widetilde{S}_2, \widetilde{R}_2^b) &= \operatorname{rank} \begin{pmatrix} \widetilde{S}_{11}^{(2)} & \widetilde{S}_{12}^{(2)} & R_{11}^{(2)} & 0 \\ 0 & \widetilde{S}_{22}^{(2)} & R_{22}^{(2)} C_2 & R_{22}^{(2)} \end{pmatrix} \\ &= \operatorname{rank} \begin{pmatrix} \widetilde{S}_{11}^{(2)} & \widetilde{S}_{12}^{(2)} & R_{11}^{(2)} & 0 \\ 0 & \widetilde{S}_{22}^{(2)} & 0 & R_{22}^{(2)} \end{pmatrix} = n. \end{aligned}$$

We especially retrieve the result

$$n - \operatorname{rank} \widetilde{R}_2^b \geq \operatorname{rank} \widetilde{S}_2 \geq \operatorname{rank}(\widetilde{S}_2, \widetilde{R}_2^b) - \operatorname{rank} \widetilde{R}_2^b = n - \operatorname{rank} \widetilde{R}_2^b,$$

i.e., $\ker \widetilde{R}_2^b = \operatorname{Im} \widetilde{S}_2^T$. We now set $\widetilde{R}_1^b := \widetilde{R}_2^b S_1 + \widetilde{S}_2$, i.e., (iii) holds. Now let $S_{11}^{(1)} \in \mathbb{R}^{r \times r}$, $S_{12}^{(1)} \in \mathbb{R}^{r \times (n-r)}$, $S_{22}^{(1)} \in \mathbb{R}^{(n-r) \times (n-r)}$ such that

$$S_1 = \begin{pmatrix} S_{11}^{(1)} & S_{12}^{(1)} \\ S_{12}^{(1)^T} & S_{22}^{(1)} \end{pmatrix}.$$

Then we have

$$\widetilde{R}_1^b = \begin{pmatrix} R_{11}^{(2)} S_{11}^{(1)} + \widetilde{S}_{11}^{(2)} & R_{11}^{(2)} S_{12}^{(1)} + \widetilde{S}_{12}^{(2)} \\ R_{22}^{(2)} (C_2 S_{11}^{(1)} + \widetilde{S}_{12}^{(2)T}) & R_{22}^{(2)} (C_2 + S_{22}^{(1)}) + \widetilde{S}_{12}^{(2)T} \end{pmatrix} = \begin{pmatrix} R_{11}^{(1)} & R_{12}^{(1)} \\ R_{22}^{(2)} C_1 & R_{22}^{(1)} \end{pmatrix}$$

with some matrices $R_{11}^{(1)} \in \mathbb{R}^{r \times r}$, $R_{12}^{(1)} \in \mathbb{R}^{r \times (n-r)}$, $R_{22}^{(1)} \in \mathbb{R}^{(n-r) \times (n-r)}$, $C_1 \in R^{(n-r) \times r}$. Furthermore, with this setting we have

$$\operatorname{rank}\left(R_{11}^{(1)}, R_{11}^{(2)}\right) = \operatorname{rank}\left(\widetilde{S}_{11}^{(2)}, R_{11}^{(2)}\right) = r$$

and
$$R_{11}^{(1)} R_{11}^{(2)T} = R_{11}^{(2)} S_{11}^{(1)} R_{11}^{(2)T}$$
is symmetric, i.e., (iv) holds. Furthermore, by applying [4, Corollary 3.1.3(i)], we have $\ker(\widetilde{R}_1^b, \widetilde{R}_2^b) = \ker(\widehat{R}_1^b, \widetilde{R}_2^b)$ so that for every $\lambda \in \mathbb{R}$ the equation

$$\begin{aligned} \ker\left(\widetilde{R}_1^b X_a(b;\lambda) + \widetilde{R}_2^b U_a(b;\lambda)\right) &= \ker\left(\widehat{R}_1^b X_a(b;\lambda) + \widetilde{R}_2^b U_a(b;\lambda)\right) \\ = \ker\left(Z^T\left(R_1^b X_a(b;\lambda) + R_2^b U_a(b;\lambda)\right)\right) &= \ker\left(R_1^b X_a(b;\lambda) + R_2^b U_a(b;\lambda)\right) \end{aligned}$$

and hence (iii) holds. ■

Proposition 3.7. Let $R_1^b = R_2^b S_1 + S_2$ with a symmetric matrix S_1 and a matrix S_2 with $\ker R_2^b = \operatorname{Im} S_2^T$. Furthermore, let $R \in \mathbb{R}^{n\times n}$ be a matrix with

$$\operatorname{Im} R = \operatorname{Im} R_2^{bT} \cap \operatorname{Im} X_a(b;\lambda+)$$

and let
$$\widetilde{n}_2(\lambda) := \operatorname{ind}\left\{R^T(S_1 + (U_a X_a^\dagger)(b;\lambda))R\right\} \text{ for } \lambda \in \mathbb{R}.$$
Then $\widetilde{n}_2(\lambda+), \widetilde{n}_2(\lambda-)$ exist for all $\lambda \in \mathbb{R}$ and we have the following equations for $\lambda \in \mathbb{R}\setminus\mathcal{M}$:

$$\widetilde{n}_2(\lambda+) = \widetilde{n}_2(\lambda) + \operatorname{def}\{\Lambda(\lambda)\mathcal{W}\}, \quad \widetilde{n}_2(\lambda-) = \widetilde{n}_2(\lambda).$$

Proof. Proposition 3.4 yields the existence of orthogonal matrices $P, Q \in \mathbb{R}^{n\times n}$ with

$$X_a(b;\lambda) = Q^T\begin{pmatrix} X_{11}(\lambda) & 0 \\ 0 & 0\end{pmatrix}P^T, \quad U_a(b;\lambda) = Q^T\begin{pmatrix} U_{11}(\lambda) & 0 \\ U_{21}(\lambda) & U_{22}\end{pmatrix}P^T$$

with an $r\times r$-matrix-valued function X_{11}, $r := \operatorname{rank} X_a(b;\lambda+)$, which is invertible on $\mathbb{R}\setminus\mathcal{M}$, and the matrix-valued functions $U_{11}(\lambda) \in \mathbb{R}^{r\times r}$, $U_{21}(\lambda) \in \mathbb{R}^{(n-r)\times r}$, $\lambda \in \mathbb{R}$, and an invertible matrix $U_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$. Let

$$\begin{aligned} \underset{\sim}{R}_1^a &:= P^T R_1^a Q^T, & \underset{\sim}{R}_2^a &:= P^T R_2^a Q^T, \\ \underset{\sim}{R}_1^b &:= P^T R_1^b Q^T, & \underset{\sim}{R}_2^b &:= P^T R_2^b Q^T, \end{aligned}$$

$$\underset{\sim}{X}_a(t;\lambda) := Q X_a(t;\lambda) P, \quad \underset{\sim}{U}_a(t;\lambda) := Q U_a(t;\lambda) P,$$

and

$$\underset{\sim}{A}(t) := QA(t)Q^T, \ \underset{\sim}{B}(t) := QB(t)Q^T, \ \underset{\sim}{C}(t) := QC(t)Q^T, \underset{\sim}{C}_0(t) := QC_0(t)Q^T$$

for $t \in \mathcal{I}$, $\lambda \in \mathbb{R}$. Then $(\underset{\sim}{X}_a, \underset{\sim}{U}_a)$ is a matrix-valued solution of the system of differential equations
$$\dot{\underset{\sim}{X}}_a = \underset{\sim}{A}\underset{\sim}{X}_a + \underset{\sim}{B}\underset{\sim}{U}_a, \quad \dot{\underset{\sim}{U}}_a = (\underset{\sim}{C} - \lambda\underset{\sim}{C}_0)\underset{\sim}{X}_a - \underset{\sim}{A}^T\underset{\sim}{U}_a$$

with $\underset{\sim}{X}_a(a;\lambda) \equiv -\underset{\sim}{R}_2^{aT}$, $U_a(a;\lambda) \equiv \underset{\sim}{R}_1^{aT}$. We further define $\underset{\sim}{R} := QR$ and $\underset{\sim}{S}_1 := QS_1Q^T$. Hence we have

$$\operatorname{Im} \underset{\sim}{R} = \operatorname{Im} QR_2^{bT} \cap \operatorname{Im} QX_a(b;\lambda+) = \operatorname{Im} \underset{\sim}{R}_2^{bT} \cap \operatorname{Im} \begin{pmatrix} I_{r\times r} \\ 0 \end{pmatrix}$$

and

$$R^T(S_1 + (U_aX_a^\dagger)(b;\lambda))R = \underset{\sim}{R}^T(\underset{\sim}{S}_1 + (\underset{\sim}{U}_a\underset{\sim}{X}_a^\dagger)(b;\lambda))\underset{\sim}{R}.$$

Thus,

$$\widetilde{n}_2(\lambda) = \operatorname{ind}\{\underset{\sim}{R}^T(\underset{\sim}{S}_1 + (\underset{\sim}{U}_a\underset{\sim}{X}_a^\dagger)(b;\lambda))\underset{\sim}{R}\}.$$

Moreover, with $\underset{\sim}{S}_2 := P^TS_2Q^T$, we have

$$R_1^b = R_2^bS_1 + S_2 \text{ iff } \underset{\sim}{R}_1^b = \underset{\sim}{R}_2^b\underset{\sim}{S}_1 + \underset{\sim}{S}_2$$

and $\ker \underset{\sim}{R}_2^b = \ker R_2^bQ^T = \operatorname{Im} QS_2^T = \operatorname{Im} \underset{\sim}{S}_2^T$. According to Proposition 3.6, there are matrices $R_{11}^{(1)}, R_{11}^{(2)} \in \mathbb{R}^{r\times r}$, $R_{12}^{(1)} \in \mathbb{R}^{r\times(n-r)}$, $R_{22}^{(1)}, R_{22}^{(2)} \in \mathbb{R}^{(n-r)\times(n-r)}$, $C_1, C_2 \in \mathbb{R}^{(n-r)\times r}$ such that with

$$\widetilde{R}_1 := \begin{pmatrix} R_{11}^{(1)} & R_{12}^{(1)} \\ R_{22}^{(2)}C_1 & R_{22}^{(1)} \end{pmatrix}, \quad \widetilde{R}_2 := \begin{pmatrix} R_{11}^{(2)} & 0 \\ R_{22}^{(2)}C_2 & R_{22}^{(2)} \end{pmatrix}$$

the following holds:

(i) $\operatorname{Im} \underset{\sim}{R}_2^{bT} = \operatorname{Im} \widetilde{R}_2^T$.

(ii) $\ker(\widetilde{R}_1\underset{\sim}{X}_a(b;\lambda) + \widetilde{R}_2\underset{\sim}{U}_a(b;\lambda)) = \ker(\underset{\sim}{R}_1^b\underset{\sim}{X}_a(b;\lambda) + \underset{\sim}{R}_2^b\underset{\sim}{X}_a(b;\lambda)) = \ker\Lambda(\lambda)$, where $\Lambda(\lambda) := R_1^bX_a(b;\lambda) + R_2^bU_a(b;\lambda)$.

(iii) There is a matrix $\widetilde{S}_2$ with

$$\widetilde{R}_1 = \widetilde{R}_2\underset{\sim}{S}_1 + \widetilde{S}_2, \ \ker \widetilde{R}_2 = \operatorname{Im} \widetilde{S}_2^T.$$

(iv) $\operatorname{rank}\left(R_{11}^{(1)}, R_{11}^{(2)}\right) = r$ and $R_{11}^{(1)}R_{11}^{(2)T}$ is symmetric.

Then

$$\operatorname{Im} \underset{\sim}{R} = \operatorname{Im} \widetilde{R}_2^T \cap \operatorname{Im} \begin{pmatrix} I_{r\times r} \\ 0 \end{pmatrix} = \operatorname{Im} \begin{pmatrix} R_{11}^{(2)T} & 0 \\ 0 & 0 \end{pmatrix}$$

and hence we can assume without loss of generality that

$$\underset{\sim}{R} := \begin{pmatrix} R_{11}^{(2)T} & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $S_{11} \in \mathbb{R}^{r\times r}$, $S_{12} \in \mathbb{R}^{r\times(n-r)}$, $S_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$ be such that

$$\underset{\sim}{S}_1 = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{pmatrix}.$$

Then we have for all $\lambda \in \mathbb{R} \setminus \mathcal{M}$

$$\underset{\sim}{R}^T(\underset{\sim}{S}_1 + (\underset{\sim}{U}_a \underset{\sim}{X}_a^\dagger)(b;\lambda))\underset{\sim}{R} = \begin{pmatrix} R_{11}^{(1)} R_{11}^{(2)T} + R_{11}^{(2)} U_{11}(\lambda) X_{11}^{-1}(\lambda) R_{11}^{(2)T} & 0 \\ 0 & 0 \end{pmatrix},$$

i.e.,

$$\widetilde{n}_2(\lambda) = \operatorname{ind}\left\{ R_{11}^{(1)} R_{11}^{(2)T} + R_{11}^{(2)} U_{11}(\lambda) X_{11}^{-1}(\lambda) R_{11}^{(2)T} \right\}$$

for all $\lambda \in \mathbb{R} \setminus \mathcal{M}$. We now may apply [4, Theorem 3.4.1] since $\operatorname{rank}\left(R_{11}^{(1)}, R_{11}^{(2)}\right) = \operatorname{rank}(X_{11}, U_{11})(\lambda) = r$, $R_{11}^{(1)} R_{11}^{(2)T}$ is symmetric and $\{X_{11}^T U_{11}\}(\lambda)$ is also symmetric for all $\lambda \in \mathbb{R}$, $X_{11}(\lambda)$ is invertible up to the discrete set $\mathcal{M}$ and because of Proposition 3.5. According to this theorem $\widetilde{n}_2(\lambda+)$, $\widetilde{n}_2(\lambda-)$ and $\operatorname{def} \widetilde{\Lambda}(\lambda+)$ exist and

$$\widetilde{n}_2(\lambda+) - \widetilde{n}_2(\lambda-) = \operatorname{def} \widetilde{\Lambda}(\lambda) - \operatorname{def} \widetilde{\Lambda}(\lambda+) - \operatorname{def} X_{11}(\lambda),$$

where $\widetilde{\Lambda}(\lambda) := R_{11}^{(1)} X_{11}(\lambda) + R_{11}^{(2)} U_{11}(\lambda)$. Furthermore, let

$$M(\lambda) := R_{11}^{(1)} R_{11}^{(2)T} + R_{11}^{(2)} U_{11}(\lambda) X_{11}^{-1}(\lambda) R_{11}^{(2)T} = \widetilde{\Lambda}(\lambda) X_{11}^{-1}(\lambda) R_{11}^{(2)}.$$

Since $M(\lambda)$ is symmetric for all $\lambda \in \mathbb{R}$ and

$$\operatorname{rank}(\widetilde{\Lambda}(\lambda) X_{11}^{-1}(\lambda), R_{11}^{(2)}) = \operatorname{rank}(R_{11}^{(1)}, R_{11}^{(2)}) = r,$$

we may apply [4, Theorem 3.1.2(ii)] which yields

$$\ker M(\lambda) = \ker \widetilde{\Lambda}^T(\lambda) \oplus \ker R_{11}^{(2)T}$$

for all $\lambda \in \mathbb{R} \setminus \mathcal{M}$. Finally, using the continuity and monotonicity of $M(\lambda)$ on every interval where $X_{11}(\cdot)$ is invertible, we obtain

$$\begin{aligned} \widetilde{n}_2(\lambda+) - \widetilde{n}_2(\lambda) &= \operatorname{ind} M(\lambda+) - \operatorname{ind} M(\lambda) = \operatorname{def} M(\lambda) - \operatorname{def} M(\lambda+) \\ &= \operatorname{def} \widetilde{\Lambda}(\lambda) - \operatorname{def} \widetilde{\Lambda}(\lambda+) \\ &= \operatorname{def} \begin{pmatrix} \widetilde{\Lambda}(\lambda) & 0 \\ R_{22}^{(2)}(C_1 X_{11}(\lambda) + C_2 U_{11}(\lambda) + U_{21}(\lambda)) & R_{22}^{(2)} U_{22} \end{pmatrix} \\ &\quad - \operatorname{def} \begin{pmatrix} \widetilde{\Lambda}(\lambda+) & 0 \\ R_{22}^{(2)}(C_1 X_{11}(\lambda+) + C_2 U_{11}(\lambda+) + U_{21}(\lambda+)) & R_{22}^{(2)} U_{22} \end{pmatrix} \\ &= \operatorname{def}\{\widetilde{R}_1 \underset{\sim}{X}_a(b;\lambda) + \widetilde{R}_2 \underset{\sim}{U}_a(b;\lambda)\} - \operatorname{def}\{\widetilde{R}_1 \underset{\sim}{X}_a(b;\lambda+) + \widetilde{R}_2 \underset{\sim}{U}_a(b;\lambda+)\} \\ &= \operatorname{def} \Lambda(\lambda) - \operatorname{def} \Lambda(\lambda+). \end{aligned}$$

Since $\ker \Lambda(\lambda) = W$ up to a discrete set by Propositions 3.2 and 3.3, $\ker \Lambda(\lambda+)$ exists and $\operatorname{def} \Lambda(\lambda+) = \dim W$ holds for all $\lambda \in \mathbb{R}$. By the definition of the matrix $\mathcal{W}$, we have $\mathbb{R}^n = W \oplus \operatorname{Im} \mathcal{W}$ and $\ker \mathcal{W} = \{0\}$. Hence

$$\operatorname{def} \Lambda(\lambda) = \dim W + \operatorname{def} \{\Lambda(\lambda)\mathcal{W}\} \text{ for all } \lambda \in \mathbb{R},$$

i.e.,

$$\widetilde{n}_2(\lambda+) - \widetilde{n}_2(\lambda) = \operatorname{def} \Lambda(\lambda) - \operatorname{def} \Lambda(\lambda+) = \operatorname{def} \{\Lambda(\lambda)\mathcal{W}\}$$

and

$$\widetilde{n}_2(\lambda+) - \widetilde{n}_2(\lambda-) = \operatorname{def} \{\Lambda(\lambda)\mathcal{W}\}$$

for all $\lambda \in \mathbb{R} \setminus \mathcal{M}$. ■

4. Proof of Theorem 1.4

In the previous two sections we have collected all the tools we require to easily retrieve the proof of Theorem 1.4.

Proof of Theorem 1.4. The set $\mathcal{M}$ is discrete according to Proposition 2.3. [5, Theorem 3] implies that $n_1(\lambda)$ is finite for all $\lambda \in \mathbb{R}$. Lemma 2.4 states that

$$n_1(\lambda+) - n_1(\lambda-) = \operatorname{def} \{X_a(b;\lambda)\mathcal{V}\}$$

for all $\lambda \in \mathbb{R}$. In particular, $n_1(\cdot)$ is nonnegative, integer-valued and monotone increasing on $\mathbb{R}$. Hence there is a $\lambda_1 \in \mathbb{R}$ such that $n_1(\cdot)$ is constant on $(-\infty, \lambda_1)$, i.e.,

$$\operatorname{def} \{X_a(b;\lambda)\mathcal{V}\} = 0 \text{ for all } \lambda < \lambda_1.$$

Therefore $n_1 = \lim_{\lambda \to -\infty} n_1(\lambda)$ is well defined. $n_2(\cdot)$ is well defined according to Propositions 3.2, 3.3 and 3.7. Furthermore, these propositions imply that $n_2(\lambda+)$, $n_2(\lambda-)$ exist and the relation

$$n_2(\lambda+) - n_2(\lambda-) = \operatorname{def} \Lambda(\lambda) - \operatorname{def} \Lambda(\lambda+) - \operatorname{def} \{X_a(b;\lambda)\mathcal{V}\}$$

holds for all $\lambda \in \mathbb{R}$. Since $0 \le n_2(\lambda) \le 2n$ by definition, $n_2(\lambda)$ is bounded for all $\lambda \in \mathbb{R}$. Moreover, $n_2(\cdot)$ is integer-valued and monotone increasing on $(-\infty, \lambda_1)$. Hence there is a $\lambda_2 \le \lambda_1$ such that $n_2(\lambda) = n_2$ for all $\lambda < \lambda_2$, and so n_2 is well defined. According to Propositions 3.2 and 3.3, $n_3(\lambda+)$ and $n_3(\lambda-)$ exist and the equation

$$n_3(\lambda+) - n_3(\lambda-) = \operatorname{def} \Lambda(\lambda) - \operatorname{def} \Lambda(\lambda+) \text{ holds for all } \lambda \in \mathbb{R}.$$

Since $n_3(\cdot)$ is an integer-valued function by definition and monotone increasing on $\mathbb{R}$, we conclude that $\lim_{\lambda \to -\infty} n_3(\lambda)$ exists. Moreover, the definition of $n_3(\cdot)$ implies that this limit is zero. Hence we have proven the following equations for all $\lambda \in \mathbb{R}$:

$$\begin{aligned}
n_1(\lambda+) - n_1(\lambda-) &= \operatorname{def} \{X_a(b;\lambda)\mathcal{V}\}, \\
n_2(\lambda+) - n_2(\lambda-) &= \operatorname{def} \Lambda(\lambda) - \operatorname{def} \Lambda(\lambda+) - \operatorname{def} \{X_a(b;\lambda)\mathcal{V}\}, \\
n_3(\lambda+) - n_3(\lambda-) &= \operatorname{def} \Lambda(\lambda) - \operatorname{def} \Lambda(\lambda+).
\end{aligned}$$

Furthermore, Lemma 2.4, Proposition 3.7 and the monotonicity of $n_3(\cdot)$ imply

$$n_\nu(\lambda+) = n_\nu(\lambda) \text{ for all } \lambda \in \mathbb{R} \setminus \mathcal{M} \text{ and all } \nu = 1, 2, 3.$$

The fact that $n_1(\lambda+) + n_2(\lambda+) - n_3(\lambda+)$ is well defined and constant on $\mathbb{R}$ finishes the proof. ■

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