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Extension of Discrete LQR-Problem to Symplectic Systems*

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Abstract

In this paper we consider a discrete linear-quadratic regulator problem in the setting of discrete symplectic systems (S). We derive minimal conditions which guarantee the solvability of this problem. The matrices appearing in these conditions have close connection to the focal point definition of conjoined bases of (S). We show that the optimal solution of this problem has a feedback form and that it is constructed from a generalized discrete Riccati equation. Several examples are provided illustrating this theory. The results of this paper extend the results obtained earlier by the authors for the special case of discrete linear Hamiltonian systems.

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1. Introduction

In this paper we study the discrete linear-quadratic regulator problem consisting of minimizing the functional

$$\mathcal{F}(x,u) := x_{N+1}^T \Gamma\, x_{N+1} + \sum_{k=0}^{N} \left\{ x_k^T \mathcal{C}_k^T \mathcal{A}_k x_k + 2\, x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{D}_k^T \mathcal{B}_k u_k \right\} \to \min, \quad \text{(P)}$$

subject to constraints

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad k \in [0,N], \qquad x_0 = x_0^*. \quad (1.1)$$

We assume that n and N are given positive integers, $[0,N]$ is the discrete interval $\{0,1,\dots,N\}$, $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$, Γ are given real $n \times n$ matrices such that the $2n \times 2n$ matrix

$$\mathcal{S}_k := \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix} \quad \text{is symplectic, i.e.,} \quad \mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J},$$

where $\mathcal{J} := \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ is the $2n \times 2n$ skew-symmetric matrix, and Γ is symmetric.

The quadratic functional $\mathcal{F}$ is associated to the *discrete symplectic system*

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k, \quad k \in [0,N]. \quad \text{(S)}$$

Such systems were introduced in the monograph [1, Chapter 3] and since then attracted full attention of many researchers, see e.g., [2–18, 20, 21, 23]. It was noted in [1] that *discrete linear Hamiltonian systems*

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k, \quad k \in [0,N], \quad \text{(H)}$$

where B_k and C_k are symmetric and $I - A_k$ is invertible, are actually symplectic. That is, the transition matrix

$$\mathcal{S}_k^H := \begin{pmatrix} \tilde{A}_k & \tilde{A}_k B_k \\ C_k \tilde{A}_k & C_k \tilde{A}_k B_k + I - A_k^T \end{pmatrix}, \quad \text{where } \tilde{A}_k := (I - A_k)^{-1},$$

from (x_k, u_k) to (x_{k+1}, u_{k+1}) in system (H) is *symplectic*. In this case, the coefficients in system (S) are then

$$\mathcal{A}_k := \tilde{A}_k, \quad \mathcal{B}_k := \tilde{A}_k B_k, \quad \mathcal{C}_k := C_k \tilde{A}_k, \quad \mathcal{D}_k := C_k \tilde{A}_k B_k + I - A_k^T. \quad (1.2)$$

We can see that in this *Hamiltonian case* the matrix $\mathcal{A}_k$ is *invertible* for all $k \in [0,N]$. On the other hand, it is also known that every symplectic system (S) with invertible $\mathcal{A}_k$ is in fact Hamiltonian, and in that case one has

$$A_k := I - \mathcal{A}_k^{-1}, \quad B_k := \mathcal{A}_k^{-1} \mathcal{B}_k, \quad C_k := \mathcal{C}_k \mathcal{A}_k^{-1}$$

with symmetric B_k and C_k.

In [19] the authors studied the discrete linear regulator problem associated with the linear Hamiltonian system (H), derived minimal assumptions which guarantee its solvability, and constructed the optimal (feedback) solution from a certain generalized discrete Riccati equation. The purpose of this paper is to extend these results to general discrete symplectic systems. This will allow the inclusion of systems for which the evolution matrix $\Phi_{k+1,j} := \mathcal{A}_k\mathcal{A}_{k-1}\dots\mathcal{A}_j$ could be *singular*, such is the case in discrete trigonometric or self-reciprocal systems that are studied e.g., in [2, 5]. We refer to the literature listed in [19] for the traditional treatment of the discrete linear-quadratic regulator problem.

In Section 2 we establish minimal assumptions for the solvability of the problem (P). They consist of two conditions, namely the minimality condition (or the so-called "$\mathcal{P}$-condition")

$$\mathcal{P}_k \geq 0 \quad \text{for all} \quad k \in [0, N], \tag{1.3}$$

where

$$\mathcal{P}_k := (\mathcal{D}_k^T - \mathcal{B}_k^T Q_{k+1})\,\mathcal{B}_k, \tag{1.4}$$

and the solvability condition

$$(I - \mathcal{P}_k\mathcal{P}_k^\dagger)\,\mathcal{B}_k^T(Q_{k+1}\mathcal{A}_k - \mathcal{C}_k) = 0 \quad \text{for all} \quad k \in [0, N], \tag{1.5}$$

in which $\mathcal{P}_k^\dagger$ denotes the Moore–Penrose generalized inverse of the indicated matrix. Conditions (1.4) and (1.5) involve a symmetric matrix Q_{k+1} which is constructed from a certain discrete Riccati matrix equation with an endpoint condition at $k = N+1$. The matrices Q_k are also (indirectly) used for the construction of the optimal feedback solution to the problem (P). In Section 3 we compare the obtained Riccati equation with a discrete Riccati equation studied in the literature in connection with the discrete symplectic system (S). Finally, In Section 4 we present several examples illustrating the applicability of the results.

2. Discrete Symplectic LQR-Problem

The property that $\mathcal{S}_k$ (and $\mathcal{S}_k^T$) is a symplectic matrix means that the coefficients satisfy

$$\begin{gathered}\mathcal{A}_k^T\mathcal{D}_k - \mathcal{C}_k^T\mathcal{B}_k = \mathcal{A}_k\mathcal{D}_k^T - \mathcal{B}_k\mathcal{C}_k^T = I,\\ \mathcal{C}_k^T\mathcal{A}_k,\ \mathcal{D}_k^T\mathcal{B}_k,\ \mathcal{A}_k\mathcal{B}_k^T,\ \mathcal{C}_k\mathcal{D}_k^T \quad \text{are symmetric.}\end{gathered} \tag{2.1}$$

These identities will be frequently used in our calculations.

For $x \in \mathbb{R}^n$ and $k \in [0, N+1]$ we define the *value function* $V(x, k)$ by the following. We set $V(x, N+1) := x^T\Gamma x$ and for $k \in [0, N]$

$$V(x,k) := \min_{u\in\mathbb{R}^n}\left\{x^T\mathcal{C}_k^T\mathcal{A}_kx + 2\,x^T\mathcal{C}_k^T\mathcal{B}_ku + u^T\mathcal{D}_k^T\mathcal{B}_ku + V(\mathcal{A}_kx + \mathcal{B}_ku, k+1)\right\}, \tag{2.2}$$

provided the minimum exists. If we denote $x_{k+1}(u) := \mathcal{A}_k x_k + \mathcal{B}_k u$, then the value function at (x_k, k) is

$$V(x_k, k) = \min_{u \in \mathbb{R}^n} \left\{ x^T \mathcal{C}_k^T \mathcal{A}_k x + 2\, x^T \mathcal{C}_k^T \mathcal{B}_k u + u^T \mathcal{D}_k^T \mathcal{B}_k u + V\big(x_{k+1}(u), k+1\big) \right\}.$$

The Bellman principle of dynamic programming says that if (x^*, u^*) is a pair satisfying (1.1), then it is optimal for problem (P) if and only if the minimum in (2.2) for $V(x_k^*, k)$ is attained at u_k^* for all $k \in [0, N]$.

The results of this paper are based on the following recursive matrix definitions. Put $Q_{N+1} := -\Gamma$ and then define for $k \in [0, N]$ the matrix $\mathcal{P}_k$ by (1.4) and matrices F_k and Q_k by

$$F_k := \mathcal{P}_k^{\dagger} \mathcal{B}_k^T (Q_{k+1} \mathcal{A}_k - \mathcal{C}_k), \tag{2.3}$$

$$Q_k := F_k^T \mathcal{P}_k F_k + \mathcal{A}_k^T (Q_{k+1} \mathcal{A}_k - \mathcal{C}_k). \tag{2.4}$$

These definitions are recursive in the sense that from Q_{N+1} we define $\mathcal{P}_N$, then F_N, then Q_N, and then back to $\mathcal{P}_{N-1}$, F_{N-1}, Q_{N-1}, and so on. Note that all the matrices $\mathcal{P}_k$, F_k, and Q_k are well defined (once $Q_{N+1} := -\Gamma$ is given) and that $\mathcal{P}_k$ and Q_k are symmetric.

Remark 2.1.

(i) Solvability condition (1.5) can now be rewritten as the identity

$$\mathcal{P}_k F_k = \mathcal{B}_k^T (Q_{k+1} \mathcal{A}_k - \mathcal{C}_k) \quad \text{for all} \quad k \in [0, N]. \tag{2.5}$$

(ii) Equation (2.4) is a *discrete Riccati equation*. Under solvability condition (1.5) it takes the form

$$Q_k = (F_k^T \mathcal{B}_k^T + \mathcal{A}_k^T)(Q_{k+1} \mathcal{A}_k - \mathcal{C}_k). \tag{2.6}$$

For its relation to another discrete Riccati equation studied in the literature in connection with discrete symplectic systems see Section 3.

One of the main results of this paper is the following characterization of conditions (1.3) and (1.5).

Theorem 2.2. Let $\mathcal{P}_k$, F_k, and Q_k be defined by (1.4), (2.3), and (2.4), respectively, with $Q_{N+1} := -\Gamma$. Then the following statements are equivalent.

(i) Minimality condition (1.3) and solvability condition (1.5) are satisfied.

(ii) For all $k \in [0, N]$ and for all $x \in \mathbb{R}^n$, the minimum in (2.2) is attained at

$$\bar{u}_k(x) = F_k x + (I - \mathcal{P}_k^{\dagger} \mathcal{P}_k)\, \gamma_k \tag{2.7}$$

for some vector $\gamma_k \in \mathbb{R}^n$, and hence the value function takes the form

$$V(x, k) = -x^T Q_k x. \tag{2.8}$$

Proof. "(i) $\Rightarrow$ (ii):" Formula (2.8) holds at $k = N + 1$ by the definition of the matrix Q_{N+1}. Fix now $k \in [0, N]$ and assume that (2.8) is satisfied at an index $k + 1$ with a symmetric matrix Q_{k+1}. Then (2.2) yields

$$V(x,k) = \min_{u \in \mathbb{R}^n} \left\{ x^T \mathcal{C}_k^T \mathcal{A}_k x + 2\, x^T \mathcal{C}_k^T \mathcal{B}_k u + u^T \mathcal{D}_k^T \mathcal{B}_k u \right. \tag{2.9}$$

$$\left. -(\mathcal{A}_k x + \mathcal{B}_k u)^T Q_{k+1} (\mathcal{A}_k x + \mathcal{B}_k u) \right\}$$

$$= \min_{u \in \mathbb{R}^n} \left\{ M_k(u) \right\} + x^T \mathcal{A}_k^T (\mathcal{C}_k - Q_{k+1} \mathcal{A}_k)\, x, \tag{2.10}$$

where we set

$$M_k(u) := u^T \mathcal{P}_k u + 2\, x^T (\mathcal{C}_k^T - \mathcal{A}_k^T Q_{k+1})\, \mathcal{B}_k u.$$

Since $\nabla^2 M_k(u) = \mathcal{P}_k \geq 0$ is assumed, the minimum in (2.10) is attained at some $\bar{u} \in \mathbb{R}^n$ whenever $\nabla M_k(\bar{u}) = 0$, i.e., whenever

$$2\, \mathcal{P}_k \bar{u} + 2\, \mathcal{B}_k^T (\mathcal{C}_k - Q_{k+1} \mathcal{A}_k)\, x = 0.$$

This means that $\bar{u}$ must solve the linear equation

$$\mathcal{P}_k \bar{u} = \mathcal{B}_k^T (Q_{k+1} \mathcal{A}_k - \mathcal{C}_k)\, x. \tag{2.11}$$

This equation has a solution if and only if its right-hand side lies in $\operatorname{Im} \mathcal{P}_k = \operatorname{Ker}(I - \mathcal{P}_k \mathcal{P}_k^\dagger)$, which is guaranteed by assuming (1.5), and in this case the solution $\bar{u} = \bar{u}(x)$ has the form

$$\bar{u} = \mathcal{P}_k^\dagger \mathcal{B}_k^T (Q_{k+1} \mathcal{A}_k - \mathcal{C}_k)\, x + (I - \mathcal{P}_k^\dagger \mathcal{P}_k)\, \gamma_k = F_k x + (I - \mathcal{P}_k^\dagger \mathcal{P}_k)\, \gamma_k$$

for some vector $\gamma_k \in \mathbb{R}^n$. This yields that $\mathcal{P}_k \bar{u} = \mathcal{P}_k F_k x$ and that

$$V(x,k) = M_k(\bar{u}) + x^T \mathcal{A}_k^T (\mathcal{C}_k^T - Q_{k+1} \mathcal{A}_k)\, x$$

$$= -x^T \left\{ F_k^T \mathcal{P}_k F_k + \mathcal{A}_k^T (Q_{k+1} \mathcal{A}_k - \mathcal{C}_k) \right\} x = -x^T Q_k x.$$

"(ii) $\Rightarrow$ (i):" Fix $k \in [0, N]$ and $x \in \mathbb{R}^n$. Let $\bar{u} = \bar{u}(x)$, defined by (2.7), be a vector for which the minimum of $M_k(u)$ is attained and formula (2.8) holds. Since $M_k(u)$ is quadratic, it follows that $\mathcal{P}_k = \nabla^2 M_k(\bar{u}) \geq 0$, i.e., minimality condition (1.3) holds, and $\nabla M_k(\bar{u}) = 0$, i.e., $\bar{u}$ solves equation (2.11). Hence,

$$(I - \mathcal{P}_k \mathcal{P}_k^\dagger)\, \mathcal{B}_k^T (Q_{k+1} \mathcal{A}_k - \mathcal{C}_k)\, x = 0.$$

Since $x \in \mathbb{R}^n$ was arbitrary, it follows that solvability condition (1.5) is satisfied as well. The proof is now complete. ■

The following characterization of optimal processes for the linear regulator problem (P) is an immediate consequence of Theorem 2.2 and Bellman's principle of dynamic programming.

Corollary 2.3. Assume that minimality condition (1.3) and solvability condition (1.5) are satisfied. Then the linear regulator problem (P) has an optimal solution (x^*, u^*) if and only if for all $k \in [0, N]$

$$u_k^* = F_k x_k^* + (I - \mathcal{P}_k^\dagger \mathcal{P}_k)\,\gamma_k \tag{2.12}$$

for some vectors $\gamma_k \in \mathbb{R}^n$, and hence $V(x_k^*, k) = -(x_k^*)^T Q_k x_k^*$.

Next we wish to analyze the form of the feedback law (2.12). For this we need one auxiliary calculation.

Lemma 2.4. Assume that solvability condition (1.5) holds. Then

$$\mathcal{B}_k F_k = \mathcal{B}_k Q_k \quad \text{for all} \quad k \in [0, N]. \tag{2.13}$$

Proof. By using Remark 2.1 (both parts), identities (2.1), and the definition of $\mathcal{P}_k$ we have

$$\begin{aligned} Q_k \mathcal{B}_k^T &= (F_k^T \mathcal{B}_k^T + \mathcal{A}_k^T)\,(Q_{k+1}\mathcal{A}_k - \mathcal{C}_k)\,\mathcal{B}_k^T \\ &= F_k^T \mathcal{B}_k^T (Q_{k+1}\mathcal{A}_k\mathcal{B}_k^T - \mathcal{C}_k\mathcal{B}_k^T) + \mathcal{A}_k^T Q_{k+1}\mathcal{A}_k\mathcal{B}_k^T - \mathcal{A}_k^T \mathcal{C}_k \mathcal{B}_k^T \\ &= F_k^T \mathcal{B}_k^T \big\{(Q_{k+1}\mathcal{B}_k - \mathcal{D}_k)\,\mathcal{A}_k^T + I\big\} + (\mathcal{A}_k^T Q_{k+1} - \mathcal{C}_k^T)\,\mathcal{B}_k \mathcal{A}_k^T \\ &= -F_k^T \mathcal{P}_k \mathcal{A}_k^T + F_k^T \mathcal{B}_k^T + F_k^T \mathcal{P}_k \mathcal{A}_k^T = F_k^T \mathcal{B}_k^T. \end{aligned}$$

This shows the desired identity. ■

Remark 2.5. Formula (2.13) implies that $\mathcal{B}_k F_k \mathcal{B}_k^T = \mathcal{B}_k Q_k \mathcal{B}_k^T$, which means that, under (1.5), the matrix F_k is symmetric on $\operatorname{Im} \mathcal{B}_k^T$. Furthermore, Riccati equation (2.6) then takes the form

$$Q_k = (\mathcal{A}_k + \mathcal{B}_k Q_k)^T (Q_{k+1}\mathcal{A}_k - \mathcal{C}_k). \tag{2.14}$$

As a consequence of Lemma 2.4 we get the "classical" results in the sense that the control law contains the Riccati equation solution Q_k. The following result is new even for the case of Hamiltonian systems.

Corollary 2.6. Assume that solvability condition (1.5) holds and $\mathcal{B}_k$ is invertible for all $k \in [0, N]$. Then $F_k = Q_k$ is symmetric and hence, the control law in (2.12) reduces to

$$u_k^* = Q_k x_k^* + (I - \mathcal{P}_k^\dagger \mathcal{P}_k)\,\gamma_k \quad \text{for all} \quad k \in [0, N]. \tag{2.15}$$

Corollary 2.7. Assume that $\mathcal{P}_k$ is invertible for all $k \in [0, N]$. Then $F_k = Q_k$ is symmetric and hence, the control law is unique and reduces to

$$u_k^* = Q_k x_k^* \quad \text{for all} \quad k \in [0, N].$$

Proof. The invertibility of $\mathcal{P}_k$ implies that $\mathcal{B}_k$ is invertible and that solvability condition (1.5) is satisfied trivially. Hence, the result follows from Corollary 2.6. ■

The following characterization of the optimality of a process (x^*, u^*) in problem (P) is a consequence of the proof of Theorem 2.2.

Corollary 2.8. A pair (x^*, u^*) is optimal for the problem (P) if and only if minimality condition (1.3) holds and

$$(I - \mathcal{P}_k \mathcal{P}_k^{\dagger})\, \mathcal{B}_k^T (Q_{k+1} \mathcal{A}_k - \mathcal{C}_k)\, x_k^* = 0 \quad \text{for all} \quad k \in [0, N]. \tag{2.16}$$

Note that the above condition involves the "state" x_k^* only. The optimal "control" u_k^* is then given via formula (2.12).

Remark 2.9.

(i) In the Hamiltonian case, i.e., when the coefficients $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, and $\mathcal{D}_k$ are given by (1.2), the results of this paper reduce to the corresponding results in [19], with the exception of Corollary 2.6 which is new even in the Hamiltonian setting.

(ii) It is interesting to note that the matrix $\mathcal{P}_k$ was previously used in the focal point definition for conjoined bases (X, U) of system (S). More precisely, a solution (X, U) of (S) with $X_k^T U_k$ symmetric and rank $\begin{pmatrix} X_k^T & U_k^T \end{pmatrix} = n$ has *no focal point* in the interval $(k, k+1]$ provided

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k \quad \text{and} \quad X_k X_{k+1}^{\dagger} \mathcal{B}_k \geq 0,$$

see e.g., [4, 18]. Then it is shown in these references that, under the above kernel condition,

$$X_k X_{k+1}^{\dagger} \mathcal{B}_k = (\mathcal{D}_k^T - \mathcal{B}_k^T Q_{k+1})\, \mathcal{B}_k = \mathcal{P}_k,$$

for a suitable choice of symmetric Q_{k+1}.

(iii) Another appearance of the matrix $\mathcal{P}_k$ is in the discrete *Picone identity*, which shows how to complete the quadratic term in the functional $\mathcal{F}$ to a "square", i.e.,

$$x_k^T \mathcal{C}_k^T \mathcal{A}_k x_k + 2\, x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{D}_k^T \mathcal{B}_k u_k = z_k^T \mathcal{P}_k z_k + \Delta(x_k^T Q_k x_k)$$

for a suitable choice of symmetric Q_k and for $z_k := u_k - Q_k x_k$, see e.g., [4, 8].

3. Discrete Riccati Equations

The discrete Riccati matrix equation traditionally associated with the symplectic system (S) and studied in the literature is of the form

$$Q_{k+1}(\mathcal{A}_k + \mathcal{B}_k Q_k) - (\mathcal{C}_k + \mathcal{D}_k Q_k) = 0, \quad k \in [0, N]. \tag{3.1}$$

See for example [4, 14, 16–18]. In this section we wish to compare discrete Riccati equation (3.1) with equations (2.4), (2.6), and (2.14).

One reason for studying equation (3.1) is that its solutions generate “good” solutions of the system (S), namely the following holds.

Proposition 3.1. Riccati equation (3.1) has a solution Q_k on $[0, N+1]$ if and only if system (S) has a solution (X, U) with X_k invertible for all $k \in [0, N+1]$.

The statement of Proposition 3.1 usually contains the invertibility of the matrix $\mathcal{A}_k + \mathcal{B}_k Q_k$ (and the symmetry of Q_k and $X_k^T U_k$), see e.g., [18, Theorem 7] in the case of zero endpoints. However this invertibility condition is a simple consequence of equation (3.1), as we shall see in Lemma 3.2 below, so that Proposition 3.1 is now in the same form as in the continuous time case, see e.g., [22, Theorem 7.1]. Note that the results below do not require the symmetry of Q_k (unless explicitly stated).

Lemma 3.2. Matrices Q_k, $k \in [0, N+1]$, solve

$$\mathcal{B}_k^T \left\{ Q_{k+1}(\mathcal{A}_k + \mathcal{B}_k Q_k) - (\mathcal{C}_k + \mathcal{D}_k Q_k) \right\} = 0 \quad \text{for all} \quad k \in [0, N] \tag{3.2}$$

if and only if $\mathcal{D}_k^T - \mathcal{B}_k^T Q_{k+1}$ and $\mathcal{A}_k + \mathcal{B}_k Q_k$ are invertible and are inverses of each other, that is,

$$(\mathcal{D}_k^T - \mathcal{B}_k^T Q_{k+1})\,(\mathcal{A}_k + \mathcal{B}_k Q_k) = I \quad \text{for all} \quad k \in [0, N]. \tag{3.3}$$

Consequently, if condition (3.3) holds and $\mathcal{B}_k$ is invertible for all $k \in [0, N]$, then Q_k solves Riccati equation (3.1).

Proof. Assume that some matrices Q_k satisfy (3.2). Then using identities (2.1) we obtain

$$\begin{aligned}(\mathcal{D}_k^T - \mathcal{B}_k^T Q_{k+1})\,(\mathcal{A}_k + \mathcal{B}_k Q_k) &= \mathcal{D}_k^T (\mathcal{A}_k + \mathcal{B}_k Q_k) - \mathcal{B}_k^T (\mathcal{C}_k + \mathcal{D}_k Q_k) \\ &= \mathcal{D}_k^T \mathcal{A}_k - \mathcal{B}_k^T \mathcal{C}_k + (\mathcal{D}_k^T \mathcal{B}_k - \mathcal{B}_k^T \mathcal{D}_k)\, Q_k = I.\end{aligned}$$

This yields that (3.3) is satisfied. Following the backstep calculation, the converse of this lemma also holds. ■

The next two lemmas show the relation between Riccati equations (2.14) and (3.1) for the case of symmetric Q_k.

Lemma 3.3. Assume that symmetric matrices Q_k, $k \in [0, N+1]$, solve Riccati equation (3.1). Then they solve Riccati equation (2.14).

Proof. From equation (3.1) we have the identity

$$Q_{k+1}\mathcal{A}_k - \mathcal{C}_k = (\mathcal{D}_k - Q_{k+1}\mathcal{B}_k)\, Q_k.$$

Then by using Lemma 3.2 and the symmetry of Q_{k+1} we get

$$Q_k = (\mathcal{D}_k - Q_{k+1}\mathcal{B}_k)^{-1}(Q_{k+1}\mathcal{A}_k - \mathcal{C}_k) = (\mathcal{A}_k + \mathcal{B}_k Q_k)^T (Q_{k+1}\mathcal{A}_k - \mathcal{C}_k),$$

i.e., Q_k solves equation (2.14). ■

Lemma 3.4. Assume that matrices Q_k, $k \in [0, N+1]$, satisfy Riccati equation (2.14). Then

$$\big\{(\mathcal{A}_k + \mathcal{B}_k Q_k)^T (\mathcal{D}_k - Q_{k+1}\mathcal{B}_k) - I\big\}\mathcal{A}_k^T = 0 \quad \text{for all} \quad k \in [0, N]. \tag{3.4}$$

Consequently, if a symmetric Q_k solves Riccati equation (2.14) and $\mathcal{A}_k$ and $\mathcal{B}_k$ are invertible for all $k \in [0, N]$, then Q_k solves Riccati equation (3.1).

Proof. If we multiply Riccati equation (2.14) by $\mathcal{B}_k^T$ from the right side, add $\mathcal{A}_k^T$ on both sides, and use properties (2.1), then we get

$$\begin{aligned} \mathcal{A}_k^T + Q_k\mathcal{B}_k^T &= \mathcal{A}_k^T + (\mathcal{A}_k + \mathcal{B}_k Q_k)^T (Q_{k+1}\mathcal{A}_k\mathcal{B}_k^T - \mathcal{C}_k\mathcal{B}_k^T) \\ &= \mathcal{A}_k^T + (\mathcal{A}_k + \mathcal{B}_k Q_k)^T (Q_{k+1}\mathcal{B}_k\mathcal{A}_k^T - \mathcal{D}_k\mathcal{A}_k^T + I). \end{aligned}$$

Hence, we obtain the identity

$$0 = \mathcal{A}_k^T + (\mathcal{A}_k + \mathcal{B}_k Q_k)^T (Q_{k+1}\mathcal{B}_k - \mathcal{D}_k)\, \mathcal{A}_k^T,$$

which yields equation (3.4). Now if $\mathcal{A}_k$ is invertible and Q_k is symmetric, then condition (3.3) holds, which together with the assumption of the invertibility of $\mathcal{B}_k$ implies the validity of Riccati equation (3.1), by Lemma 3.2. ■

4. Examples

In this section we present examples illustrating the applicability of the results.

Example 4.1. Let $n = 1$ and $\mathcal{S}_k \equiv \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}$, i.e., $\mathcal{A}_k = -\mathcal{B}_k \equiv 1$, $\mathcal{C}_k \equiv 4$, $\mathcal{D}_k \equiv -3$, and $\Gamma = -2$. Then $\mathcal{A}_k\mathcal{D}_k - \mathcal{B}_k\mathcal{C}_k \equiv 1$, so that these coefficients define a discrete symplectic system. It follows that $\mathcal{P}_k = 3 - Q_{k+1}$, $F_k = (Q_{k+1} - 4)/(Q_{k+1} - 3)$, and $Q_k = F_k$. Hence, given that $Q_{N+1} := -\Gamma = 2$ we get

$$\mathcal{P}_k \equiv 1 > 0, \quad Q_k = F_k \equiv 2 \quad \text{for all} \quad k \in [0, N]$$

and solvability condition (1.5) holds, since $\mathcal{P}_k$ is invertible. Hence, by Corollary 2.7, the optimal feedback law takes the form $u_k^* = 2\,x_k^*$ and $V(x_k^*, k) = -2\,(x_k^*)^2$.

In the previous example the matrix $\mathcal{A}_k$ is *invertible*, so that the corresponding discrete symplectic system can be written as a linear Hamiltonian system (H) and the optimal solution of the associated problem (P) can be obtained from [19]. Hence, it is more appropriate to give examples with $\mathcal{A}_k$ *singular* to which the results of [19] cannot be applied.

First note that if $\mathcal{A}_k = 0$, then coefficient identities (2.1) imply that $\mathcal{B}_k$ and $\mathcal{C}_k$ are invertible with $\mathcal{C}_k = -\mathcal{B}_k^{T-1}$, and $\mathcal{D}_k^T\mathcal{B}_k = \mathcal{B}_k^T\mathcal{D}_k$. Consequently, solvability condition (1.5) yields that $I - \mathcal{P}_k\mathcal{P}_k^\dagger = 0$, i.e., the matrix $\mathcal{P}_k$ is necessarily *invertible*, and thus

$Q_k = F_k = \mathcal{P}_k^{-1}$. On the other hand, the invertibility of $\mathcal{P}_k$ always implies the solvability condition. Hence, when $\mathcal{A}_k = 0$, the solvability condition and the invertibility of the matrices $\mathcal{P}_k$ are equivalent.

Example 4.2. Let $\mathcal{A}_k \equiv 0$, $\mathcal{B}_k = -\mathcal{C}_k \equiv I$, $\mathcal{D}_k \equiv 2I$, and $\Gamma = -I$. Then these coefficients define a discrete symplectic system. It follows that $\mathcal{P}_k = 2I - Q_{k+1}$, $F_k = \mathcal{P}_k^{-1}$, and $Q_k = F_k$. We know from the above considerations that the invertibility of $\mathcal{P}_k$ is a necessary condition for solvability condition (1.5). Hence, given that $Q_{N+1} := -\Gamma = I$, we get

$$\mathcal{P}_k \equiv I > 0, \quad Q_k = F_k \equiv I \quad \text{for all} \quad k \in [0, N].$$

By Corollary 2.7, the optimal feedback law yields $u_k^* = x_k^*$ and $V(x_k^*, k) = -\|x_k^*\|^2$.

From the above, one can see that the most interesting examples arise when $\mathcal{A}_k$ is *singular* but never a zero matrix, $\mathcal{P}_k$ is positive semidefinite but never positive definite, and solvability condition (1.5) holds. In the following example we show that such situation can indeed be treated and solved by the results of this paper.

Example 4.3. Let $n = 2$ and

$$\mathcal{A}_k \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B}_k \equiv \begin{pmatrix} 0 & -bd \\ 0 & b \end{pmatrix}, \quad \mathcal{C}_k \equiv \begin{pmatrix} 0 & 0 \\ -dq & -\frac{1}{b} \end{pmatrix},$$

$$\mathcal{D}_k \equiv \begin{pmatrix} 1 & -bdq \\ d & b + \frac{1}{b} \end{pmatrix}, \quad \Gamma = \begin{pmatrix} -q & 0 \\ 0 & -1 \end{pmatrix},$$

where $b, d, q \in \mathbb{R}$ are given real numbers and $b \neq 0$. It follows that these coefficients define a discrete symplectic system and with $Q_{N+1} := -\Gamma$ we have

$$\mathcal{P}_k = \mathcal{D}_k^T \mathcal{B}_k - \mathcal{B}_k^T Q_{k+1} \mathcal{B}_k \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{P}_k^\dagger = \mathcal{P}_k,$$

$$F_k = \mathcal{P}_k^\dagger \mathcal{B}_k^T (Q_{k+1} \mathcal{A}_k - \mathcal{C}_k) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$Q_k = F_k^T \mathcal{P}_k F_k + \mathcal{A}_k^T (Q_{k+1} \mathcal{A}_k - \mathcal{C}_k) = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix},$$

and solvability condition (1.5) is

$$(I - \mathcal{P}_k \mathcal{P}_k^\dagger)\, \mathcal{B}_k^T (Q_{k+1} \mathcal{A}_k - \mathcal{C}_k) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -bd & b \end{pmatrix} \begin{pmatrix} q & 0 \\ dq & \frac{1}{b} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, by Corollary 2.3, the optimal feedback law takes the form

$$u_k^* = F_k x_k^* + (I - \mathcal{P}_k^\dagger \mathcal{P}_k)\, \gamma_k = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x_k^* + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \gamma_k$$

for some vectors $\gamma_k \in \mathbb{R}^2$, and

$$V(x_k^*, k) = -(x_k^*)^T \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} x_k^*.$$

Note that even though in Example 4.2 the matrix $\mathcal{A}_k$ is singular and in Example 4.3 both $\mathcal{A}_k$ and $\mathcal{B}_k$ are singular, the corresponding matrices Q_k solve both Riccati equations (2.14) and (3.1), and condition (3.3) holds.

References

[1] C.D. Ahlbrandt and A.C. Peterson, Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations, Kluwer Academic Publishers, Boston, 1996.

[2] D.R. Anderson, Discrete trigonometric matrix functions, *PanAmer. Math. J.*, 7(1):39–54, 1997.

[3] D.R. Anderson, Normalized prepared bases for discrete symplectic matrix systems, In: "Discrete and Continuous Hamiltonian Systems", R. P. Agarwal and M. Bohner, editors, *Dynam. Systems Appl.*, 8(3–4):335–344, 1999.

[4] M. Bohner and O. Došlý, Disconjugacy and transformations for symplectic systems, *Rocky Mountain J. Math.*, 27(3):707–743, 1997.

[5] M. Bohner and O. Došlý, Trigonometric transformation of symplectic difference systems, *J. Differential Equations*, 163(1):113–129, 2000.

[6] M. Bohner and O. Došlý, The discrete Prüfer transformation, *Proc. Amer. Math. Soc.*, 129(9):2715–2726, 2001.

[7] M. Bohner, O. Došlý, R. Hilscher, and W. Kratz, Diagonalization approach to discrete quadratic functionals, *Arch. Inequal. Appl.*, 1(2):261–274, 2003.

[8] M. Bohner, O. Došlý, and W. Kratz, Positive semidefiniteness of discrete quadratic functionals, *Proc. Edinburgh Math. Soc.*, 46:627–636, 2003.

[9] M. Bohner, O. Došlý, and W. Kratz, An oscillation theorem for discrete eigenvalue problems, *Rocky Mountain J. Math.*, 33(4):1233–1260, 2003.

[10] Z. Došlá and D. Škrabáková, Phases of second order linear difference equations and symplectic systems, *Math. Bohem.*, 128(3):293–308, 2003.

[11] O. Došlý, Discrete quadratic functionals and symplectic difference systems, *Funct. Differ. Equ.*, 11(1–2):49–58, 2004.

[12] O. Došlý, R. Hilscher, and V. Zeidan, Nonnegativity of discrete quadratic functionals corresponding to symplectic difference systems, *Linear Algebra Appl.*, 375:21–44, 2003.

[13] O. Došlý and W. Kratz, A Sturmian separation theorem for symplectic difference systems, *J. Math. Anal. Appl.*, 325(1):333–341, 2007.

[14] Y.V. Eliseeva, An algorithm for solving the matrix difference Riccati equation, *Comput. Math. Math. Phys.*, 39 (1999), no. 2, 177–184. Translated from *Zh. Vychisl. Mat. Mat. Fiz.*, 39(2):187–194, 1999.

[15] R. Hilscher, Disconjugacy of symplectic systems and positivity of block tridiagonal matrices, *Rocky Mountain J. Math.*, 29(4):1301–1319, 1999.

[16] R. Hilscher and V. Růžičková, Riccati inequality and other results for discrete symplectic systems, *J. Math. Anal. Appl.*, 322(2):1083–1098, 2006.

[17] R. Hilscher and V. Růžičková, Implicit Riccati equations and quadratic functionals for discrete symplectic systems, *Internat. J. Difference Equ.*, 1(1):135–154, 2006.

[18] R. Hilscher and V. Zeidan, Symplectic difference systems: variable stepsize discretization and discrete quadratic functionals, *Linear Algebra Appl.*, 367:67–104, 2003.

[19] R. Hilscher and V. Zeidan, Solvability of the discrete LQR-problem under minimal assumptions, In: "Difference Equations and Discrete Dynamical Systems", Proceedings of the Ninth International Conference on Difference Equations and Applications (Los Angeles, 2004), L. Allen, B. Aulbach, S. Elaydi, and R. Sacker, editors, pp. 273–282, World Scientific Publishing Co., London, 2005.

[20] R. Hilscher and V. Zeidan, Coupled intervals for discrete symplectic systems, *Linear Algebra Appl.*, 419(2–3):750–764, 2006.

[21] W. Kratz, Discrete oscillation, *J. Difference Equ. Appl.*, 9(1):135–147, 2003.

[22] W.T. Reid, Riccati Differential Equations, Academic Press, New York – London, 1972.

[23] V. Růžičková, Discrete Symplectic Systems and Definiteness of Quadratic Functionals, PhD dissertation. Masaryk University, Brno, 2006.