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On the Existence of Almost Periodic Solutions of a Nonlinear Volterra Difference Equation

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Abstract

The existence of almost periodic solutions of a nonlinear Volterra difference equation with infinite delay is obtained by using some restrictive conditions of the equation.

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1. Introduction

For ordinary differential equations and integro-differential equations, the existence of almost periodic solutions of almost periodic systems has been studied by many authors [2, 3, 5, 6]. Hamaya [3] has investigated the existence of almost periodic solutions of functional difference equations with infinite delay by using stability properties of bounded solutions. Y. Song, C.T.H. Baker and H. Tian [5] have studied the existence of periodic and almost periodic solutions for nonlinear Volterra difference equations by means of certain stability conditions. However, it is not easy to know whether or not there exists bounded solutions for nonlinear Volterra difference equations. In this paper, we discuss the existence of almost periodic solutions of nonlinear Volterra difference equations motivated by C. Feng [1] and S. Kato and M. Imai [4]. Our restrictive conditions to guarantee the existence of almost periodic solutions are weaker than those of Hamaya [3], and Y. Song, C.T.H. Baker and H. Tian [5], and moreover, this article is to extend results in [1] to discrete nonlinear Volterra difference equations.

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We first introduce an almost periodic function

$$f(n,x) : \mathbb{Z} \times D \to \mathbb{R}^l,$$

where D is an open set in $\mathbb{R}^l$, and $f(n,\cdot)$ is continuous for each $n \in \mathbb{Z}$.

Definition 1.1. $f(n,x)$ is said to be almost periodic in n uniformly for $x \in D$, if for any $\epsilon > 0$ and any compact set K in D, there exists a positive integer $L^*(\epsilon, K)$ such that any interval of length $L^*(\epsilon, K)$ contains an integer τ for which

$$|f(n+\tau,x) - f(n,x)| \leq \epsilon \tag{1.1}$$

for all $n \in \mathbb{Z}$ and all $x \in K$. Such a number τ in (1.1) is called an ϵ-translation number of $f(n,x)$.

In order to formulate a property of almost periodic functions, which is equivalent to the above definition, we discuss the concept of normality of almost periodic functions. Namely, let $f(n,x)$ be almost periodic in n uniformly for $x \in D$. Then, for any sequence $\{h_k^{'}\} \subset \mathbb{Z}$, there exists a subsequence $\{h_k\}$ of $\{h_k^{'}\}$ such that the sequence of functions $\{f(n+h_k, x_n)\}$ converges uniformly on $\mathbb{Z} \times S$, where S is a compact set in D. In other words, for any sequence $\{h_k^{'}\} \subset \mathbb{Z}$, there exists a subsequence $\{h_k\}$ of $\{h_k^{'}\}$ and a function $g(n,x)$ such that

$$f(n+h_k,x) \to g(n,x) \tag{1.2}$$

uniformly on $\mathbb{Z} \times K$ as $k \to \infty$, where K is any compact set in D. There are many properties of discrete almost periodic functions [5], which are corresponding properties of continuous almost periodic functions $f(t,x) \in C(\mathbb{R} \times D, \mathbb{R}^l)$, (cf., [2,6]).

We shall denote by $T(f)$ the function space consisting of all translates of f, that is, $f_\tau \in T(f)$, where

$$f_\tau(n,x) = f(n+\tau,x), \qquad \tau \in \mathbb{Z}. \tag{1.3}$$

Let $H(f)$ denote the uniform closure of $T(f)$ in the sense of (1.3). $H(f)$ is called the hull of f. By (1.2), if $f : \mathbb{Z} \times D \to \mathbb{R}^l$ is almost periodic in n uniformly for $x \in D$, so is a function in $H(f)$. Moreover, for $g \in H(f)$, $T_\alpha f = g$ means that

$$g(n,x) = \lim_{k\to\infty} f(n+\alpha_k, x)$$

for some sequence $\{\alpha_k\} \in \mathbb{Z}$, and is written only when the limit exists.

The following concept of asymptotic almost periodicity was introduced by Fréchet in the case of continuous functions (cf., [2,6]).

Definition 1.2. $u(n)$ is said to be asymptotically almost periodic if it is a sum of an almost periodic function $p(n)$ and a function $q(n)$ defined on $I^* = [a,\infty) \subset \mathbb{Z}^+$ which tends to zero as $n \to \infty$, that is,

$$u(n) = p(n) + q(n).$$

$u(n)$ is asymptotically almost periodic if and only if for any sequence $\{n_k\}$ such that $n_k \to \infty$ as $k \to \infty$ there exists a subsequence $\{n_{k_j}\}$ for which $u(n + n_{k_j})$ converges uniformly on I^*.

We consider the Volterra difference equation with delay

$$\Delta x(n) = f(n, x(n)) + \sum_{m=-\infty}^{n} F(n, m)x(m) + p(n), \quad n \geq 0, \tag{1.4}$$

where $f(n, x)$ is almost periodic in n uniformly for x, $p(n)$ is an almost periodic function and $F(n, m)$ is almost periodic, that for any $\epsilon > 0$ and any compact set K, there exists an integer $L^{**} = L^{**}(\epsilon, K) > 0$ such that any interval of length L^{**} contains a τ for which

$$|F(n + \tau, m + \tau) - F(n, m)| \leq \epsilon.$$

Let $\mathbb{R}^l$ be Euclidean space with norm denoted by $\|\cdot\|$. We define the functional $[\cdot, \cdot] : \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ by

$$[x, y] = h^{-1}(\|x + hy\| - \|x\|) \quad \text{for} \quad h > 0.$$

Let x, y and z be in $\mathbb{R}^l$. From [4], the functional $[\cdot, \cdot]$ has the following properties:

(i) $|[x, y]| \leq \|y\|$,

(ii) $[x, y + z] \leq [x, y] + [x, z]$,

(iii) $\Delta_h^+\|u(n)\| = [u(n), \Delta_h u(n)]$, where u is a function and $\Delta_h^+\|u(n)\|$ denotes the h-differences of $\|u(n)\|$, that is $\Delta_h^+\|u(n)\| = h^{-1}(\|u(n + h)\| - \|u(n)\|)$.

2. Main Results

We assume the following conditions:

(I) $\|p(n) + f(n, 0)\| \leq L$ for all $n \in \mathbb{Z}$, where L is a positive constant.

(II) There exists a positive constant γ such that

$$\lim_{n-m\to\infty} \frac{1}{n-m} \sum_{j=m}^{n-1} e(j) = -\gamma < 0,$$

where the function $e(n) : \mathbb{Z} \to \mathbb{R}$.

(III) For all $(n, x) \in \mathbb{Z} \times \mathbb{R}^l$,

$$\left[x - y, f(n, x) - f(n, y) + \sum_{m=-\infty}^{n} F(n, m)x(m) - \sum_{m=-\infty}^{n} F(n, m)y(m)\right] \leq e(n)\|x - y\|.$$

We have two preliminary lemmas.

Lemma 2.1. Suppose that $e(n)$ is an almost periodic function. Then, for any $b \in \mathbb{Z}$, the following equality holds:

$$\lim_{n-m\to\infty} \frac{1}{n-m} \sum_{j=m}^{n-1} e(j) = \lim_{n-m\to\infty} \frac{1}{n-m} \sum_{j=m}^{n-1} e(j+b).$$

By discrete analogy of [2, Corollary 3.2], we can prove this lemma. So, we will omit it.

Lemma 2.2. Suppose that (II) is satisfied. Then, for any $b \in \mathbb{Z}$, the following inequality holds:

$$\exp\left(\sum_{j=m}^{n-1} e(j+b)\right) \le \beta \exp(-\alpha(n-m)), \tag{2.1}$$

where α and β are positive constants independent of b.

Proof. From Lemma 2.1, for $\epsilon = \frac{\gamma}{4} > 0$, there exists a sufficiently large $T = T(\epsilon)$ such that $\left| \frac{1}{n-m} \sum_{j=m}^{n-1} e(j+b) - (-\gamma) \right| < \epsilon$, namely,

$$\sum_{j=m}^{n-1} e(j+b) < -\frac{3\gamma(n-m)}{4} \quad \text{for} \quad n-m \ge T. \tag{2.2}$$

Since $e(n)$ is an almost periodic function, there exists a positive constant c such that inequality $|e(j+b)| < c$ holds for any $j, b \in \mathbb{Z}$. Thus, when $|n-m| \le T$, we have

$$\left|\sum_{j=m}^{n-1} e(j+b)\right| \le \sum_{j=m}^{n-1} |e(j+b)| \le cT. \tag{2.3}$$

Let $\alpha = \frac{3\gamma}{4}$ and $\beta = \exp(cT + \alpha T)$. Inequality (2.2) and (2.3) imply that (2.1) holds. ∎

Now, we present the main result of this paper.

Theorem 2.3. Suppose that assumptions (I), (II) and (III) hold. Then equation (1.4) has a unique almost periodic solution.

Proof. We first show that equation (1.4) has a bounded solution. Let $x(n)$ be a solution of equation (1.4). Then we have

$$\begin{aligned}
\Delta^{+}\|x(n)\| &= [x(n), \Delta x(n)] \\
&= \left[x(n), f(n, x(n)) + \sum_{m=-\infty}^{n} F(n, m)x(m) + p(n)\right] \\
&\le \left[x(n), f(n, x(n)) - f(n, 0) + \sum_{m=-\infty}^{n} F(n, m)x(m) - 0\right] \\
&\qquad + [x(n), f(n, 0) + p(n)] \\
&\le \left[x(n), f(n, x(n)) - f(n, 0) + \sum_{m=-\infty}^{n} F(n, m)x(m)\right] + \|f(n, 0) + p(n)\| \\
&\le e(n)\|x(n)\| + L \qquad \text{(by (I) and (III))}.
\end{aligned}$$

Solving this difference inequality, from Lemma 2.2 and

$$\prod_{i=0}^{n-1}(1 + |e(i)|) \le \prod_{i=0}^{n-1} \exp(|e(i)|) = \exp\left(\sum_{i=0}^{n-1} |e(i)|\right) \le \beta \exp(-\alpha n),$$

we obtain

$$\|x(n)\| \le \beta\|x(0)\| + \frac{L\beta}{\alpha^*} \quad n \ge 0, \tag{2.4}$$

where $\alpha^* = \dfrac{\exp(\alpha) - 1}{\exp(\alpha)}$. This implies that $x(n)$ $(n \ge 0)$ is bounded. Since equation (1.4) is almost periodic, from our assumptions, there exists a subsequence $\{n_k\}$, $n_k \to +\infty$ as $k \to +\infty$ such that $x(n + n_k)$ converges locally uniformly to $\hat{x}$ for $n \ge 0$. Moreover, $x(n + n_k)$ is a solution of the equation

$$\Delta x(n+n_k) = f(n+n_k, x(n+n_k)) + \sum_{m=-\infty}^{n} F(n+n_k, m+n_k)x(m+n_k) + p(n+n_k).$$

For $k \to \infty$, we have

$$\Delta\hat{x}(n) = f(n, \hat{x}(n)) + \sum_{m=-\infty}^{n} F(n, m)\hat{x}(m) + p(n),$$

that is, $\hat{x}(n)$ is a solution of equation (1.4). On the other hand, $x(n+n_k) \le \beta\|x(0)\| + \dfrac{L\beta}{\alpha^*}$ for $n > -n_k$. For $k \to \infty$, we have $\hat{x}(n) \le \beta\|x(0)\| + \dfrac{L\beta}{\alpha^*}$ for $n \in \mathbb{Z}$. This means that equation (1.4) has a bounded solution.

Now, we will show that equation (1.4) has a unique bounded solution. Suppose that $\hat{x}_1(n)$ and $\hat{x}_2(n)$ for $n \in \mathbb{Z}$ are two bounded solutions of equation (1.4). Then we have

$$\Delta^+\|\hat{x}_1(n) - \hat{x}_2(n)\| = [\hat{x}_1(n) - \hat{x}_2(n), \Delta(\hat{x}_1(n) - \hat{x}_2(n)] \le e(n)\|\hat{x}_1(n) - \hat{x}_2(n)\|.$$

Solving this difference inequality, we obtain

$$\|\hat{x}_1(n) - \hat{x}_2(n)\| \le \|\hat{x}_1(0) - \hat{x}_2(0)\| \exp\Bigg(\sum_{i=0}^{n-1} e(i)\Bigg).$$

Then $\|\hat{x}_1(n) - \hat{x}_2(n)\| \to 0$ as $n \to \infty$ by Lemma 2.2. This implies that there is a unique bounded solution of equation (1.4). According to [3, 5, 6], if there exists an asymptotically almost periodic solution for an almost periodic difference equation, then there exists an almost periodic solution of the equation. Therefore, in order to show that there exists an almost periodic solution of the equation (1.4), we prove that there exists an asymptotically almost periodic solution of this equation.

Suppose that $x(n)$ is a bounded solution of equation (1.4). Then for the sequence $\{r_k\}$, $r_k \to \infty$ as $k \to \infty$, $\{x(r_k)\}$ is a bounded sequence. Thus, there exists a convergent subsequence $\{x(n_k)\}$ of $\{x(r_k)\}$. Consider the sequence $\{x(n + n_k)\}$. Then we have

$$\begin{aligned}
&\Delta^+\|x(n + n_k) - x(n + n_j)\| \\
&\quad \le [x(n + n_k) - x(n + n_j), \Delta(x(n + n_k) - x(n + n_j))] \\
&\quad \le \Bigg[x(n + n_k) - x(n + n_j), f(n + n_k, x(n + n_k)) - f(n + n_j, x(n + n_j)) \\
&\qquad + \sum_{m=-\infty}^{n} F(n + n_k, m + n_k)x(m + n_k) - \sum_{m=-\infty}^{n} F(n + n_j, m + n_j)x(m + n_j)\Bigg] \\
&\qquad + \|f(n + n_k, 0) - f(n + n_j, 0)\| + \|p(n + n_k) - p(n + n_j)\| \\
&\quad \le e(n)\|x(n + n_k) - x(n + n_j)\| + \|f(n + n_k, 0) - f(n + n_j, 0)\| \\
&\qquad + \|p(n + n_k) - p(n + n_j)\|.
\end{aligned}$$

Solving the above difference inequality, we obtain

$$\begin{aligned}
\|x(n + n_k) - x(n + n_j)\| \le \beta\|x(n_k) - x(n_j)\| + \frac{\beta}{\alpha^*}\|f(n + n_k, 0) - f(n + n_j, 0)\| \\
+ \frac{\beta}{\alpha^*}\|p(n + n_k) - p(n + n_j)\|.
\end{aligned}$$

Since $\{x(n_k)\}$ is convergent, there exists N_1 sufficiently large such that for any $\epsilon > 0$, if $k, j > N_1$, then

$$\|x(n_k) - x(n_j)\| < \frac{\epsilon}{3\beta}.$$

Note that $f(n, x(n))$ is an almost periodic in n uniformly for x, and $p(n)$ is an almost periodic function. Therefore, there exists N_2 sufficiently large, such that if $k, j > N_2$, we have $\|f(n+n_k,0)-f(n+n_j,0)\| < \dfrac{\epsilon\alpha^*}{3\beta}$ and similar, if $k, j > N_3$, we have $\|p(n+n_k)-p(n+n_j)\| < \dfrac{\epsilon\alpha^*}{3\beta}$. Let $N_0 = \max\{N_1, N_2, N_3\}$. Then if $k, j > N_0$, we have

$$\|x(n+n_k)-x(n+n_j)\| < \beta\cdot\frac{\epsilon}{3\beta}+\frac{\beta}{\alpha^*}\cdot\frac{\epsilon\alpha^*}{3\beta}+\frac{\beta}{\alpha^*}\cdot\frac{\epsilon\alpha^*}{3\beta}=\epsilon.$$

This means $x(n)$ is an asymptotically almost periodic solution of equation (1.4). Note that since any almost periodic solution is bounded, equation (1.4) has one and only one bounded solution, thus, there exists a unique almost periodic solution of equation (1.4). The proof is completed. ■

Next, we consider the equation

$$\Delta x(n) = f(n, x(n)) + \sum_{m=-\infty}^{n} F(n,m)x(m) + g(n, x(n-\tau)) + p(n), \quad n \ge 0, \tag{2.5}$$

where $g(n,\phi)$ is almost periodic in n uniformly for ϕ and time delay $\tau > 0$.

We assume the following conditions hold in addition to conditions (II) and (III):

(I′) $\|p(n)+f(n,0)+g(n,0)\| \le L$ for all $n \in \mathbb{Z}$, where L is a positive constant.

(IV) $g(n,\phi)$ satisfies $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sup_{\|\phi\|\le n}\|g(n,\phi)\| = 0$.

Now, we have the following theorem.

Theorem 2.4. Suppose that assumptions (I′), (II), (III) and (IV) hold. Then equation (2.5) has an almost periodic solution.

Proof. We consider a set $B = \{u(n)|u(n)$ is an almost periodic function$\}$. It is obvious that if the set B is equipped with norm $\|u\| = \sup_{n\in Z}|u(n)|$, then B is a Banach space. For any $u \in B$, consider the equation

$$\Delta x(n) = f(n, x(n)) + \sum_{m=-\infty}^{n} F(n,m)x(m) + g(n, u(n-\tau)) + p(n), \quad n \ge 0. \tag{2.6}$$

From Theorem 2.3, equation (2.6) has an almost periodic solution denoted by $x_u(n)$ for each u. Thus, we can define a mapping S such that $Su(n) = x_u(n)$. Let $B_r := \{u(n)|u(n) \in B, \|u\| \le r\}$, where r is a natural number. We first show that there exists a natural number N such that $S : B_N \to B_N$. If this conclusion is not true, then for any natural number r, there is $u_r(n) \in B_r$ such that $\|Su_r\| > r$. From (IV), for $0 < \epsilon < \frac{1}{4}$, there exists a natural number $N > \max\left\{4\beta\|u_r(0)\|, \dfrac{4\beta L}{\alpha}\right\}$ such that if $r \ge N$, we have

$$\frac{1}{r}\sup_{\|\phi\|\le n}\|g(n,\phi)\| < \epsilon < \frac{1}{4}.$$

Similar to (2.4), we have

$$\frac{1}{r}\|Su_r(n)\| \le \frac{\beta}{r}\|u_r(0)\|\frac{\beta L}{\alpha r} + \frac{1}{r}\sup_{\|\phi\|\le n}\|g(n,\phi)\| < \frac{3}{4}.$$

This is a contradiction for $\|Su_r\| > r$. Therefore, there exists a natural number N sufficiently large such that $S : B_N \to B_N$.

For $u(n), v(n) \in B_N$, we consider $w(n) = \lambda u(n) + (1-\lambda)v(n)$ for some $\lambda \in [0, 1]$. Then

$$\begin{aligned}\|w(n)\| &= \|\lambda u(n) + (1-\lambda)v(n)\| \\ &\le \lambda\|u(n)\| + (1-\lambda)\|v(n)\| \\ &\le \lambda N + (1-\lambda)N = N.\end{aligned}$$

Thus, $w(n) \in B_N$. This implies that B_N is a convex set. Now, we show that S is a compact operator. Since $SB_N \subseteq B_N$, thus $\{Su(n)|u \in B_N\}$ is uniformly bounded. Note that $f(n, x(n))$ is almost periodic in n uniformly for x, so for $\|u\| \le N$, $f(n,u)$ is bounded. From condition (IV), we obtain that $\|g(n,\phi)\|$ is bounded for $\|\phi\| \le N(\phi)$. Let

$$\begin{aligned}h_1 &= \sup_{n\in Z, \|x\|\le N}\|f(n,x)\|, \quad h_2 = \sup_{n\in Z}\sum_{m=-\infty}^{n}\|F(n,m)\|, \\ h_3 &= \sup_{n\in Z, \|x\|\le N}\|g(n,\phi)\|, \quad h_4 = \sup_{n\in Z}\|p(n)\|.\end{aligned}$$

Then

$$\begin{aligned}\Delta Su(n) &= \Delta x_u(n) \\ &= f(n, x_u(n)) + \sum_{m=-\infty}^{n} F(n,m)x_u(m) + g(n, x_u(n-\tau)) + p(n), \quad n \ge 0.\end{aligned}$$

Therefore, $\|\Delta Su(n)\| \le h_1 + h_2 + h_3 + h_4$. This implies that $\{Su(n)|u(n) \in B_N\}$ is uniformly bounded and equicontinuous, so we can assert that there is a sequence $u_k(n)$ satisfying $\|u_k(n)\| \le N$ such that $\{Su_k(n)\}$ locally uniformly converges on $\mathbb{Z}$ by using the Arzelà–Ascoli theorem. This proves that S is a compact operator.

Next, we show that S is a continuous operator. Let $H_1 = \sup_{n\in Z}\{Su(n)|u(n) \in B_N\}$. Note that $f(n, x(n))$ is almost periodic in n uniformly for x, $g(n,\phi)$ is almost periodic in n uniformly for ϕ, so $f(n, x(n))$ and $g(n, x(n))$ both are continuous functions. For any $u(n), v(n) \in B_N$, we have

$$\Delta^+\|Su(n) - Sv(n)\| \le e(n)\|Su(n) - Sv(n)\| + \|g(n,u_n) - g(n,v_n)\|, \tag{2.7}$$

where $u_n = u(n-\tau)$, $v_n = v(n-\tau)$. For any $\epsilon > 0$, from (II) there exists a positive constant K sufficiently large such that $\exp\left(\sum_{j=n-K}^{n-1} e(j)\right) < \frac{\epsilon}{4H_1}$. Similar to [4] we obtain that $\sum_{j=n-K}^{n-1} \exp\left(\sum_{i=j}^{n-1} e(i)\right)$ is bounded. Let $H_2 = \sum_{j=n-K}^{n-1} \exp\left(\sum_{i=j}^{n-1} e(i)\right)$. Since $g(n, x(n))$ is continuous, there exists $\delta = \delta(\epsilon) > 0$ such that if $\|u_n - v_n\| < \delta$ we have $\|g(n, u_n) - g(n, v_n)\| < \frac{\epsilon}{2H_2}$. From (2.7) we get

$$\|Su(n) - Sv(n)\| \le \|Su(n-K) - Sv(n-K)\| \exp\left(\sum_{j=n-K}^{n-1} e(j)\right)$$
$$+ \|g(n, u_n - g(n, v_n)\| \sum_{j=n-K}^{n-1} \exp\left(\sum_{i=j}^{n-1} e(i)\right) < 2H_1 \cdot \frac{\epsilon}{4H_1} + \frac{\epsilon}{2H_2} \cdot H_2 = \epsilon.$$

This proves that S is a continuous operator. From Schauder's fixed point theorem, S has a fixed point on B_N. Namely, equation (2.5) has an almost periodic solution. It is easy to obtain that this almost periodic solution is unique. ■

We consider two specific examples for Theorems 2.3 and 2.4.

Example 2.5. Let $F(n, m) = 0$. We shall consider the almost periodic equation

$$\Delta x(n) = -\left(\frac{1}{3} - \sin n\right)x(n) - (1 + \sin\sqrt{2}n)\sqrt[3]{x(n)} + p(n), \tag{2.8}$$

where $p(n)$ is an almost periodic function. Note that since $e(n) = -\frac{1}{3} + \sin n$, we cannot find a $T_0 > 0$ such that $e(n) < 0$ for $n < -T_0$. Thus, a discrete analogy of Kato's theorem [4] cannot deal with equation (2.8). However, since

$$\lim_{n-m\to\infty} \frac{1}{n-m} \sum_{j=m}^{n-1} \left(-\frac{1}{3} + \sin j\right) < \infty,$$

we set

$$q(n) = -1 + \sin\sqrt{2}n,\ f(n, x) = -\left(\frac{1}{3} - \sin n\right)x(n) - \left(1 + \sin\sqrt{2}n\right)\sqrt[3]{x(n)}.$$

Then we have

$$|x - y + h(f(n, x) - f(n, y))|$$
$$\le \left|1 + he(n) + \frac{hq(n)}{\sqrt[3]{x^2(n)} + \sqrt[3]{x(n)y(n)} + \sqrt[3]{y^2(n)}}\right| \cdot |x - y|, \qquad x^2 + y^2 \ne 0.$$

Since $\sqrt[3]{x^2} + \sqrt[3]{y^2} \ge \sqrt[3]{xy}$ for $x^2 + y^2 \ne 0$, and $q(n) \le 0$, there exists a sufficiently small positive constant h_0 such that

$$|x - y + h(f(n, x) - f(n, y))| \le (1 + he(n))|x - y|, \qquad 0 < h < h_0.$$

Then, we have

$$[x - y, f(n, x) - f(n, y)] \le e(n)|x - y|.$$

From Theorem 2.3, equation (2.8) has a unique almost periodic solution.

Example 2.6. We consider the almost periodic equation

$$\begin{aligned}\Delta x(n) = &-\Big(\frac{2}{3} + \frac{x(n)\sin n}{\sqrt{1 + x^2(n)}}\Big)x(n) \\ &+ \frac{1}{4}\sum_{m=-\infty}^{n} x(m)e^{-(n-m)}\cos n + \sqrt{|x(n - k_0)|}\sin\sqrt{2}n + 2\sin n. \quad (2.9)\end{aligned}$$

We do not know whether or not this equation (2.9) has a bounded solution. Even if we suppose that equation (2.9) has a bounded solution, since $\sqrt{|x(n - k_0)|}\sin\sqrt{2}n$ cannot be sufficiently small as $n \to \infty$, Song, Baker and Tian's theorem [5] also fails for equation (2.9). We note that $F(n, m) = \frac{1}{4}\exp(-(n - m))\cos n$ and $\frac{1}{4}\sum_{m=-\infty}^{n} |F(n, m)| < \infty$. Obviously, equation (2.9) satisfies all assumptions of Theorem 2.4. So there exists a unique almost periodic solution of equation (2.9).

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