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Oscillation of Fourth Order Nonlinear Difference Equations

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Abstract

In this paper we shall establish some new criteria for the oscillation of nonlinear fourth order difference equations of the form

$$\Delta^2\left(\frac{1}{a(k)}(\Delta^2 x(k))^\alpha\right) = q(k)f(x[g(k)]) + p(k)h(x[\sigma(k)]),$$

where α is the ratio of two positive odd integers.

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1. Introduction

Consider the fourth order nonlinear difference equation

$$\Delta^2\left(\frac{1}{a(k)}(\Delta^2 x(k))^\alpha\right) = q(k)f(x[g(k)]) + p(k)h(x[\sigma(k)]), \tag{1.1}$$

where $k \in \mathbb{N}_0 = \{n_0, n_0+1, \cdots\}$, n_0 is a nonnegative integer, Δ is the forward difference operator defined by $\Delta x(k) = x(k+1) - x(k)$, and

(i). α is the ratio of two positive odd integers,

(ii). $\{a(k)\}$, $\{p(k)\}$ and $\{q(k)\}$ are positive sequences,

(iii). $\{g(k)\}$ and $\{\sigma(k)\}$ are nondecreasing sequences of real constants with $g(k) < k$ and $\sigma(k) > k$ for $k \in \mathbb{N}_0$ and $\lim_{k\to\infty} g(k) = \infty$,

(iv). $f, h : C(\mathbb{R}, \mathbb{R})$ satisfying $xf(x) > 0$, $f'(x) \geq 0$, $xh(x) > 0$ and $h'(x) \geq 0$ for $x \neq 0$.

By a solution of equation (1.1), we mean a real sequence $\{x(k)\}$ satisfying equation (1.1) for all large $k \geq n_0 \in \mathbb{N}_0$. A nontrivial solution $\{x(k)\}$ of (1.1) is said to be *nonoscillatory* if it is either eventually positive or eventually negative, and it is *oscillatory* otherwise. The equation (1.1) is said to be *oscillatory* if all its solutions are oscillatory.

The problem of determining the nonoscillation and oscillation of all solutions of second order half-linear difference equations of the form

$$\Delta\left(\frac{1}{a(k)}(\Delta x(k))^\alpha\right) + q(k)x^\alpha[g(k)] = 0$$

and

$$\Delta\left(\frac{1}{a(k)}(\Delta x(k))^\alpha\right) = q(k)x^\alpha[g(k)] + p(k)x^\alpha[\sigma(k)]$$

has been a very active area of research in the last two decades, and for surveys of the recent contributions, we refer the reader to the books and the papers [1–10] and the references cited therein. A question naturally arises as the possibility of generalizing some of the results in [2, Section 6.2.2] or [9, Section 16] to equation (1.1). The main objective of this paper is to give an affirmative answer to the above question and to establish some new criteria for the oscillation of equation (1.1) via comparison with first and second order difference equations whose oscillatory characteristics are known.

2. Main Results

We define the operators

$$L_0x(k) = x(k), \quad L_1x(k) = \Delta L_0x(k), \quad L_2x(k) = \frac{1}{a(k)}(\Delta L_1x(k))^\alpha,$$
$$L_3x(k) = \Delta L_2x(k) \quad \text{and} \quad L_4x(k) = \Delta L_3x(k).$$

Clearly, equation (1.1) takes the form

$$L_4 x(k) = q(k) f(x[g(k)]) + p(k) h(x[\sigma(k)]). \tag{2.1}$$

In what follows we shall assume that

$$\sum^{\infty} a^{1/\alpha}(j) = \infty, \tag{2.2}$$

$$-f(-xy) \geq f(xy) \geq f(x) f(y) \quad \text{for} \quad xy > 0 \tag{2.3}$$

and

$$-h(-xy) \geq h(xy) \geq h(x) h(y) \quad \text{for} \quad xy > 0. \tag{2.4}$$

In the following results, we shall use the following notation. For all large $k \geq m+2 \geq n_0$, we let

$$A[k, m] = \sum_{\ell=m}^{k-2} (k - \ell - 1)(\ell - m)^{1/\alpha} a^{1/\alpha}(\ell)$$

and

$$B[k, n_0] = \sum_{\ell=n_0}^{k-2} (k - \ell - 1)(k + 1 - \ell)^{1/\alpha} a^{1/\alpha}(\ell).$$

Also, we let

$$C[k, m] = \sum_{\ell=m}^{k-1} (\ell - m + 1)(k + 1 - \ell)^{1/\alpha} a^{1/\alpha}(\ell).$$

We are now ready to prove the following result.

Theorem 2.1. Let conditions (i)–(iv), (2.2)–(2.4) hold,

$$\frac{h(u^{1/\alpha})}{u} \geq c_1 > 0, \quad c_1 \text{ is a constant,} \quad \text{for} \quad u \neq 0 \tag{2.5}$$

and

$$\frac{f(u^{1/\alpha})}{u} \geq c > 0, \quad c \text{ is a constant,} \quad \text{for} \quad u \neq 0. \tag{2.6}$$

If

$$\limsup_{n\to\infty} \sum_{j=k}^{\sigma(k)-1} p(j) h(A[\sigma(j), \sigma(k)]) > \frac{1}{c_1}, \tag{2.7}$$

$$\limsup_{n\to\infty} \sum_{j=g(k)}^{k-1} q(j) f(B[g(j), n_0]) > \frac{1}{c}, \tag{2.8}$$

and

$$\limsup_{n\to\infty} \sum_{j=g(k)}^{k-1} q(j) f(B[g(k), g(j)]) > \frac{1}{c}, \tag{2.9}$$

then equation (1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (1.1). Then, $L_4x(k) \geq 0$ eventually and hence $L_ix(k)$, $i = 1, 2, 3$ are eventually of one sign. Now we shall distinguish the following three cases:

(I). $L_ix(k) > 0$, $i = 1, 2, 3$ eventually,

(II). $L_ix(k) > 0$, $i = 1, 2$ and $L_3x(k) < 0$ eventually, and

(III). $(-1)^i L_ix(k) > 0$, $i = 1, 2, 3$ eventually.

Case (I). Let $L_ix(k) > 0$, $i = 1, 2, 3$ for $k \geq n_0 \in \mathbb{N}_0$. Then for $\ell \geq m + 1 \geq n_0 + 1$, we have

$$L_2x(\ell) - L_2x(m) = \sum_{j=m}^{\ell-1} L_3x(j)$$

and so

$$L_2x(\ell) \geq (\ell - m)L_3x(m),$$

or

$$\Delta^2 x(\ell) \geq (\ell - m)^{1/\alpha} a^{1/\alpha}(\ell) L_3^{1/\alpha} x(m). \tag{2.10}$$

From the discrete Taylor formula it follows that x satisfies the equality

$$x(k) = x(n_0) + (k - n_0)\Delta x(n_0) + \sum_{\ell=n_0}^{k-2} (k - \ell - 1)\Delta^2 x(\ell), \quad k \geq n_0 + 2. \tag{2.11}$$

Using (2.10) in (2.11), we have

$$\begin{aligned} x(k) &\geq \left(\sum_{\ell=n_0}^{k-2} (k - \ell - 1)(\ell - m)^{1/\alpha} a^{1/\alpha}(\ell) \right) L_3^{1/\alpha} x(m), \quad k \geq m + 2 \geq n_0 + 2 \\ &\geq \left(\sum_{\ell=m}^{k-2} (k - \ell - 1)(\ell - m)^{1/\alpha} a^{1/\alpha}(\ell) \right) L_3^{1/\alpha} x(m) \\ &= A[k, m] L_3^{1/\alpha} x(m) \quad \text{for} \quad k \geq m + 2 \geq n_0 + 2. \end{aligned} \tag{2.12}$$

Letting $k = \sigma(j)$ and $m = \sigma(k)$ in (2.12), we have

$$x[\sigma(j)] \geq A[\sigma(j), \sigma(k)] L_3^{1/\alpha} x[\sigma(k)] \quad \text{for} \quad \sigma(j) \geq \sigma(k) + 2 \geq n_0 + 2. \tag{2.13}$$

From equation (1.1), we see that

$$L_4x(j) \geq p(j)h(x[\sigma(j)]). \tag{2.14}$$

Using (2.13) and (2.4) in (2.14), we obtain

$$L_4x(j) \geq p(j)h(A[\sigma(j), \sigma(k)])h(L_3^{1/\alpha}x[\sigma(k)]) \quad \text{for} \quad \sigma(j) \geq \sigma(k) + 2 \geq n_0 + 2.$$

Summing both sides of the above inequality from k to $\sigma(k) - 1$, we have

$$L_3x[\sigma(k)] - L_3x(k) \geq \left(\sum_{j=k}^{\sigma(k)-1} p(j)h(A[\sigma(j), \sigma(k)]) \right) \left(h\left(L_3^{1/\alpha}x[\sigma(k)] \right) \right),$$

or

$$\frac{L_3x[\sigma(k)]}{h(L_3^{1/\alpha}x[\sigma(k)])} \geq \sum_{j=k}^{\sigma(k)-1} p(j)h(A[\sigma(j), \sigma(k)]).$$

Taking lim sup of both sides of the above inequality as $k \to \infty$, we obtain a contradiction to condition (2.7).

Case (II). Let $L_ix(k) > 0$, $i = 1, 2$ and $L_3x(k) < 0$ for $k \geq n_0 \in \mathbb{N}_0$. Then for $k + 1 \geq \ell \geq n_0$, we have

$$L_2x(k+1) - L_2x(\ell) = \sum_{j=\ell}^{k} L_3x(j)$$

and so,

$$L_2x(\ell) \geq (k + 1 - \ell)(-L_3x(k)),$$

or

$$\Delta^2x(\ell) \geq (k + 1 - \ell)^{1/\alpha}a^{1/\alpha}(\ell)\left(-L_3^{1/\alpha}x(k)\right). \tag{2.15}$$

From Taylor's formula (2.11), we obtain

$$x(k) \geq \sum_{\ell=n_0}^{k-2}(k - \ell - 1)\Delta^2x(\ell) \quad \text{for} \quad k \geq n_0 + 2. \tag{2.16}$$

Using (2.15) in (2.16), we have

$$\begin{aligned} x(k) &\geq \left(\sum_{\ell=n_0}^{k-2}(k - \ell - 1)(k + 1 - \ell)^{1/\alpha}a^{1/\alpha}(\ell) \right) \left(-L_3^{1/\alpha}x(k)\right) \\ &= B[k, n_0]\left(-L_3^{1/\alpha}x(k)\right) \quad \text{for} \quad k \geq n_0 + 2. \end{aligned} \tag{2.17}$$

Letting $k = g(j)$ in (2.17), we get

$$x[g(j)] \geq B[g(j), n_0]\left(-L_3^{1/\alpha}x[g(j)]\right) \quad \text{for} \quad g(j) \geq n_0 + 2. \tag{2.18}$$

From equation (1.1), we have

$$L_4x(j) \geq q(j)f(x[g(j)]) \quad \text{for} \quad g(j) \geq n_0 + 2. \tag{2.19}$$

Using (2.3) and (2.18) in (2.19), we obtain

$$L_4x(j) \geq q(j)f(B[g(j), n_0])f\left(-L_3^{1/\alpha}x[g(j)]\right).$$

The rest of the proof is similar to that of Case (I) and hence omitted.

Case (III). Let $(-1)^i L_i x(k) > 0$, $i = 1, 2, 3$ for $k \geq n_0$. Then as in the proof of Case (II) we obtain (2.15). From Taylor's formula and for $k - 1 \geq m \geq n_0$ one can easily see that

$$x(m) \geq \sum_{\ell=m}^{k-1}(\ell + 1 - m)\Delta^2 x(\ell). \tag{2.20}$$

Using (2.15) in (2.20), we have

$$\begin{aligned} x(m) &\geq \left(\sum_{\ell=m}^{k-1}(\ell + 1 - m)(k + 1 - \ell)^{1/\alpha}a^{1/\alpha}(\ell)\right)\left(-L_3^{1/\alpha}x(k)\right) \\ &= C[k, m]\left(-L_3^{1/\alpha}x(k)\right) \quad \text{for} \quad k - 1 \geq m \geq n_0. \end{aligned} \tag{2.21}$$

Substituting $g(j)$ and $g(k)$ for m and k, respectively, in the above inequality, we get

$$x[g(j)] \geq C[g(k), g(j)]\left(-L_3^{1/\alpha}x[g(k)]\right) \quad \text{for} \quad g(k) - 1 \geq g(j) \geq n_0. \tag{2.22}$$

From equation (1.1), we have

$$L_4x(j) \geq q(j)f(x[g(j)]), \quad j \geq n_0.$$

Using (2.3) and (2.22) in the above inequality, we obtain

$$\begin{aligned} L_4x(j) \geq q(j)f(C[g(k), g(j)])f\left(-L_3^{1/\alpha}x[g(k)]\right) \quad &\text{for} \quad g(k) - 1 \\ \geq g(j) \geq n_0. \end{aligned} \tag{2.23}$$

Summing both sides of (2.23) from $g(k)$ to $k - 1$, we have

$$-L_3x[g(k)] \geq \left(\sum_{j=g(k)}^{k-1} q(j)f(C[g(k), g(j)])\right) f\left(-L_3^{1/\alpha}x[g(k)]\right),$$

or

$$\frac{-L_3x[g(k)]}{f(-L_3^{1/\alpha}x[g(k)])} \geq \sum_{j=g(k)}^{k-1} q(j)f(C[g(k), g(j)]).$$

Taking lim sup of both sides of the above inequality as $k \to \infty$, we arrive at the desired contradiction. This completes the proof. ■

Next, we shall replace conditions (2.3) and (2.6) by

$$\frac{f^{1/\alpha}(u)}{u} \geq b > 0, \quad b \quad \text{is a constant,} \quad \text{for} \quad u \neq 0 \tag{2.24}$$

and conditions (2.4) and (2.5) by

$$\frac{h^{1/\alpha}(u)}{u} \geq b_1 > 0, \quad b_1 \quad \text{is a constant,} \quad \text{for} \quad u \neq 0 \tag{2.25}$$

and prove the following result.

Theorem 2.2. Let conditions (i)–(iv), (2.2), (2.24) and (2.25) hold. If

$$\limsup_{k\to\infty} \sum_{s_3=k}^{\sigma(k)-1} \sum_{s_2=k}^{s_3-1} \left(a(s_2) \sum_{s_1=k}^{s_2-1} \sum_{s=k}^{s_1-1} p(s) \right)^{1/\alpha} > \frac{1}{b_1}, \tag{2.26}$$

$$\limsup_{k\to\infty} B[g(k), n_0] \left(\sum_{s=k}^{\infty} q(s) \right)^{1/\alpha} > \frac{1}{b}, \quad n_0 \in \mathbb{N}_0 \tag{2.27}$$

and

$$\limsup_{k\to\infty} \sum_{v_3=g(k)}^{k-1} \sum_{v_3=v_2}^{k-1} \left(a(v_2) \sum_{v_2=v_1}^{k-1} \sum_{v_1=v}^{k-1} q(v) \right)^{1/\alpha} > \frac{1}{b}, \tag{2.28}$$

then equation (1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (1.1). As in the proof of Theorem 2.1, we shall consider the three Cases (I), (II) and (III). Suppose (I) holds. For $s_1 - 1 \geq k \geq n_0 \in \mathbb{N}_0$, it follows from equation (1.1) that

$$L_3 x(s_1) \geq \sum_{s=k}^{s_1-1} p(s) h(x[\sigma(s)]) \geq \left(\sum_{s=k}^{s_1-1} p(s) \right) h(x[\sigma(k)]).$$

Summing the above inequality from k to $s_2 - 1$, $s_2 - 1 \geq s_1 - 1 \geq k \geq n_0$, we have

$$L_2 x(s_2) \geq \left(\sum_{s_1=k}^{s_2-1} \sum_{s=k}^{s_1-1} p(s) \right) h(x[\sigma(k)]),$$

or

$$\Delta^2 x(s_2) \geq \left(a(s_2) \sum_{s_1=k}^{s_2-1} \sum_{s=k}^{s_1-1} p(s) \right)^{1/\alpha} h^{1/\alpha}(x[\sigma(k)]).$$

Next, summing the above inequality from k to s_3-1, $s_3-1 \geq s_2-1 \geq s_1-1 \geq k \geq n_0$, we obtain

$$\Delta x(s_3) \geq \sum_{s_2=k}^{s_3-1}\left(a(s_2)\sum_{s_1=k}^{s_2-1}\sum_{s=k}^{s_1-1}p(s)\right)^{1/\alpha} h^{1/\alpha}(x[\sigma(k)]).$$

Once again, summing the above inequality from k to $\sigma(k)-1$, we obtain

$$x[\sigma(k)] \geq h^{1/\alpha}(x[\sigma(k)])\sum_{s_3=k}^{\sigma(k)-1}\sum_{s_2=k}^{s_3-1}\left(a(s_2)\sum_{s_1=k}^{s_2-1}\sum_{s=k}^{s_1-1}p(s)\right)^{1/\alpha},$$

or

$$\frac{x[\sigma(k)]}{h^{1/\alpha}(x[\sigma(k)])} \geq \sum_{s_3=k}^{\sigma(k)-1}\sum_{s_2=k}^{s_3-1}\left(a(s_2)\sum_{s_1=k}^{s_2-1}\sum_{s=k}^{s_1-1}p(s)\right)^{1/\alpha}, \quad k \geq n_0.$$

Taking lim sup of both sides of the above inequality as $k \to \infty$, we arrive at a contradiction to condition (2.26).

Suppose (II) holds. From equation (1.1), it is easy to see that

$$-L_3x(k) \geq \left(\sum_{s=k}^{\infty}q(s)\right) f(x[g(k)]) \quad \text{for} \quad k \geq n_0.$$

Using inequality (2.18) with $j = k$ and the fact that $-L_3x(k)$ is nonincreasing, we have

$$\begin{aligned} x[g(k)] &\geq B[g(k), n_0]\left(-L_3^{1/\alpha}x[g(k)]\right) \\ &\geq B[g(k), n_0]\left(-L_3^{1/\alpha}x(k)\right) \\ &\geq B[g(k), n_0]\left(\sum_{s=k}^{\infty}q(s)\right)^{1/\alpha} f^{1/\alpha}(x[g(k)]), \quad k \geq n_0. \end{aligned}$$

The rest of the proof is similar to that of Case (I) and hence omitted.

Suppose (III) holds. For $k-1 \geq v_1 \geq n_0$, we have

$$-L_3x(v_1) \geq \left(\sum_{v=v_1}^{k-1}q(v)\right) f(x[g(k)]).$$

Summing the above inequality from v_2 to $k-1$, $k-1 \geq v_2 \geq v_1 \geq n_0$, we obtain

$$L_2x(v_2) \geq \left(\sum_{v_1=v_2}^{k-1}\sum_{v=v_1}^{k-1}q(v)\right) f(x[g(k)]),$$

or

$$\Delta^2 x(v_2) \geq \left(a(v_2) \sum_{v_1=v_2}^{k-1} \sum_{v=v_1}^{k-1} q(v) \right)^{1/\alpha} f^{1/\alpha}(x[g(k)]).$$

Summing the above inequality from v_3 to $k-1$, $k-1 \geq v_3 \geq v_2 \geq v_1 \geq n_0$, we get

$$\Delta x(v_3) \geq \sum_{v_2=v_3}^{k-1} \left(a(v_2) \sum_{v_1=v_2}^{k-1} \sum_{v=v_1}^{k-1} q(v) \right)^{1/\alpha} f^{1/\alpha}(x[g(k)]).$$

Finally, we sum the above inequality from $g(k)$ to $k-1$, to obtain

$$\frac{x[g(k)]}{f^{1/\alpha}(x[g(k)])} \geq \sum_{v_3=g(k)}^{k-1} \sum_{v_2=v_3}^{k-1} \left(a(v_2) \sum_{v_1=v_2}^{k-1} \sum_{v=v_1}^{k-1} q(v) \right)^{1/\alpha}.$$

Taking lim sup of both sides of the above inequality as $k \to \infty$, we obtain a contradiction to condition (2.28). This completes the proof. ■

Next for $k \geq m+1 \geq n_0 \in \mathbb{N}_0$, we let

$$C[k, m] = \sum_{\ell=m}^{k-1} \sum_{j=m}^{\ell-1} a^{1/\alpha}(j)$$

and present the following comparison results.

Theorem 2.3. Let conditions (i)–(iv), (2.2)–(2.4) hold. If all the unbounded solutions of the advanced second order equation

$$\Delta^2 y(k) - p(k) h\left(C\left[\sigma(k), \frac{k+\sigma(k)}{2} \right] \right) h\left(y^{1/\alpha}\left[\frac{k+\sigma(k)}{2} \right] \right) = 0, \tag{2.29}$$

and all the bounded solutions of the second order delay equations

$$\Delta^2 z(k) - q(k) f(C[g(k), n_0]) f(z^{1/\alpha}[g(k)]) = 0 \tag{2.30}$$

and

$$\Delta^2 w(k) - q(k) f\left(C\left[\frac{k+g(k)}{2}, g(k) \right] \right) f\left(w^{1/\alpha}\left[\frac{k+g(k)}{2} \right] \right) = 0 \tag{2.31}$$

are oscillatory, then equation (1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (1.1). As in the proof of Theorem 2.1, we shall consider the three Cases (I), (II) and (III).

Case (I). Suppose $L_i x(k) > 0$, $i = 1, 2, 3$ for $k \geq n_0 \in \mathbb{N}_0$. Then for $\ell \geq m+1 \geq n_0$, we have

$$L_1 x(\ell) - L_1 x(m) = \sum_{j=m}^{\ell-1} a^{1/\alpha}(j) L_2^{1/\alpha} x(j),$$

or

$$\Delta x(\ell) = L_1 x(\ell) \geq \left(\sum_{j=m}^{\ell-1} a^{1/\alpha}(j) \right) L_2^{1/\alpha} x(m). \tag{2.32}$$

Summing both sides of the above inequality from m to $k-1$, we obtain

$$x(k) \geq C[k, m] L_2^{1/\alpha} x(m) \quad \text{for} \quad k \geq m+1 \geq n_0. \tag{2.33}$$

Let $y(k) = L_2 x(k)$. Substituting $\sigma(k)$ and $(k + \sigma(k))/2$ for k and m, respectively, in (2.33), we get

$$x[\sigma(k)] \geq C\left[\sigma(k), \frac{k+\sigma(k)}{2}\right] y^{1/\alpha}\left[\frac{k+\sigma(k)}{2}\right] \quad \text{for} \quad k \geq n_0. \tag{2.34}$$

Using (2.4) and (2.33) in equation (1.1), we obtain

$$\begin{aligned} \Delta^2 y(k) &\geq p(k) h(x[\sigma(k)]) \\ &\geq p(k) h\left(C\left[\sigma(k), \frac{k+\sigma(k)}{2}\right]\right) h\left(y^{1/\alpha}\left[\frac{k+\sigma(k)}{2}\right]\right). \end{aligned}$$

By applying a result in [2, 10], we see that the equation (2.29) has an eventually positive solution which is a contradiction.

Case (II). Suppose $L_i x(k) > 0$, $i = 1, 2$ and $L_3 x(k) < 0$ for $k \geq n_0$. Then for $k \geq m+1 \geq n_0$, we get

$$\Delta x(\ell) = L_1 x(\ell) \geq \left(\sum_{j=m}^{\ell-1} a^{1/\alpha}(j) \right) L_2^{1/\alpha} x(\ell). \tag{2.35}$$

Summing the above inequality from n_0 to $k-1$, we find

$$x(k) \geq C[k, n_0] L_2^{1/\alpha} x(k). \tag{2.36}$$

Let $z(k) = L_2 x(k)$. Substituting $g(k)$ for k in (2.36), we get

$$x[g(k)] \geq C[g(k), n_0] z^{1/\alpha}[g(k)], \quad k \geq n_0 + 1. \tag{2.37}$$

Using (2.3) and (2.37) in equation (1.1), we obtain

$$\begin{aligned} \Delta^2 z(k) &\geq q(k) f(x[g(k)]) \\ &\geq q(k) f(C[g(k), n_0]) f(z^{1/\alpha}[g(k)]). \end{aligned}$$

Now by a result in [2], one can easily see that equation (2.30) has an eventually positive solution, which is a contradiction.

Case (III). Suppose $L_1x(k) < 0$, $L_2x(k) > 0$ and $L_3x(k) < 0$ for $k \ge n_0$. Then as in Case (II) we obtain (2.35). Summing (2.35) from m to $k-1 \ge m \ge n_0$, we have

$$x(m) \ge C[k, m]L_2^{1/\alpha}x(k). \tag{2.38}$$

Let $w(k) = L_2x(k)$. Substituting $(k + g(k))/2$ and $g(k)$ for k and m respectively, in (2.38), we find

$$x[g(k)] \ge C\left[\frac{k+g(k)}{2}, g(k)\right] w^{1/\alpha}\left[\frac{k+g(k)}{2}\right]. \tag{2.39}$$

Using (2.3) and (2.39) in equation (1.1), we find

$$\begin{aligned}\Delta^2 w(k) &\ge q(k)f(x[g(k)])\\ &\ge q(k)f\left(C\left[\frac{k+g(k)}{2}, g(k)\right]\right) f\left(w^{1/\alpha}\left[\frac{k+g(k)}{2}\right]\right).\end{aligned}$$

The rest of the proof is similar to that of Case (II) and hence omitted. This completes the proof. ■

Theorem 2.4. Let conditions (i)–(iv), (2.2)–(2.4) hold. If the first order advanced equation

$$\Delta y(k) - p(k)h\left(A\left[\sigma(k), \frac{k+\sigma(k)}{2}\right]\right) h\left(y^{1/\alpha}\left[\frac{k+\sigma(k)}{2}\right]\right) = 0 \tag{2.40}$$

and the first order delay equations

$$\Delta z(k) + q(k)f(B[g(k), n_0])f(z^{1/\alpha}[g(k)]) = 0, \quad k \ge n_0 \ge 0 \tag{2.41}$$

and

$$\Delta w(k) + q(k)f\left(C\left[\frac{k+g(k)}{2}, g(k)\right]\right) f\left(w^{1/\alpha}\left[\frac{k+g(k)}{2}\right]\right) = 0 \tag{2.42}$$

are oscillatory, then equation (1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (1.1). As in the proof of Theorem 2.1, we consider the Cases (I), (II) and (III).

Case (I). If $L_ix(k) > 0, i = 1, 2, 3$ for $k \ge n_0 \in \mathbb{N}_0$, then (2.12) holds for $k \ge m+2 \ge n_0$. Let $y(k) = L_3x(k)$. Replacing k and m by $\sigma(k)$ and $(k + \sigma(k))/2$ respectively, in (2.12) we obtain

$$x[\sigma(k)] \ge A\left[\sigma(k), \frac{k+\sigma(k)}{2}\right] y^{1/\alpha}\left[\frac{k+\sigma(k)}{2}\right], \quad k-1 \ge n_1 \ge n_0. \tag{2.43}$$

Using (2.4) and (2.43) in equation (1.1), we get

$$\begin{aligned}\Delta y(k) &\geq p(k)h(x[\sigma(k)])\\ &\geq p(k)h\left(A\left[\sigma(k), \frac{k+\sigma(k)}{2}\right]\right)h\left(y^{1/\alpha}\left[\frac{k+\sigma(k)}{2}\right]\right), \quad k \geq n_1 \geq n_0.\end{aligned}$$

By a result in [2, 10], one can easily see that equation (2.40) has an eventually positive solution, which contradicts the hypothesis.

Case (II). If $L_i x(k) > 0$, $i = 1, 2$ and $L_3 x(k) < 0$ for $k \geq n_0 \in \mathbb{N}_0$, then (2.18) holds. Letting $j = k$ and $-L_3 x(j) = z(j)$ in (2.18), we have

$$x[g(k)] \geq f(B[g(k), n_0])f(z^{1/\alpha}[g(k)]), \quad k \geq n_1 \geq n_0. \tag{2.44}$$

Using (2.3) and (2.44) in equation (1.1), we have

$$\begin{aligned}-\Delta z(k) &\geq q(k)f(x[g(k)])\\ &\geq q(k)f(C[g(k), n_0])f(z^{1/\alpha}[g(k)]),\end{aligned}$$

or

$$\Delta z(k) + q(k)f(C[g(k), n_0])f(z^{1/\alpha}[g(k)]) \leq 0 \quad \text{for} \quad k \geq n_1.$$

By a known result in [2, 10], equation (2.41) has an eventually positive solution, which is a contradiction.

Case (III). If $(-1)^i L_i x(k) > 0$, $i = 1, 2, 3$ for $k \geq n_0 \in \mathbb{N}_0$, then (2.21) holds. Let $w(k) = -L_3 x(k)$. Substituting $(k + g(k))/2$ and $g(k)$ for k and m respectively, in (2.21) we find

$$x[g(k)] \geq C\left[\frac{k+g(k)}{2}, g(k)\right] w^{1/\alpha}\left[\frac{k+g(k)}{2}\right] \quad \text{for} \quad k \geq n_1 \geq n_0.$$

Using (2.3) and the above inequality in equation (1.1), we have

$$\begin{aligned}-\Delta w(k) &\geq q(k)f(x[g(k)])\\ &\geq q(k)f\left(C\left[\frac{k+g(k)}{2}, g(k)\right]\right)f\left(w^{1/\alpha}\left[\frac{k+g(k)}{2}\right]\right), \quad k \geq n_1.\end{aligned}$$

The rest of the proof is similar to that of Case (II) and hence omitted. This completes the proof. ■

The following corollary is immediate.

Corollary 2.5. Let conditions (i)–(iv), (2.2)–(2.4) hold,

$$\int^{\pm\infty} \frac{du}{h(u^{1/\alpha})} < \infty \tag{2.45}$$

and

$$\int_{\pm 0} \frac{du}{f(u^{1/\alpha})} < \infty. \tag{2.46}$$

If

$$\sum^{\infty} p(j)h\left(A\left[\sigma(j), \frac{j+\sigma(j)}{2}\right]\right) = \infty, \tag{2.47}$$

$$\sum^{\infty} q(j)f(B[g(j), n_0]) = \infty, \quad k \geq n_0 \tag{2.48}$$

and

$$\sum^{\infty} q(j)f\left(C\left[\frac{j+g(j)}{2}, g(j)\right]\right) = \infty, \tag{2.49}$$

then equation (1.1) is oscillatory.

Example 2.6. As an illustrative example, we consider the equation

$$\Delta^2\left(\frac{1}{a(k)}(\Delta^2 x(k))^\alpha\right) = q(k)x^\beta[g(k)] + p(k)x^\gamma[\sigma(k)], \tag{2.50}$$

where β and γ are the ratios of two positive odd integers with $0 < \beta < \alpha < \gamma$. Equation (2.50) is oscillatory if

$$\sum^{\infty} p(j)A^\gamma\left[\sigma(j), \frac{j+\sigma(j)}{2}\right] = \infty,$$

$$\sum^{\infty} q(j)B^\beta[g(j), n_0] = \infty \quad \text{and} \quad \sum^{\infty} q(j)B^\beta\left[\frac{j+g(j)}{2}, g(j)\right] = \infty.$$

Next for a special case of equation (1.1), namely, the equation

$$\Delta^2\left(\frac{1}{a(k)}(\Delta^2 x(k))^\alpha\right) = q(k)x^\alpha[k-\tau+1] + p(k)x^\alpha[k+\sigma+1], \tag{2.51}$$

where τ and $\sigma > 2$ are integers, we have the following corollary.

Corollary 2.7. Equation (2.51) is oscillatory if

$$\liminf_{k\to\infty} \sum_{j=k+1}^{k-1+(\sigma/2)} p(j)A^\alpha\left[j+\sigma, j+\frac{\sigma}{2}\right] > \left(\frac{\sigma-2}{\sigma}\right)^{\sigma/2}, \tag{2.52}$$

$$\liminf_{k\to\infty} \sum_{j=k-\tau}^{k-1} q(j)B^\alpha[j-\tau, n_0] > \left(\frac{\tau}{\tau+1}\right)^{\tau+1} \tag{2.53}$$

and

$$\liminf_{k\to\infty} \sum_{j=k-\tau/2}^{k-1} q(j)C^\alpha\left[j-\frac{\tau}{2}, j-\tau\right] > \left(\frac{\tau}{\tau+2}\right)^{(\tau+2)/2}. \tag{2.54}$$

Remark 2.8.

1. From Theorems 2.3 and 2.4 we see that the oscillation of equation (1.1) is related to those of first or second order equations whose oscillatory characters are studied intensively in [1,2,5,10]. These criteria in [2,10] may be applied to the equations given in Theorems 2.3 and 2.4 and produce many results for the oscillation of equation (1.1). The details are left to the reader.

2. The results of this paper can be extended to neutral type equations of the form
$$\Delta^2\left(\frac{1}{a(k)}(\Delta^2(x(k)+c(k)x[\tau(k)]))^\alpha\right)=q(k)f(x[g(k)])+p(k)h(x[\sigma(k)]), \tag{2.55}$$
where $\{c(k)\}$ is a sequence of real numbers and $\{\tau(k)\}$ is an increasing sequence of real numbers with $\lim_{k\to\infty}\tau(k)=\infty$. The details are left to the reader and for similar results we refer to our results in [6] for second order nonlinear equations.

3. We note that the bounded oscillation of a special case of equation (1.1), namely, the equation
$$\Delta^2\left(\frac{1}{a(k)}(\Delta^2 x(k))^\alpha\right)=q(k)f(x[g(k)]), \tag{2.56}$$
can be investigated from the following result.

Corollary 2.9. Let conditions (i)–(iv) hold with $p(k)\equiv 0$ and $h(x)\equiv 0$. Then all bounded solutions of equation (2.56) are oscillatory, if one of the following hold:

(I_1). Conditions (2.3), (2.6) and (2.9) hold,

(I_2). Conditions (2.25) and (2.28) hold,

(I_3). Condition (2.3) holds and all bounded solutions of equation (2.31) are oscillatory,

(I_4). Condition (2.3) holds and equation (2.42) is oscillatory.

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