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Oscillation and Asymptotic Stability of Difference Equations Depending on Parameters*

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Abstract

In this paper we study oscillation and asymptotic stability of the difference equation of the form

$$u_{n+1} = au_n + bu_{n-\tau} + cu_{n-\sigma},$$

where a, b and c are real parameters, τ and σ are positive integers. We find the asymptotically stable domain of this equation. When the tuple (a, b, c) belongs to this domain, the equation is asymptotically stable and the inverse is also true. In addition, we provide necessary and sufficient conditions for the equation to be oscillatory.

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1. Introduction

The aim of this work is to investigate oscillation and asymptotic stability of the difference equation

$$u_{n+1} = au_n + bu_{n-\tau} + cu_{n-\sigma}. \tag{1.1}$$

The asymptotic stability and oscillation are main topics for difference equations, and many researchers have been attracted. In [8] Lin considered asymptotic stability for the difference equation (1.1) when $c = 0$, and obtained the asymptotically stable domain of the equation he focused on. When $c \neq 0$, is the result of [8] also true? In this paper, we shall answer this question. We find that when $c \neq 0$, things are different to some extend.

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We not only consider the asymptotically stable domain of (1.1), but also necessary and sufficient conditions of oscillation for (1.1). Other works related to difference equations can be found in [2–10, 12] and references therein.

Definition 1.1. The difference equation (1.1) is asymptotically stable if each solution converges to 0 as $n \to \infty$. For given $\tau, \sigma \in \mathbb{N}$, the set of all tuples (a, b, c) such that the equation (1.1) is asymptotically stable is called the asymptotically stable set of (1.1), and denoted by $D(a, b, c|\tau, \sigma)$. The set $\bigcap_{\tau,\sigma\in\mathbb{N}} D(a, b, c|\tau, \sigma)$ is called the asymptotically stable domain of (1.1).

It is well known that (1.1) is asymptotically stable if and only if all roots of the equation

$$\lambda^{n+1} = a\lambda^n + b\lambda^{n-\tau} + c\lambda^{n-\sigma}$$

or

$$1 = a\lambda^{-1} + b\lambda^{-\tau-1} + c\lambda^{-\sigma-1} \tag{1.2}$$

are in the unit ball, i.e., $|\lambda| < 1$ [8].

For convenience, we list some conditions which will be used later.

(L1) $|a| + |b| + |c| < 1$.

(L2) $|a| > |b| + |c| + 1$.

(L3) $|b| > |a| + |c| + 1$.

(L4) $|c| > |a| + |b| + 1$.

(L5) $a > 0$, $b < 0$, $c \leq 0$.

(L6) $a > 0$, $bc < 0$.

(L7) $a > 0$, $b = 0$, $c < 0$.

Lemma 1.2. Suppose that a, b and c are real numbers with $a^2 + b^2 + c^2 \neq 0$. Then the equation

$$|a|x^{-1} + |b|x^{-\tau-1} + |c|x^{-\sigma-1} = 1, \tag{1.3}$$

has only one root in the interval $(0, \infty)$.

Proof. Set

$$f(x) = |a|x^{-1} + |b|x^{-\tau-1} + |c|x^{-\sigma-1}, \quad x > 0.$$

By continuity of f in $(0, \infty)$, $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 0$, and the fact

$$f'(x) = -|a|x^{-2} - (\tau + 1)|b|x^{-\tau-2} - (\sigma + 1)|c|x^{-\sigma-2},$$

together with the intermediate value theorem, there exists a number $\xi \in (0, \infty)$ such that $f(\xi) = 1$. The proof is complete. ∎

Lemma 1.3. Suppose that a, b and c are real numbers with $a^2 + b^2 + c^2 \neq 0$, and that $\tau, \sigma \in \mathbb{N}$. Then for the equation

$$az + bz^{\tau+1} + cz^{\sigma+1} = 1, \quad z \in \mathbb{C}, \tag{1.4}$$

where $\mathbb{C}$ is the set of all complex numbers, the following results are true:

(1) If (L1) holds, then (1.4) has no roots satisfying $|z| < 1$.

(2) If (L2) holds, then (1.4) has one root satisfying $|z| < 1$.

(3) If (L3) holds, then (1.4) has $\tau + 1$ roots satisfying $|z| < 1$.

(4) If (L4) holds, then (1.4) has $\sigma + 1$ roots satisfying $|z| < 1$.

The proof is just based on the fundamental theorem of complex function theory, so here is omitted.

2. Asymptotically Stable Domain

Throughout this section, we always assume a, b and c are real numbers such that $a^2 + b^2 + c^2 \neq 0$.

Theorem 2.1. Suppose that

$$|a| + |b| + |c| < 1.$$

Then the difference equation (1.1) is asymptotically stable.

Proof. Note that all roots of the equation (1.4) are not in the set $\{z \in \mathbb{C} : |z| < 1\}$. An application of Lemma 1.3 concludes the proof. ■

Theorem 2.2. Suppose that the condition (L2), (L3), and (L4) is satisfied, respectively. Then the equation (1.1) has one, $\tau + 1$ and $\sigma + 1$ roots outside of the unit ball $\{z \in \mathbb{C} : |z| < 1\}$, respectively. Furthermore, the equation (1.1) is not asymptotically stable.

The proof is a direct result of Lemma 1.3.

Theorem 2.3. Suppose that $(a, b, c) \in D(a, b, c|\tau, \sigma)$ are such that (L5)–(L7) are not satisfied. Then the equation (1.1) is asymptotically stable if and only if

$$\bigcap_{\tau,\sigma \in \mathbb{N}} D(a, b, c|\tau, \sigma) = \{(a, b, c) : |a| + |b| + |c| < 1\}.$$

Proof. Sufficiency is obvious from Theorem 2.1.

Necessity. Suppose that (1.1) is asymptotically stable and μ is a root of (1.2) with $|\mu| < 1$. Set $\mu = re^{i\theta}$. The equation (1.2) is equivalent to the two equations

$$ar^{-1} \cos\theta + br^{-\tau-1} \cos(\tau + 1)\theta + cr^{-\sigma-1} \cos(\sigma + 1)\theta = 1 \tag{2.1}$$

and

$$ar^{-1}\sin\theta + br^{-\tau-1}\sin(\tau+1)\theta + cr^{-\sigma-1}\sin(\sigma+1)\theta = 1. \tag{2.2}$$

We divide the proof into four parts: (1) $a = 0$; (2) $b = 0$; (3) $c = 0$; (4) $abc \neq 0$. The proof of part (1) can be seen in [8]. For the part (2), we consider the following four cases: (i) $b = 0, c = 0$ and $a \neq 0$. Taking $\theta = 0$, by (2.1), we have

$$|a\mu^{-1}\cos\theta| = |a\mu^{-1}| = 1,$$

which implies that $|a| = |\mu| < 1$. When $b = 0, a = 0$ and $c \neq 0$, $|c| < 1$ can be obtained similarly. (ii) $b = 0$, $a > 0, c > 0$. By Lemma 1.2, the equation $a\mu^{-1} + c\mu^{-\sigma-1} = 1$ has a unique positive root ρ_1. Pick a solution of the equations (2.1) and (2.2) $(r,\theta) = (\rho_1, 0)$. Then

$$1 = a\rho_1^{-1} + c\rho_1^{-\sigma-1} > a + c = |a| + |c|.$$

(iii) $b = 0$, $a < 0, c > 0$. By Lemma 1.2, the equation $-a\mu^{-1} + c\mu^{-\sigma-1} = 1$ has a unique positive root ρ_2. Pick a solution of the equations (2.1) and (2.2) $(r,\theta) = (\rho_2, \pi)$. Then

$$1 = -a\rho_2^{-1} + c\rho_2^{-\sigma-1} > -a + c = |a| + |c|.$$

(iv) $b = 0$, $a < 0, c < 0$. In this case, the proof is just similar to the case (iii). A similar argument can prove the part (3).

Next, we prove part (4). We consider five cases. (i) $a, b, c > 0$; (ii) $a < 0, b, c > 0$; (iii) $a, c < 0, b > 0$; (iv) $a, b < 0, c > 0$; (v) $a, b, c < 0$. We only consider the case (i) and case (ii), since the others can be verified in similar ways. (i) Under this condition, by Lemma 1.2, the equation $a\mu^{-1} + b\mu^{-\tau-1} + c\mu^{-\sigma-1} = 1$ has a root $\rho_4 > 0$, which implies that $(r,\theta) = (\rho_4, 0)$ is a solution of (2.1) and (2.2). Thus

$$1 = a\rho_4^{-1} + b\rho_4{-\tau-1} + c\rho_4^{-\sigma-1} > a + b + c = |a| + |b| + |c|.$$

(ii) In this case, the equation $-a\mu^{-1} + b\mu^{-\tau-1} + c\mu^{-\sigma-1} = 1$ has a root $\rho_5 > 0$, which implies that $(r,\theta) = (\rho_5, \pi)$ is a solution of (2.1) and (2.2). Thus

$$1 = -a\rho_5^{-1} + b\rho_5{-\tau-1} + c\rho_5^{-\sigma-1} > -a + b + c = |a| + |b| + |c|.$$

The proof is complete. ■

3. Oscillation

In this section, we discuss oscillation of the equation (1.1). It is known that (1.1) is oscillatory if and only if the equation (1.2) has no positive solutions.

Theorem 3.1. If $b, c < 0$, then (1.1) is oscillatory if and only if

$$\lambda_0 - a - b\lambda_0^{-\tau} - c\lambda_0^{-\sigma} > 0,$$

where λ_0 is the unique positive root of the equation

$$1 + b\tau\lambda^{-\tau-1} + c\sigma\tau\lambda^{-\sigma-1} = 0. \tag{3.1}$$

Proof. Sufficiency. Let $F(\lambda) = \lambda - a - b\lambda^{-\tau} - c\sigma\lambda^{-\sigma}$. Note that $F'(\lambda) = 1 + b\tau\lambda^{-\tau-1} + c\sigma\tau\lambda^{-\sigma-1}$ and $F''(\lambda) = -b\tau(\tau+1)\lambda^{-\tau-2} - c\sigma(\sigma+1)\lambda^{-\sigma-2} > 0$, which imply that $F'(\lambda)$ is continuous and increasing in $(0,\infty)$. It follows from $\lim_{\lambda\to 0^+} F'(\lambda) = -\infty$ and $\lim_{\lambda\to\infty} F''(\lambda) = 1$ that there exists $\lambda_0 > 0$ such that $F'(\lambda_0) = 0$, i.e., λ_0 is the unique root of the equation (3.1). Since $F'(\lambda_0) = 0$ and $F''(\lambda_0) > 0$, it follows that $F(\lambda)$ attains its minimum at λ_0. So, $\lambda_0 - a - b\lambda_0^{-\tau} - c\sigma\lambda_0^{-\sigma} > 0$, i.e., $F(\lambda_0) > 0$ leads to the fact $F(\lambda) > 0$ for $\lambda \in (0,\infty)$.

Necessity. Assume (1.1) is oscillatory and

$$\lambda_0 - a - b\lambda_0^{-\tau} - c\sigma\lambda_0^{-\sigma} \le 0,$$

where λ_0 is the unique root of the equation (3.1). Since $F(\lambda_0) \le 0$ and $\lim_{\lambda\to\infty} F(\lambda) = \infty$, there exists $\lambda_* > 0$ such that $F(\lambda_*) = 0$. That is, (1.2) has a positive root λ_*, which is a contradiction. The proof is complete. ■

Theorem 3.2. If $b, c < 0$, then (1.1) is oscillatory if and only if there exist A_1 and A_2 with $A_1, A_2 \in [0,1]$ and $A_1 + A_2 = 1$ such that

$$\left(-bA_1^{\tau}\frac{(1+\tau)^{\tau+1}}{\tau^{\tau}}\right)^{\frac{1}{1+\tau}} + \left(-cA_2^{\sigma}\frac{(1+\sigma)^{\sigma+1}}{\sigma^{\sigma}}\right)^{\frac{1}{1+\sigma}} > a. \tag{3.2}$$

Proof. Sufficiency. Let $f_1(\lambda) = A_1\lambda - b\lambda^{-\tau}$, and $f_2(\lambda) = A_2\lambda - c\lambda^{-\sigma}$. Then $f_1(\lambda), f_2(\lambda) > 0$ for $b, c < 0$ and $\lambda > 0$. We know that $f_1(\lambda)$, $f_2(\lambda)$ attains its minimum $\left(-bA_1^{\tau}\frac{(1+\tau)^{\tau+1}}{\tau^{\tau}}\right)^{\frac{1}{1+\tau}}$ and $\left(-cA_2^{\sigma}\frac{(1+\sigma)^{\sigma+1}}{\sigma^{\sigma}}\right)^{\frac{1}{1+\sigma}}$ at $\lambda_1 = \left(-\frac{b\tau}{A_1}\right)^{\frac{1}{1+\tau}}$ and $\lambda_2 = \left(-\frac{c\sigma}{A_2}\right)^{\frac{1}{1+\sigma}}$, respectively. Thus

$$\begin{aligned} F(\lambda) &= \lambda - a + f_1(\lambda) + f_2(\lambda) - (A_1 + A_2)\lambda \\ &= -a + f_1(\lambda) + f_2(\lambda) \\ &\ge -a + f_1(\lambda_1) + f_2(\lambda_2) \\ &\ge -a + \left(-bA_1^{\tau}\frac{(1+\tau)^{\tau+1}}{\tau^{\tau}}\right)^{\frac{1}{1+\tau}} \left(-cA_2^{\sigma}\frac{(1+\sigma)^{\sigma+1}}{\sigma^{\sigma}}\right)^{\frac{1}{1+\sigma}} \\ &> -a + a = 0, \end{aligned}$$

which implies that $F(\lambda)$ has no positive roots.

Necessity. Assume that (1.1) is oscillatory and λ_0 is the unique positive root of the equation (3.1). In view of Theorem 3.1,

$$\lambda_0 - a - b\lambda_0^{-\tau} - c\sigma\lambda_0^{-\sigma} > 0.$$

Taking $A_1 = -b\tau\lambda_0^{-\tau-1}$ and $A_2 = -c\sigma\lambda_0^{-\sigma-1}$, we have $A_1 + A_2 = 1$. Furthermore,

$$\begin{aligned}
&\left(-bA_1^{\tau}\frac{(1+\tau)^{\tau+1}}{\tau^{\tau}}\right)^{\frac{1}{1+\tau}} + \left(-cA_2^{\sigma}\frac{(1+\sigma)^{\sigma+1}}{\sigma^{\sigma}}\right)^{\frac{1}{1+\sigma}} \\
&= \left((-b(\tau\lambda_0^{-\tau-1})^{\tau}(-b)\frac{(1+\tau)^{\tau+1}}{\tau^{\tau}}\right)^{\frac{1}{1+\tau}} \\
&\quad + \left((-c\sigma\lambda_0^{-\sigma-1})^{\sigma}(-c)\frac{(1+\sigma)^{\sigma+1}}{\sigma^{\sigma}}\right)^{\frac{1}{1+\sigma}} \\
&= -b\lambda_0^{-\tau}(1+\tau) - c\lambda_0^{-\sigma}(1+\sigma) \\
&= (-b\lambda_0^{-\tau} - c\lambda_0^{-\sigma}) + (-b\tau\lambda_0^{-\tau} - c\sigma\lambda_0^{-\sigma}) \\
&> a - \lambda_0 + \lambda_0 = a.
\end{aligned}$$

The proof is complete. ■

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