International Journal of Difference Equations.
ISSN 0973-6069 Volume 1 Number 2 (2006), pp. 241–247

http://www.ripublication.com/ijde.htm

Inverse Nodal Problems for Second Order Differential Operators with a Regular Singularity

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Abstract

The inverse nodal problem for the Sturm–Liouville operator is the problem of finding the potential function q and the boundary conditions using the nodal points. The purpose of this paper is to present a method for solving the inverse nodal problem for a singular differential operator on a finite interval. We find asymptotic formulas for nodal points and the nodal lengths for differential operators having singularity type $\dfrac{l(l+1)}{x^2} + \dfrac{2}{x}$ at the point 0. We also determine the potential function from the position of nodes.

AMS subject classification: 34B24, 34B99.
Keywords: Singular differential equation, eigenfunction, nodal point.

1. Introduction

The inverse Sturm–Liouville problem is primarily a model problem. Typically, in an inverse eigenvalue problem, one measures the frequencies of a vibration system and tries to infer some physical properties of the system. There are several versions of the inverse Sturm–Liouville problem. An early important result in this direction, which gave vital impetus for the further development of inverse problem theory, was obtained [1]. At present, inverse problems are studied for certain special classes of ordinary differential operators. An effective method of constructing a regular and singular Sturm–Liouville operator from a spectral function or from two spectra are given [10,11,20]. We note that

Received March 11, 2006; Accepted June 5, 2006

details of inverse problems for singular equations have been given in [6, 7, 15, 18, 23] and the references therein.

In some recent interesting works [4, 12], Hald and McLaughlin, Browne and Sleeman have taken a new approach to inverse spectral theory for the Sturm–Liouville problem. The novelty of this works lies in the use of nodal points as the given spectral data. In later years, inverse nodal problems have been studied by several authors [4, 5, 14, 17, 22].

Before giving the main results of this paper, we mention some known results. We will consider the equation

$$R'' + \frac{2}{x}R' - \frac{l(l+1)}{x^2}R + \left(E + \frac{2}{x}\right)R = 0, \tag{1.1}$$

where $l \in \mathbb{N}_0$.

In quantum mechanics, the study of the energy levels of the hydrogen atom leads to this equation. The substitution $R = \dfrac{y}{x}$ reduces this equation to the form

$$y'' + \left\{E + \frac{2}{x} - \frac{l(l+1)}{x^2}\right\} y = 0. \tag{1.2}$$

As known [3, 9, 16] the solution of (1.2) is bounded at zero.

We consider the singular Sturm–Liouville problems

$$-y'' + \left[\frac{l\,(l+1)}{x^2} - \frac{2}{x} + q\,(x)\right] y = \lambda y \quad (0 < x \le \pi)\,, \tag{1.3}$$

$$y(0) = 0, \quad y'(\pi) + Hy(\pi) = 0, \tag{1.4}$$

and

$$-y'' + \left[\frac{l\,(l+1)}{x^2} - \frac{2}{x} + \tilde{q}\,(x)\right] y = \lambda y \quad (0 < x \le \pi)\,, \tag{1.5}$$

$$y\,(0) = 0, \quad y'\,(\pi) + \widetilde{H} y\,(\pi) = 0, \tag{1.6}$$

in which the functions $q(x)$ and $\tilde{q}(x)$ are assumed to be real valued and square integrable. H and $\tilde{H}$ are finite real numbers. Next, we denote by $\varphi\,(x, \lambda)$ the solution of (1.3), and we denote by $\tilde{\varphi}\,(x, \lambda)$ the solution of (1.5) satisfying the initial conditions (1.4) and (1.6), respectively.

It is well known that there exists a function $K(x, s)$ such that

$$\tilde{\varphi}\,(x, \lambda) = \varphi\,(x, \lambda) + \int_0^x K\,(x, s)\,\varphi\,(s, \lambda)\,ds. \tag{1.7}$$

The function $K(x, s)$ satisfies the equation

$$\frac{\partial^2 K}{\partial x^2} - \left[\frac{2}{x} - \frac{l\,(l+1)}{x^2} + \tilde{q}\,(x)\right] K = \frac{\partial^2 K}{\partial s^2} - \left[\frac{2}{s} - \frac{l\,(l+1)}{s^2} + q\,(s)\right] K \tag{1.8}$$

and the conditions

$$K(x,x) = \frac{1}{2}\int\limits_0^x [\tilde{q}(t) - q(t)]\,dt, \tag{1.9}$$

$$K(x,0) = 0. \tag{1.10}$$

After the transformations

$$z = \frac{1}{4}(x+s)^2, \quad w = \frac{1}{4}(x-s)^2, \quad K(x,s) = (z-w)^{-\nu+\frac{1}{2}}u(z,w),$$

we obtain the following problem $(\alpha = -l)$:

$$\frac{\partial^2 u}{\partial z \partial w} - \frac{\alpha}{z-w}\frac{\partial u}{\partial z} + \frac{\alpha}{z-w}\frac{\partial u}{\partial w} = \frac{(\tilde{q}-q)\,u}{4\sqrt{zw}} + \frac{u}{\sqrt{z(z-w)}},$$

$$u(z, z-\delta) = 0,$$

$$\frac{\partial u}{\partial z} + \frac{\alpha}{z}u = \frac{1}{4}\left(\tilde{q}\left(\sqrt{z}\right) - q\left(\sqrt{z}\right)\right)\left(\sqrt{z}\right) z^{l-\frac{1}{2}},$$

for a constant δ. This problem can be solved by using the Riemann method [21].

Let $\lambda_0(q,H) < \lambda_1(q,H) < \cdots \to \infty$ be the eigenvalues of the problem (1.3), (1.4) and $0 < x_1^n < \cdots < x_n^j < \pi$, $j = 1,2,\ldots,n-1$ be nodal points of the nth eigenfunction. It is shown that the set of all nodals point $\left\{x_n^j\right\}$ is dense in $(0,\pi)$; in fact, a judicious choice of one nodal point x_n^j for each y_n, $n > 1$ also forms a dense set in $(0,\pi)$.

2. Main Results

In this section, our purpose is to develop asymptotic expressions for the points x_n^j and l_j^n $(j = 1,2,\ldots,n-1, n \in \mathbb{N})$ at y_n the eigenfunction corresponding to the eigenvalue λ_n.

Theorem 2.1. We consider the equation

$$-y'' + \left[\frac{l(l+1)}{x^2} - \frac{2}{x} + q(x)\right] y = \lambda y \tag{2.1}$$

with the boundary conditions

$$y(0) = 0, \tag{2.2}$$

$$y'(\pi,\lambda) + Hy(\pi,\lambda) = 0. \tag{2.3}$$

Then, the nodal points and nodal lengths of the problem (2.1)–(2.3) are

$$x_n^j = \left(\frac{2j+l-1}{2n}\right)\pi + O\left(\frac{1}{n^2}\right), \tag{2.4}$$

$$l_n^j = \frac{\pi}{n} + O\left(\frac{1}{n^2}\right). \tag{2.5}$$

Proof. As known [2], the eigenvalues and eigenfunctions of the problem (2.1)–(2.3) satisfy the asymptotics

$$\sqrt{\lambda_n} = \left(n + \frac{l}{2}\right) + \frac{1}{\pi}\frac{\ln(n+\frac{l}{2})}{(n+\frac{l}{2})} + O\left(\frac{1}{n^2}\right), \quad n \to \infty. \tag{2.6}$$

The eigenfunctions of the problem (2.1)–(2.3) are in the asymptotic form given as

$$\varphi_n(x) = \cos\left[\left(n + \frac{l}{2}\right)x - \frac{l\pi}{2}\right] + O\left(\frac{1}{n}\right).$$

Hence, we use the classical estimate

$$\left|\varphi_n(x) - \cos\left[\left(n + \frac{l}{2}\right)x - \frac{l\pi}{2}\right]\right| < \frac{M}{n},$$

where M is a constant. Thus $\varphi_n(x)$ will vanish in the intervals whose end points are solutions to

$$\cos\left[\left(n + \frac{l}{2}\right)x - \frac{l\pi}{2}\right] = \pm\frac{M}{n}.$$

This equation can also be written as

$$\left(n + \frac{l}{2}\right)x - \frac{l\pi}{2} = \arccos\left(\pm\frac{M}{n}\right).$$

Using the Taylor expansion for $\arccos\left(\frac{M}{n}\right)$, we get

$$\left(n + \frac{l}{2}\right)x = \left(j - \frac{1}{2} + \frac{l}{2}\right)\pi + \frac{M}{n} + O\left(\frac{1}{n^2}\right),$$

$$x_n^j = \left(\frac{2j + l - 1}{2n}\right)\pi + O\left(\frac{1}{n^2}\right), \ (j = 1, 2, \ldots, n-1, \ n \in \mathbb{N}).$$

The nodal length is

$$l_n^{j+1} = x_n^{j+1} - x_n^j,$$

$$l_n^j = \left(\frac{2\,(j+1) + l - 1}{2n}\right)\pi - \left(\frac{2j + l - 1}{2n}\right)\pi + O\left(\frac{1}{n^2}\right),$$

$$l_n^j = \frac{\pi}{n} + O\left(\frac{1}{n^2}\right).$$

This completes the proof. ∎

Now, we will give a uniqueness theorem. It states that the potential function $q(x)$ for a singular Sturm–Liouville problem is uniquely determined by a dense subset of the nodes. We mention that this theorem was given for regular Sturm–Liouville problems by McLaughlin [17], Hald and McLaughlin [12], Browne and Sleeman [4].

Theorem 2.2. Suppose that q is integrable. Then, H and $q - \int\limits_0^\pi q$ are uniquely determined by any dense set of nodal points.

Proof. Assume that we have two problems of the type (2.1)–(2.3) with H, $\tilde{H}$ and q, $\tilde{q}$. Let the nodal points x_n^j, $\tilde{x}_n^j$ satisfying $x_n^j = \tilde{x}_n^j$ form a dense set in $(0, \pi)$. We take solutions of (2.1)–(2.3) w_n for (H, q) and $\tilde{w}_n$ for $(\tilde{H}, \tilde{q})$. It follows from (2.1) that

$$(w_n' \tilde{w}_n - w_n \tilde{w}_n')' = \left(q - \tilde{q} + \tilde{\lambda}_n - \lambda_n + \frac{l(l+1)}{x^2} + \frac{2}{x} - \frac{l(l+1)}{x^2} - \frac{2}{x}\right) w_n \tilde{w}_n. \tag{2.7}$$

Recall that $(\tilde{\lambda}_n - \lambda_n)$ are uniformly bounded in n and that $w_n \tilde{w}_n$ are uniformly bounded in n and $x \in (0, \pi)$. We select a subsequence of nodes from the dense set. To show that $H = \tilde{H}$, we integrate both sides of (2.7) from x_n^j to π and choose a subsequence that tends to π to obtain

$$(H - \tilde{H}) w_n(\pi) \tilde{w}_n(\pi) = \int\limits_{x_n^j}^{\pi} (q - \tilde{q} + \tilde{\lambda}_n - \lambda_n) w_n \tilde{w}_n dx.$$

From this results we can conclude $H = \tilde{H}$.

Let $x_n^j = \tilde{x}_n^j$ and integrate both sides of (2.7) from 0 to x_n^j to obtain

$$\int\limits_0^{x_n^j} \left(w_n' \tilde{w}_n - w_n \tilde{w}_n'\right)' dx = \int\limits_0^{x_n^j} (q - \tilde{q} + \tilde{\lambda}_n - \lambda_n) w_n \tilde{w}_n dx,$$

$$\left[w_n'\left(x_n^j\right) \tilde{w}_n\left(x_n^j\right) - w_n\left(x_n^j\right) \tilde{w}_n'\left(x_n^j\right)\right] = \int\limits_0^{x_n^j} (q - \tilde{q} + \tilde{\lambda}_n - \lambda_n) w_n \tilde{w}_n dx,$$

$$0 = \int\limits_0^{x_n^j} (q - \tilde{q} + \tilde{\lambda}_n - \lambda_n) w_n \tilde{w}_n dx.$$

From the asymptotic forms of $\tilde{\lambda}_n$ and λ_n, we have

$$0 = \int\limits_0^{x_n^j} (q - \tilde{q} + \tilde{a} - a + H - \tilde{H}) w_n \tilde{w}_n dx,$$

where

$$\begin{aligned} a &= \frac{1}{\pi}\left[-H+\ln\pi+2+\frac{1}{2}\int_0^\pi q(t)dt\right], \\ \tilde{a} &= \frac{1}{\pi}\left[-\tilde{H}+\ln\pi+2+\frac{1}{2}\int_0^\pi \tilde{q}(t)dt\right]. \end{aligned}$$

We take a sequence x_n^j accumulating at an arbitrary $x \in (0,\pi)$. Hence,

$$0=\int_0^x \left(q-\tilde{q}-\int_0^\pi (\tilde{q}-q)\,ds\right) w_n\tilde{w}_n dt,$$

and this holds for all x. We can therefore conclude that $q-\int_0^\pi q(s)ds$ is uniquely determined by a dense set of nodes. ■

Corollary 2.3. For the problem (2.1)–(2.3), the potential q is uniquely determined by a dense set of nodes and the constant a.

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