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Oscillation Criteria for Higher-Order Sublinear Neutral Delay Difference Equations with Oscillating Coefficients

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Abstract

In this paper we are concerned with the oscillation of solutions of higher-order sublinear neutral type difference equation with oscillating coefficients. We obtain some comparison criteria for oscillatory behaviour.

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1. Introduction

We consider the higher-order sublinear difference equation of the form

$$\Delta^n[y(k) + p(k)y(\tau(k))] + q(k)y^\alpha(\sigma(k)) = 0, n \in \mathbb{N} \setminus \{1\}, k \in \mathbb{N} \tag{1.1}$$

where $\alpha \in (0, 1)$ is a ratio of positive odd integers. Throughout this work, we assume that

i) p is an oscillating function with $p(k) \to 0$ as $k \to \infty$,

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ii) $q(k) \geq 0$ for $k \geq k_0$,

iii) $\tau(k) < k$ with $\tau(k) \to \infty$ as $k \to \infty$ and $\sigma(k) < k$ with $\sigma(k) \to \infty$ as $k \to \infty$.

By a solution of equation (1.1) we mean a real sequence $\{y(k)\}$ which is defined for all $k \geq \min\limits_{i \geq 0}\{\tau(i), \sigma(i)\}$ and satisfies equation (1.1) for sufficiently large k. We consider only such solutions which are nontrivial for all large k. As it is customary, a solution $y(k)$ is said to be oscillatory if the terms $y(k)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is called nonoscillatory, real-valued solution. A difference equation is called oscillatory if all of its solutions oscillate. Otherwise, it is nonoscillatory. In this paper, we restrict our attention to real-valued solutions $y(k)$.

Neutral difference equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines. Recently, much research has been done on the oscillatory and asymptotic behaviour of solutions of higher order delay and neutral delay type difference equations. But there are very few results in the case when the coefficient p is an oscillating function.

The purpose of this paper is to study oscillatory behaviour of solutions of equation (1.1). For the general theory of difference equations, one can refer to [1–5]. Many references for the applications of difference equations can be found in [4, 5].

For the sake of convenience, the function $z(k)$ is defined as

$$z(k) = y(k) + p(k)y(\tau(k)). \tag{1.2}$$

2. Some Auxiliary Lemmas

Lemma 2.1. ([2]) Let $y(k)$ be defined for $k \geq k_0 \in \mathbb{N}$, and $y(k) > 0$ with $\Delta^n y(k)$ of constant sign for $k \geq k_0$, $n \in \mathbb{N}$ and not identically zero. Then, there exists an integer $m, 0 \leq m \leq n$ with $(n+m)$ even for $\Delta^n y(k) \geq 0$ or $(n+m)$ odd for $\Delta^n y(k) \leq 0$ such that

i) $m \leq n-1$ implies $(-1)^{m+i}\Delta^i y(k) > 0$ for all $k \geq k_0, m \leq i \leq n-1$

ii) $m \geq 1$ implies $\Delta^i y(k) > 0$ for all large $k \geq k_0, 1 \leq i \leq m-1$.

Lemma 2.2. ([2]) Let $y(k)$ be defined for $k \geq k_0$, and $y(k) > 0$ with $\Delta^n y(k) \leq 0$ for $k \geq k_0$ and not identically zero. Then, there exist a large $k_1 \geq k_0$ such that

$$y(k) \geq \frac{1}{(n-1)!}(k-k_1)^{n-1}\Delta^{n-1}y(2^{n-m-1}k), \quad k \geq k_1$$

where m is defined as in Lemma 2.1. Further, if $y(k)$ is increasing, then

$$y(k) \geq \frac{1}{(n-1)!}\left(\frac{k}{2^{n-1}}\right)^{n-1}\Delta^{n-1}y(k), \quad k \geq 2^{n-1}k_1.$$

Lemma 2.3. ([6]) Let $\alpha \in (0,1)$ and $l \in \mathbb{N}$. Then the difference inequality

$$\Delta u(n) + q(n)u^{\alpha}(n-l) \leq 0$$

does not have any eventually positive solutions if

$$\sum_{s=0}^{\infty} q(s) = \infty.$$

3. Main Results

Theorem 3.1. Assume that n is even and inequality

$$\Delta z(k) + \frac{1}{(2(2^{n-1})^{n-1}(n-1)!)^{\alpha}} q(k)\sigma^{\alpha(n-1)}(k)z^{\alpha}(\sigma(k)) \leq 0 \tag{3.1}$$

has not any positive bounded solution for all sufficiently large k. Then every bounded solution of equation (1.1) is either oscillatory or tends to zero as $k \to \infty$.

Proof. Assume that equation (1.1) has a bounded nonoscillatory solution $y(k)$. Without loss of generality, assume that $y(k)$ is eventually positive (the proof is similar when $y(k)$ is eventually negative). That is, $y(k) > 0, y(\tau(k)) > 0$ and $y(\sigma(k)) > 0$ for all $k \geq k_1 \geq k_0$. Further, suppose that $y(k)$ does not tend to zero as $k \to \infty$. By (1.1) and (1.2) we have

$$\Delta^n z(k) = -q(k)y^{\alpha}(\sigma(k)), \quad (0 < \alpha < 1, \ \ k \geq k_1). \tag{3.2}$$

That is $\Delta^n z(k) < 0$. It follows that $\Delta^a z(k)$ $(a = 0, 1, 2, \ldots, n-1)$ is strictly monotone and eventually of constant sign. Since $y(k)$ is bounded, by virtue of (i) and (1.2) there is a $k_2 \geq k_1$ such that $z(k) > 0$. Because n is even, by Lemma 2.1, since $m = 1$ (otherwise, $z(k)$ is not bounded) there exists $k_3 \geq k_2$ such that for $k \geq k_3$

$$(-1)^{i+1}\Delta^i z(k) > 0 \quad (i = 0, 1, 2, \ldots, n-1). \tag{3.3}$$

In particular, since $\Delta z(k) > 0$ for $k \geq k_3$, $z(k)$ is increasing. Since $y(k)$ is bounded, $\lim\limits_{k\to\infty} p(k)y(\tau(k)) = 0$ by (i). Then there exists a $k_4 \geq k_3$ by (1.2) such that

$$y(k) = z(k) - p(k)y(\tau(k)) \geq \frac{1}{2}z(k) > 0$$

for $k \geq k_4$. We may find a $k_5 \geq k_4$ such that for $k \geq k_5$ we have $y(\sigma(k)) \geq \dfrac{1}{2}z(\sigma(k)) > 0$ and

$$y^{\alpha}(\sigma(k)) \geq \left(\frac{1}{2}z(\sigma(k))\right)^{\alpha} > 0, \quad 0 < \alpha < 1. \tag{3.4}$$

From (3.2) and (3.4) we obtain the result of

$$\Delta^n z(k) + q(k)\left(\frac{1}{2}z(\sigma(k))\right)^{\alpha} \leq 0, \quad 0 < \alpha < 1 \tag{3.5}$$

for all large $k \geq k_5$. By Lemma 2.2, inequality (3.5) can be written as

$$\Delta^n z(k) + \frac{1}{(2(2^{n-1})^{n-1}(n-1)!)^{\alpha}} q(k)\sigma^{\alpha(n-1)}(k)(\Delta^{n-1}z(\sigma(k)))^{\alpha} \leq 0 \tag{3.6}$$

for all $k \geq k_5$. Let us take $u(k)$ as $\Delta^{n-1}z(k)$, i.e., $u(k) = \Delta^{n-1}z(k)$ in (3.6). Thus $u(k)$ satisfies for large enough k

$$\Delta u(k) + \frac{1}{(2(2^{n-1})^{n-1}(n-1)!)^{\alpha}} q(k)\sigma^{\alpha(n-1)}(k)u^{\alpha}(\sigma(k)) \leq 0. \tag{3.7}$$

Inequality (3.7) does not have any eventually positive solutions by Lemma 2.3 and (3.1). This contradicts the fact that $\Delta^{n-1}z(k) > 0$ by (3.3). In the case where $y(k)$ is an eventually negative solution, then $-y(k)$ will be an eventually positive solution. The proof of Theorem 3.1 is completed. ■

Theorem 3.2. Assume that n is odd and inequality

$$\Delta z(k) + \frac{(k-k_3)^{\alpha(n-1)}}{(2(n-1)!)^{\alpha}} q(k)z^{\alpha}(\sigma(k)) \leq 0 \tag{3.8}$$

has not any positive bounded solution for all sufficiently large k. Then every bounded solution of equation (1.1) is either oscillatory or tends to zero as $k \to \infty$.

Proof. Assume that equation (1.1) has a bounded nonoscillatory solution $y(k)$. Without loss of generality, assume that $y(k)$ is eventually positive (the proof is similar when $y(k)$ is eventually negative). That is, $y(k) > 0$, $y(\tau(k)) > 0$ and $y(\sigma(k)) > 0$ for all $k \geq k_1 \geq k_0$. Further, suppose that $y(k)$ does not tend to zero as $k \to \infty$. As in the proof of Theorem 3.1, we can find that $z(k)$ is bounded. Because n is odd, by Lemma 2.1 since $m = 0$ (otherwise, $z(k)$ is not bounded) there exists $k_1 \geq k_0$ such that

$$(-1)^i \Delta^i y(k) > 0, \ (i = 0, 1, 2, \ldots, n-1) \tag{3.9}$$

for all $k \geq k_1$. In particular, since $\Delta z(k) < 0$ for $k \geq k_1$, $z(k)$ is decreasing. Since $y(k)$ is bounded, $\lim_{k\to\infty} p(k)y(\tau(k)) = 0$ by (i). Then there exists a $k_2 \geq k_1$ by (1.2) such that $y(k) = z(k) - p(k)y(\tau(k)) \geq \frac{1}{2}z(k) > 0$ for $k \geq k_2$. We may find a $k_3 \geq k_2$ such that for $k \geq k_3$ we have $y(\sigma(k)) \geq \frac{1}{2}z(\sigma(k)) > 0$ and

$$y^{\alpha}(\sigma(k)) \geq \left(\frac{1}{2}z(\sigma(k))\right)^{\alpha} > 0, \ 0 < \alpha < 1. \tag{3.10}$$

From (3.2) and (3.10) we can obtain the result of $\Delta^n z(k) + \frac{1}{2^\alpha} q(k) z^\alpha(\sigma(k)) \le 0$ for all large $k \ge k_3$. Since $z(k)$ is decreasing, we can write this last inequality in the form

$$\Delta^n z(k) + \frac{1}{2^\alpha} q(k) z^\alpha(k) \le 0. \tag{3.11}$$

By Lemma 2.2, inequality (3.11) can be written as

$$\Delta^n z(k) + \frac{(k-k_3)^{\alpha(n-1)}}{(2(n-1)!)^\alpha} q(k) \Delta^{n-1} z^\alpha(k) \le 0. \tag{3.12}$$

Let us take $u(k)$ as $\Delta^{n-1} z(k)$, i.e., $u(k) = \Delta^{n-1} z(k)$ in the inequality (3.12). Thus $u(k)$ satisfies for large enough k, $\Delta u(k) + \frac{(k-k_3)^{\alpha(n-1)}}{[2(n-1)!]^\alpha} q(k) u^\alpha(k) \le 0$ which does not have any eventually positive solutions by Lemma 2.3 and (3.8). This contradicts the fact that $\Delta^{n-1} z(k) > 0$ by (3.9). In the case where $y(k)$ is an eventually negative solution, then $-y(k)$ will be an eventually positive solution. The proof of Theorem 3.2 is completed. ■

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