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Discrete q-Hermite Polynomials are Linked by the Integral and Finite Fourier Transforms

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Abstract

It is shown that the classical Fourier integral transform links the discrete q-Hermite polynomials $h_n(x;q)$ of type I and $\widetilde{h}_n(x;q)$ of type II. This Fourier integral transform is then used to prove that the same two q-families of polynomials are also interrelated by the finite Fourier transform.

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1. Introduction

The finite Fourier transform is defined as an action of the operator (or the equivalent $N \times N$ unitary matrix)

$$A_{jk}^{(N)} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi \mathrm{i}}{N}\, jk\right), \qquad 0 \le j,k \le N-1. \tag{1.1}$$

Given a complex valued function $f(k)$, one can compute another function $\widetilde{f}(j)$,

$$\widetilde{f}(j) := \sum_{k=0}^{N-1} A_{jk}^{(N)}\, f(k),$$

which is called *the finite Fourier transform* of the function $f(k)$. Those N functions $f^{(l)}(k)$, $l = 0, 1, 2, \ldots, N-1$, which satisfy the relation

$$\sum_{k=0}^{N-1} A_{jk}^{(N)}\, f^{(l)}(k) = \lambda_l\, f^{(l)}(j), \tag{1.2}$$

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are then eigenvectors of the finite Fourier transform $A^{(N)}$, associated with the eigenvalues λ_l. Since the fourth power of $A^{(N)}$ is the identity operator (or matrix), the only distinct eigenvalues among λ_l's are ± 1 and $\pm \mathrm{i}$.

The finite Fourier transform (1.1) has deep roots in classical pure mathematics and we briefly recall here only one aspect of this close connection.

It is easy to evaluate a sum of the first N terms of the geometric series, $S_N(x) := \sum_{k=0}^{N-1} x^k$, since the characteristic form of this series allows us to represent $S_N(x)$ in two different ways: as $S_N(x) = S_{N-1}(x) + x^{N-1}$ or $S_N(x) = 1 + x\, S_{N-1}(x)$. By equating then these two expressions, we deduce that

$$S_N(x) = \frac{1 - x^N}{1 - x}.$$

But if one tries to deal with the sequence $\{x^{k^n}\}_{k=0}^{\infty}$, where n is an integer greater than one, then the task of evaluating the sum $S_N^{(n)}(x) := \sum_{k=0}^{N-1} x^{k^n}$ becomes considerably more difficult.

For simplicity let us discuss here how one can treat the case $n = 2$. The trace of the matrix $A_{jk}^{(N)}$ is equal to

$$\mathrm{tr}\, A_{jk}^{(N)} := \sum_{k=0}^{N-1} A_{kk}^{(N)} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left(\frac{2\pi \mathrm{i}}{N}\, k^2\right) = \frac{1}{\sqrt{N}}\, S_N^{(2)}(e^{2\pi \mathrm{i}/N}). \tag{1.3}$$

On the other hand, if we know N eigenvalues λ_j of the matrix $A_{jk}^{(N)}$, then the trace of $A_{jk}^{(N)}$ is equal to the sum $\sum_{k=0}^{N-1} \lambda_j$ of these eigenvalues. So in this way one expresses the quadratic sum $S_N^{(2)}\left(e^{2\pi \mathrm{i}/N}\right)$ through the eigenvalues λ_j of the finite Fourier transform matrix $A_{jk}^{(N)}$.

Gauss had studied and solved the problem of the eigenvalues of the finite Fourier transform. He established that

$$S_N^{(2)}\left(e^{2\pi \mathrm{i}/N}\right) = \begin{cases} \sqrt{N}, & \text{if} \quad N \equiv 1\,(mod\ 4), \\ \mathrm{i}\sqrt{N}, & \text{if} \quad N \equiv 3\,(mod\ 4), \end{cases} \tag{1.4}$$

where N is an odd positive integer. Observe that formulas (1.3) and (1.4) provide a lucid illustration of how it is convenient to express some properties of the finite Fourier transform in terms of basic notions of number theory (see, for example, [2]).

Later Dirichlet who employed a version of the Poisson summation formula, Cauchy who offered a proof based on the transformation formula of the classical theta-function, Kronecker who used contour integration, and Schur who provided an elementary proof by evaluating determinants of matrices with elements being roots of unity, have essentially contributed into a better understanding of this intimate connection between quadratic Gauss sums and the finite Fourier transform. Many valuable mathematical and historical details may be found in [10, 12, 13, 20].

Mehta studied in [18] the eigenvalue problem (1.2) and found explicitly a set of eigenvectors $f^{(l)}(j) = F_{jl}^{(N)}$ of the form

$$F_{jk}^{(N)} := \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{N}(nN+j)^2} \, H_k \left(\sqrt{\frac{2\pi}{N}} \, (nN+j) \right), \tag{1.5}$$

where $H_k(x)$ is the Hermite polynomial of degree k in the variable x. These eigenvectors $F_{jl}^{(N)}$ correspond to the eigenvalues $\lambda_l = \mathrm{i}^l$, that is,

$$\sum_{l=0}^{N-1} A_{jl}^{(N)} \, F_{lk}^{(N)} = \mathrm{i}^k \, F_{jk}^{(N)}, \qquad 0 \le j, k \le N-1. \tag{1.6}$$

So Mehta in fact explicitly constructed the discrete analogue of the well-known continuum case where the Hermite functions $H_k(x) \exp\left(-x^2/2\right)$ are constant multiples of their own Fourier transforms:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}xy - x^2/2} \, H_n(x) \, dx = \mathrm{i}^n \, H_n(y) \; e^{-y^2/2}. \tag{1.7}$$

But the point is that there are also q-extensions of the Fourier integral transform (1.7), which interrelate certain q-polynomial families (see [7] and references therein). For instance, the continuous q-Hermite polynomials $H_n(x|q)$ of Rogers [1, 5, 19] possess the transformation property [9]

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}xy - x^2/2} \, H_n(\sin \kappa x|\, q) \; dx = \mathrm{i}^n \, q^{n^2/4} \, h_n(\sinh \kappa y|\, q) \; e^{-y^2/2} \tag{1.8}$$

with respect to the Fourier integral transform (here $q := \exp(-2\kappa^2),\ 0 \le \kappa < \infty$, and $h_n(x|q) := \mathrm{i}^{-n} \, H_n(\mathrm{i}x|\, q^{-1})$ [6]). Therefore in [8] it was shown that one may use Mehta's technique of constructing the eigenvectors $F_{jk}^{(N)}$ of the finite Fourier transform (1.1) in order to find discrete analogues of some q-extensions of (1.7) of the type (1.8). This led the authors of [8] to the understanding that the finite Fourier transform (1.1) provides also a link between continuous q-Hermite and q^{-1}-Hermite polynomials of Rogers, as well as between families of Rogers–Szegő and Stieltjes–Wigert polynomials.

It was conjectured in [8] that probably such connection exists also between discrete q-Hermite polynomials of types I and II. This work is aimed at proving that this conjecture does indeed hold. So we show in Section 2 that these two families of q-polynomials are linked by the Fourier *integral* transform. This result is then used in Section 3 in order to derive an explicit form of the *finite* Fourier transform, which connects the same two families. Section 4 concludes the paper with a brief discussion of some further research directions of interest.

Throughout our exposition we employ standard notations of the theory of special functions (see, for example, [3, 15]).

2. Fourier Integral Transform

We begin this section with the explicit formula (see [16, p. 118])

$$h_n(x;q) := x^n\,{}_2\phi_0\left(\begin{matrix} q^{-n},\, q^{1-n} \\ - \end{matrix}\;\middle|\; q^2\,;\frac{q^{2n-1}}{x^2}\right) = \sum_{k=0}^{[n/2]} c_k^{(n)}(q)\,x^{n-2k} \tag{2.1}$$

for the discrete q-Hermite polynomials $h_n(x;q)$ of type I with $0 < q \le 1$. The symbol $[a]$ in (2.1) stands for the largest integer not greater than a and the coefficients $c_k^{(n)}(q)$ in the above expansion in powers of x are equal to

$$c_k^{(n)}(q) = (-1)^k\,q^{k(2n-k)}\,\frac{(q^{-n};q)_{2k}}{(q^2;q^2)_k}. \tag{2.2}$$

To consider the values $1 \le q < \infty$, it proves convenient [6], similar to other cases of q-polynomials, to introduce the discrete q-Hermite polynomials $\widetilde{h}_n(x;q)$ of type II as (see [16, p. 119])

$$\begin{aligned}\widetilde{h}_n(x;q) &:= \mathrm{i}^{-n}\,h_n\left(\mathrm{i}\,x;q^{-1}\right) = x^n\,{}_2\phi_1\left(\begin{matrix} q^{-n},\, q^{1-n} \\ 0 \end{matrix}\;\middle|\; q^2;-\frac{q^2}{x^2}\right) \\ &= \sum_{k=0}^{[n/2]} q^{k(k+2-2n)}\,c_k^{(n)}(q)\,x^{n-2k}.\end{aligned} \tag{2.3}$$

A glance at (2.1) and (2.3) shows that the explicit expansion for $\widetilde{h}_n(x;q)$ in powers of x differs from the one for $h_n(x;q)$ only by the extra factor $q^{k(k+2-2n)}$ in the expansion coefficients for the former q-polynomials. As we shall see in what follows, this circumstance slightly simplifies our task of evaluating the Fourier integral transform between the discrete q-Hermite polynomials $h_n(x;q)$ and $\widetilde{h}_n(x;q)$.

Proposition 2.1. The discrete q-Hermite polynomials $h_n(x;q)$ and $\widetilde{h}_n(x;q)$, defined in (2.1) and (2.3), respectively, are interrelated by the integral Fourier transform of the following form:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h_n\left(\alpha\,e^{\mathrm{i}\kappa s};q\right)\,e^{\mathrm{i}st-s^2/2}\,ds = q^{n(3n-4)/4}\,\widetilde{h}_n\left(\alpha\,q^{1-\frac{n}{2}}\,e^{-\kappa t};q\right)\,e^{-t^2/2}. \tag{2.4}$$

Proof. To prove this statement, let us evaluate the integral

$$I_n(t;q) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_n\left(\alpha\, e^{\mathrm{i}\kappa s};q\right)\, e^{\mathrm{i}st-s^2/2}\, ds, \tag{2.5}$$

where α is an arbitrary complex number. From the definition (2.1) it is plain that

$$\begin{aligned} I_n(t;q) &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{[n/2]} \alpha^{n-2k}\, c_k^{(n)}(q) \int_{-\infty}^{\infty} e^{\mathrm{i}[t+\kappa(n-2k)]s-s^2/2}\, ds \\ &= \sum_{k=0}^{[n/2]} \alpha^{n-2k}\, c_k^{(n)}(q)\, \exp\left(-\frac{1}{2}\left[t+\kappa(n-2k)\right]^2\right). \end{aligned} \tag{2.6}$$

The expression on the second line in (2.6) follows from the well-known property of the exponential function $\exp\left(-\,x^2/2\right)$ of being its own Fourier transform (i.e., formula (1.7) for $n=0$). An easy computation using (2.3) shows that

$$\begin{aligned} I_n(t;q) &= q^{n(3n-4)/4}\, e^{-\,t^2/2} \sum_{k=0}^{[n/2]} q^{k(k+2-2n)}\, c_k^{(n)}(q)\, \left(\alpha\, q^{1-\frac{n}{2}}\, e^{-\kappa t}\right)^{n-2k} \\ &= q^{n(3n-4)/4}\, \widetilde{h}_n\left(\alpha\, q^{1-\frac{n}{2}}\, e^{-\kappa t};q\right)\, e^{-\,t^2/2}. \end{aligned} \tag{2.7}$$

Combining now relations (2.5) and (2.7), results in the desired Fourier integral transform (2.4), which links the discrete q-Hermite polynomials $h_n\,(x;q)$ and $\widetilde{h}_n\,(x;q)$. This completes the proof of the proposition. ■

Observe that in the limit as the parameter q tends to 1 (or the parameter κ vanishes), we have

$$\lim_{q\to 1} h_n\left(\alpha\, e^{\mathrm{i}\kappa\, x};q\right) = \lim_{q\to 1} \widetilde{h}_n\left(\alpha\, q^{1-\frac{n}{2}}\, e^{-\kappa y};q\right) = \alpha^n. \tag{2.8}$$

Therefore (2.4) simply reduces to the Fourier integral transform (1.7) for $n=0$ in the limit as $q\to 1$.

3. Finite Fourier Transform

Now we are in a position to define two q-extensions of Mehta's "lowest" eigenvector $F_{j0}^{(N)}$ of the form

$$\begin{aligned} F_{jk}^{(N)}\,(\alpha;q) &:= \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{N}(nN+j)^2}\, h_k\left(\alpha\, e^{\mathrm{i}\kappa\sqrt{\frac{2\pi}{N}}(nN+j)};q\right), \\ \Phi_{jk}^{(N)}\,(\beta;q) &:= \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{N}(nN+j)^2}\, \widetilde{h}_k\left(\beta\, q^{1-\frac{k}{2}}\, e^{\kappa\sqrt{\frac{2\pi}{N}}(nN+j)};q\right), \end{aligned} \tag{3.1}$$

where, as before, $0 < q := \exp\left(-2\kappa^2\right) \le 1$. In view of (2.8), in the $q \to 1$ limit we have

$$\lim_{q\to 1} F_{jk}^{(N)}(\alpha; q) = \alpha^k F_{j0}^{(N)}, \qquad \lim_{q\to 1} \Phi_{jk}^{(N)}(\beta; q) = \beta^k F_{j0}^{(N)}. \tag{3.2}$$

From the definitions (3.1) it is evident that both of $F_{jk}^{(N)}(\alpha; q)$ and $\Phi_{jk}^{(N)}(\beta; q)$ are periodic functions of j with period N. Therefore one may write the Fourier expansion

$$F_{jk}^{(N)}(\alpha; q) = \sum_{l=-\infty}^{\infty} a_{lk}^{(N)}(\alpha; q)\, \exp\left(\frac{2\pi \mathrm{i}}{N}\, lj\right), \tag{3.3}$$

with the Fourier coefficients

$$a_{lk}^{(N)}(\alpha; q) = \frac{1}{N}\int_0^N e^{-\frac{2\pi \mathrm{i}}{N}\, lx}\, F_{xk}^{(N)}(\alpha; q)\, dx. \tag{3.4}$$

Lemma 3.1. The coefficients $a_{lk}^{(N)}(\alpha; q)$ of the Fourier expansion (3.3) can be expressed in terms of the discrete q-Hermite polynomials $\widetilde{h}_k(x; q)$ of type II, defined by (2.3), as

$$a_{lk}^{(N)}(\alpha; q) = \frac{1}{\sqrt{N}}\, q^{k(3k-4)/4}\, \widetilde{h}_k\left(\alpha\, q^{1-\frac{k}{2}}\, e^{\kappa\sqrt{\frac{2\pi}{N}}\, l}; q\right) e^{-\frac{\pi}{N} l^2}. \tag{3.5}$$

Proof. To prove this Lemma, substitute the explicit form of $F_{xk}^{(N)}(\alpha; q)$ from (3.1) into (3.4) and evaluate step by step that

$$\begin{aligned}
a_{lk}^{(N)}(\alpha; q) &= \frac{1}{N}\int_0^N e^{-\frac{2\pi \mathrm{i}}{N}\, lx} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{N}(nN+x)^2}\, h_k\left(\alpha\, e^{\mathrm{i}\kappa\sqrt{\frac{2\pi}{N}}(nN+x)}; q\right) dx \\
&= \frac{1}{N}\sum_{n=-\infty}^{\infty}\int_0^N e^{-\frac{2\pi \mathrm{i}}{N}\, lx - \frac{\pi}{N}(nN+x)^2}\, h_k\left(\alpha\, e^{\mathrm{i}\kappa\sqrt{\frac{2\pi}{N}}(nN+x)}; q\right) dx \\
&= \frac{1}{\sqrt{2\pi N}}\sum_{n=-\infty}^{\infty}\int_{n\sqrt{2\pi N}}^{(n+1)\sqrt{2\pi N}} e^{-\mathrm{i}\sqrt{\frac{2\pi}{N}}\, l y_n - y_n^2/2}\, h_k\left(\alpha\, e^{\mathrm{i}\kappa\, y_n}; q\right) dy_n \\
&= \frac{1}{\sqrt{2\pi N}}\int_{-\infty}^{\infty} e^{-\mathrm{i}\sqrt{\frac{2\pi}{N}}\, l\, y - y^2/2}\, h_k\left(\alpha\, e^{\mathrm{i}\kappa\, y}; q\right) dy \\
&= \frac{1}{\sqrt{N}}\, q^{k(3k-4)/4}\, \widetilde{h}_k\left(\alpha\, q^{1-\frac{k}{2}}\, e^{\kappa\sqrt{\frac{2\pi}{N}}\, l}; q\right) e^{-\frac{\pi}{N} l^2}.
\end{aligned} \tag{3.6}$$

Thus we arrived at the desired relation (3.5) and the Lemma is proved. ■

Having expressed the Fourier coefficients $a_{lk}^{(N)}(\alpha; q)$ through the discrete q-Hermite polynomials (2.3) of type II, we are led to the following proposition.

Proposition 3.2. Two q-extensions of Mehta's "lowest" eigenvector, defined by (3.1), are interrelated by the finite Fourier transform of the form:

$$F_{jk}^{(N)}(\alpha;q) = q^{k(3k-4)/4} \sum_{m=0}^{N-1} A_{jm}^{(N)} \, \Phi_{mk}^{(N)}(\alpha;q) \,. \tag{3.7}$$

Proof. Employ the explicit form (3.5) of the Fourier coefficients $a_{lk}^{(N)}(\alpha;q)$ from Lemma 3.1 in order to show that

$$\begin{aligned} F_{jk}^{(N)}(\alpha;q) &= \frac{1}{\sqrt{N}} \, q^{k(3k-4)/4} \sum_{l=-\infty}^{\infty} e^{\frac{2\pi \mathrm{i}}{N} lj - \frac{\pi}{N} l^2} \, \widetilde{h}_k\left(\alpha \, q^{1-\frac{k}{2}} \, e^{\sqrt{\frac{2\pi}{N}} \kappa l}; q\right) \\ &= \frac{1}{\sqrt{N}} \, q^{k(3k-4)/4} \sum_{m=0}^{N-1} \sum_{n=-\infty}^{\infty} e^{\frac{2\pi \mathrm{i}}{N} (nN+m)j - \frac{\pi}{N} (nN+m)^2} \\ &\qquad \times \widetilde{h}_k\left(\alpha \, q^{1-\frac{k}{2}} \, e^{\sqrt{\frac{2\pi}{N}} \kappa (nN+m)}; q\right) \\ &= \frac{1}{\sqrt{N}} \, q^{k(3k-4)/4} \sum_{m=0}^{N-1} e^{\frac{2\pi \mathrm{i}}{N} mj} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{N} (nN+m)^2} \\ &\qquad \times \widetilde{h}_k\left(\alpha \, q^{1-\frac{k}{2}} \, e^{\sqrt{\frac{2\pi}{N}} \kappa (nN+m)}; q\right) \\ &= q^{k(3k-4)/4} \sum_{m=0}^{N-1} A_{jm}^{(N)} \, \Phi_{mk}^{(N)}(\alpha;q) \,, \end{aligned} \tag{3.8}$$

where the q-extension $\Phi_{mk}^{(N)}(\alpha;q)$ is defined in (3.1). So the proposition is proved. ∎

Evidently, one can repeat the same steps, leading from (3.3) to (3.7), but applying them this time to the $\Phi_{jk}^{(N)}(\alpha;q)$. This results in the relation

$$\Phi_{jk}^{(N)}(\alpha;q) = q^{-k(3k-4)/4} \sum_{m=0}^{N-1} \left(A^{(N)}\right)^{-1}_{jm} F_{mk}^{(N)}(\alpha;q) \,, \tag{3.9}$$

where $\left(A^{(N)}\right)^{-1}$ is the inverse operator with respect to the $A^{(N)}$. It is clear that (3.9) represents the inverse finite Fourier transform with respect to (3.7).

Thus we conclude that relations (3.7) and (3.9), written as

$$\begin{aligned} \sum_{m=0}^{N-1} A_{jm}^{(N)} \, \Phi_{mk}^{(N)}(\alpha;q) &= q^{-k(3k-4)/4} \, F_{jk}^{(N)}(\alpha;q) \,, \\ \sum_{m=0}^{N-1} \left(A^{(N)}\right)^{-1}_{jm} F_{mk}^{(N)}(\alpha;q) &= q^{k(3k-4)/4} \, \Phi_{jk}^{(N)}(\alpha;q) \,, \end{aligned} \tag{3.10}$$

respectively, describe the simple transformation properties of the discrete q-Hermite polynomials $h_n(x;q)$ of type I and $\widetilde{h}_n(x;q)$ of type II with respect to the finite Fourier transform operator $A^{(N)}$ and its inverse. In the continuous limit (i.e., when the parameter N tends to infinity) these relations coincide with the Fourier integral transform (2.4) and its inverse.

4. Concluding Remarks

We have demonstrated that the discrete q-Hermite polynomials $h_n(x;q)$ of type I and $\widetilde{h}_n(x;q)$ of type II are interrelated by the classical Fourier integral transform (2.4). Then the technique of constructing the eigenvectors of (1.1), developed by Mehta [18], has been employed in order to prove that the same two q-families of polynomials are also linked by the finite Fourier transform (3.10).

At least two directions for further study deserve attention. Perhaps the most important one is to find out whether there are other families of q-polynomials, which also possess such simple transformations properties with respect to the finite Fourier transform. The point is that all q-polynomial families, which have been already considered in [8] and in the present paper, belong to the lower level in the Askey hierarchy of basic hypergeometric polynomials [16]. Therefore it will be of interest to attempt to apply the same technique to the study of other q-families, which depend on some additional parameters (and therefore occupy the higher levels in the Askey q-scheme). For instance, Al-Salam–Carlitz polynomials $U_n^{(a)}(x;q)$ of type I and $V_n^{(a)}(x;q)$ of type II are known to depend on the parameter a and when $a=-1$ they reduce to the discrete q-Hermite polynomials $h_n(x;q)$ of type I and $\widetilde{h}_n(x;q)$ of type II, respectively (see [16, pp. 113–115]). So it will be probably natural to test the case of Al-Salam–Carlitz polynomials first.

Finally, another direction for further study is connected with q-extensions of the harmonic oscillator in quantum mechanics [4, 11, 14, 17]. We remind the reader that for proving the fundamental formula (1.6) for Mehta's eigenvectors $F_{jk}^{(N)}$ of the finite Fourier transform operator (1.1) it is vital to use the simple transformation property (1.7) of the Hermite functions $\mathcal{H}_k(x) := H_k(x)\exp(-x^2/2)$ with respect to the Fourier integral transform. Moreover, these eigenvectors $F_{jk}^{(N)}$ are actually built in terms of these Hermite functions $\mathcal{H}_k(x)$, taken at the infinite set of discrete points $x_j^{(n)} := \sqrt{\dfrac{2\pi}{N}}\,(nN+j)\,,\ 0\le j\le N-1,\ n\in\mathbb{Z}$ (cf. (1.5)). In other words, Mehta's technique of proving (1.6) is based on introducing a discrete analogue of the quantum harmonic oscillator. It seems that instances of q-polynomial families, considered in [8] and in the foregoing sections, can be similarly viewed as discrete analogues of the q-harmonic oscillator of Macfarlane and Biedenharn.

Work in both of these directions is in progress.

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