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On Absolute Summability Factors for a Triangular Matrix*

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Abstract

In this paper we obtain an absolute summability factor theorem for lower triangular matrices.

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Bor [1] obtained sufficient conditions for $\sum a_n\lambda_n$ to be $|\overline{N}, p_n|_k$ summable, $k \in \mathbb{N}$. Unfortunately he used an inappropriate definition of absolute summability (see, e.g., [3]). In this paper we obtain the corresponding summability factor theorem for a lower triangular matrix, and obtain the correct form of [1] as a special case. Let A be a lower triangular matrix, $\{s_n\}$ a sequence. Then we put

$$A_n := \sum_{\nu=0}^{n} a_{n\nu} s_\nu.$$

A series $\sum a_n$ is said to be $|A|_k$ summable, $k \in \mathbb{N}$ if

$$\sum_{n=1}^{\infty} n^{k-1}|A_n - A_{n-1}|^k < \infty.$$

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We may associate with A two lower triangular matrices $\overline{A}$ and $\hat{A}$ defined as follows:

$$\bar{a}_{n\nu} = \sum_{r=\nu}^{n} a_{nr}, \qquad n, \nu \in \mathbb{N}_0$$

and

$$\hat{a}_{n\nu} = \bar{a}_{n\nu} - \bar{a}_{n-1,\nu}, \qquad n \in \mathbb{N}.$$

Also we shall define

$$\begin{aligned} y_n &:= \sum_{i=0}^{n} a_{ni} s_i = \sum_{i=0}^{n} a_{ni} \sum_{\nu=0}^{i} \lambda_\nu a_\nu \\ &= \sum_{\nu=0}^{n} \lambda_\nu a_\nu \sum_{i=\nu}^{n} a_{ni} = \sum_{\nu=0}^{n} \bar{a}_{n\nu} \lambda_\nu a_\nu \end{aligned}$$

and

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n} (\bar{a}_{n\nu} - \bar{a}_{n-1,\nu}) \lambda_\nu a_\nu = \sum_{\nu=0}^{n} \hat{a}_{n\nu} \lambda_\nu a_\nu.$$

Theorem 1.1. Let A be a lower triangular matrix with nonnegative entries satisfying

(i) $\bar{a}_{n0} = 1$, $n \in \mathbb{N}_0$,

(ii) $a_{n-1,\nu} \geq a_{n\nu}$ for $n \geq \nu + 1$, and

(iii) $na_{nn} \asymp O(1)$,

(iv) $\sum_{\nu=1}^{n-1} a_{\nu\nu} \hat{a}_{n,\nu+1} = O(a_{nn})$.

If $\{X_n\}$ is a positive nondecreasing sequence such that

(v) $\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m)$, where $t_n := \frac{1}{n+1} \sum_{k=1}^{n} k a_k$,

(vi) $\sum_{n=1}^{m} (nX_n) |\Delta^2 \lambda_n| = O(1)$, and

(vii) $\lambda_n \to 0$ as $n \to \infty$,

then the series $\sum a_n \lambda_n$ is $|A|_k$ summable, $k \in \mathbb{N}$.

We shall need the following lemma for the proof of Theorem 1.1, where Δ denotes the forward difference operator defined by $\Delta u_n = u_n - u_{n+1}$ for any sequence $\{u_n\}$.

Lemma 1.2. [1] Under the conditions of Theorem 1.1 we have that

(1) $\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty$,

(2) $nX_n|\Delta\lambda_n| = O(1)$, and

(3) $X_n|\lambda_n| = O(1)$.

We now prove Theorem 1.1.

Proof. From (i) it follows that $\hat{a}_{n,0} = 0$. Using (2) we may write

$$\begin{aligned}
Y_n &= \sum_{\nu=1}^{n} \Big(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\Big)\nu a_\nu \\
&= \sum_{\nu=1}^{n} \Big(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\Big)\Big[\sum_{r=1}^{\nu} r a_r - \sum_{r=1}^{\nu-1} r a_r\Big] \\
&= \sum_{\nu=1}^{n-1} \Delta_\nu\Big(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\Big)\sum_{r=1}^{\nu} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n}\sum_{\nu=1}^{n} \nu a_\nu \\
&= \sum_{\nu=1}^{n-1}(\Delta_\nu \hat{a}_{n\nu})\lambda_\nu \frac{\nu+1}{\nu} t_\nu + \sum_{\nu=1}^{n-1} \hat{a}_{n,\nu+1}(\Delta\lambda_\nu)\frac{\nu+1}{\nu} t_\nu \\
&\quad + \sum_{\nu=1}^{n-1} \hat{a}_{n,\nu+1}\lambda_{\nu+1}\frac{1}{\nu} t_\nu + \frac{(n+1)a_{nn}\lambda_n t_n}{n} \\
&= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \quad \text{say}.
\end{aligned}$$

To prove the theorem it will be sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty} n^{k-1}|T_{nr}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4.$$

From the definition of $\hat{A}$ and using (i) and (ii),

$$\begin{aligned}
\hat{a}_{n,\nu+1} &= \bar{a}_{n,\nu+1} - \bar{a}_{n-1,\nu+1} \\
&= \sum_{i=\nu+1}^{n} a_{ni} - \sum_{i=\nu+1}^{n-1} a_{n-1,i} \\
&= 1 - \sum_{i=0}^{\nu} a_{ni} - 1 + \sum_{i=0}^{\nu} a_{n-1,i} \\
&= \sum_{i=0}^{\nu} \Big(a_{n-1,i} - a_{n,i}\Big) \geq 0.
\end{aligned}$$

Using Hölder's inequality and (iii) and (v),

$$\begin{aligned}
I_1 :&= \sum_{n=1}^{m} n^{k-1}|T_{n1}|^k = \sum_{n=1}^{m} n^{k-1}\left|\sum_{\nu=1}^{n-1}\Delta_\nu \hat{a}_{n\nu}\lambda_\nu \frac{\nu+1}{\nu}t_\nu\right|^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{\nu=1}^{n-1}|\Delta_\nu \hat{a}_{n\nu}||\lambda_\nu||t_\nu|\right)^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{\nu=1}^{n-1}|\Delta_\nu \hat{a}_{n\nu}||\lambda_\nu|^k|t_\nu|^k\right)\left(\sum_{\nu=1}^{n-1}|\Delta_\nu \hat{a}_{n\nu}|\right)^{k-1}.
\end{aligned}$$

From (ii)

$$\begin{aligned}
\Delta_\nu \hat{a}_{n\nu} &= \hat{a}_{n\nu} - \hat{a}_{n,\nu+1} \\
&= \bar{a}_{n\nu} - \bar{a}_{n-1,\nu} - \bar{a}_{n,\nu+1} + \bar{a}_{n-1,\nu+1} \\
&= a_{n\nu} - a_{n-1,\nu} \le 0.
\end{aligned}$$

Thus from (i)

$$\sum_{\nu=0}^{n-1}|\Delta_\nu \hat{a}_{n\nu}| = \sum_{\nu=0}^{n-1}(a_{n-1,\nu} - a_{n\nu}) = 1 - 1 + a_{nn} = a_{nn}.$$

Using the fact that, from (vii), $\{\lambda_n\}$ is bounded, and (1) and (3) of Lemma 1.2,

$$\begin{aligned}
I_1 &= O(1)\sum_{n=1}^{m+1}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}|\lambda_\nu|^k|t_\nu|^k|\Delta_\nu \hat{a}_{n\nu}| \\
&= O(1)\sum_{n=1}^{m+1}(na_{nn})^{k-1}\left(\sum_{\nu=1}^{n-1}|\lambda_\nu|^{k-1}|\lambda_\nu||\Delta_\nu \hat{a}_{n\nu}||t_\nu|^k\right) \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_\nu||t_\nu|^k\sum_{n=\nu+1}^{m+1}(na_{nn})^{k-1}|\Delta_\nu \hat{a}_{n\nu}| \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_\nu||t_\nu|^k\sum_{n=\nu+1}^{m+1}|\Delta_\nu \hat{a}_{n\nu}| \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_\nu|a_{\nu\nu}|t_\nu|^k \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_\nu|\left[\sum_{r=1}^{\nu}a_{rr}|t_r|^k - \sum_{r=1}^{\nu-1}a_{rr}|t_r|^k\right]
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\left[\sum_{\nu=1}^{m}|\lambda_\nu|\sum_{r=1}^{\nu}a_{rr}|t_r|^k - \sum_{\nu=0}^{m-1}|\lambda_{\nu+1}|\sum_{r=1}^{\nu}a_{rr}|t_r|^k\right]\\
&= O(1)\left[\sum_{\nu=1}^{m-1}\Delta(|\lambda_\nu|)\sum_{r=1}^{\nu}a_{rr}|t_r|^k + |\lambda_m|\sum_{r=1}^{m}a_{rr}|t_r|^k\right]\\
&= O(1)\left[\sum_{\nu=1}^{m-1}\Delta(|\lambda_\nu|)\sum_{r=1}^{\nu}\frac{1}{r}|t_r|^k + |\lambda_m|\sum_{r=1}^{m}\frac{1}{r}|t_r|^k\right]\\
&= O(1)\sum_{\nu=1}^{m-1}|\Delta\lambda_\nu|X_\nu + O(1)|\lambda_m|X_m\\
&= O(1).
\end{aligned}
$$

Using Hölder's inequality, (iii) and (iv),

$$
\begin{aligned}
I_2 &:= \sum_{n=2}^{m+1}n^{k-1}|T_{n2}|^k = \sum_{n=2}^{m+1}n^{k-1}\left|\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}(\Delta\lambda_\nu)\frac{\nu+1}{\nu}t_\nu\right|^k\\
&\le \sum_{n=2}^{m+1}n^{k-1}\left[\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}|\Delta\lambda_\nu|\frac{\nu+1}{\nu}|t_\nu|\right]^k\\
&= O(1)\sum_{n=2}^{m+1}n^{k-1}\left[\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}|\Delta\lambda_\nu||t_\nu|\right]^k\\
&= O(1)\sum_{n=2}^{m+1}n^{k-1}\left[\sum_{\nu=1}^{n-1}(\nu)|\Delta\lambda_\nu||t_\nu|a_{\nu\nu}\hat{a}_{n,\nu+1}\right]^k\\
&= O(1)\sum_{n=2}^{m+1}n^{k-1}\left[\sum_{\nu=1}^{n-1}(\nu|\Delta\lambda_\nu|)^k|t_\nu|^k a_{\nu\nu}\hat{a}_{n,\nu+1}\right]\left[\sum_{\nu=1}^{n-1}a_{\nu\nu}\hat{a}_{n,\nu+1}\right]^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}(\nu|\Delta\lambda_\nu|)^k|t_\nu|^k a_{\nu\nu}\hat{a}_{n,\nu+1}\\
&= O(1)\sum_{n=2}^{m+1}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}(\nu|\Delta\lambda_\nu|)^{k-1}(\nu|\Delta\lambda_\nu|)a_{\nu\nu}\hat{a}_{n,\nu+1}|t_\nu|^k.
\end{aligned}
$$

Conclusion (2) of Lemma 1.2 implies that $\nu|\Delta\lambda_\nu| = O(1)$. Therefore, using (iii)

$$
\begin{aligned}
I_2 &= O(1)\sum_{\nu=1}^{m}\nu|\Delta\lambda_\nu|a_{\nu\nu}|t_\nu|^k\sum_{n=\nu+1}^{m+1}(na_{nn})^{k-1}\hat{a}_{n,\nu+1}\\
&= O(1)\sum_{\nu=1}^{m}\nu|\Delta\lambda_\nu|a_{\nu\nu}|t_\nu|^k\sum_{n=\nu+1}^{m+1}\hat{a}_{n,\nu+1}.
\end{aligned}
$$

From (3)

$$\sum_{n=\nu+1}^{m+1}\left(\sum_{i=0}^{\nu}(a_{n-1,i}-a_{ni})\right)=\sum_{i=0}^{\nu}\sum_{n=\nu+1}^{m+1}(a_{n-1,i}-a_{ni})$$
$$=\sum_{i=0}^{\nu}(a_{\nu,i}-a_{m+1,i})$$
$$\leq\sum_{i=0}^{\nu}a_{\nu,i}=1.$$

Therefore

$$I_2=O(1)\sum_{\nu=1}^{m}\nu|\Delta\lambda_\nu|a_{n\nu}|t_\nu|^k.$$

Using summation by parts, (v) and conclusion (1) of Lemma 1.2,

$$I_2=O(1)\sum_{\nu=1}^{m}\nu|\Delta\lambda_\nu|\left[\sum_{r=1}^{\nu}a_{rr}|t_r|^k-\sum_{r=1}^{\nu-1}a_{rr}|t_r|^k\right]$$
$$=O(1)\left[\sum_{\nu=1}^{m}\nu|\Delta\lambda_\nu|\sum_{r=1}^{\nu}a_{rr}|t_r|^k-\sum_{\nu=0}^{m-1}(\nu+1)|\Delta\lambda_{\nu+1}|\sum_{r=1}^{\nu}a_{rr}|t_r|^k\right]$$
$$=O(1)\left[\sum_{\nu=1}^{m-1}\Delta(\nu|\Delta\lambda_\nu|)\sum_{r=1}^{\nu}\frac{1}{r}|t_r|^k+m|\Delta\lambda_m|\sum_{r=1}^{m}\frac{1}{r}|t_r|^k\right]$$
$$=O(1)\sum_{\nu=1}^{m-1}|\Delta(\nu\Delta\lambda_\nu)|X_\nu+O(1)m|\Delta\lambda_m|X_m.$$

But

$$\Delta(\nu\Delta\lambda_\nu)=\nu\Delta\lambda_\nu-(\nu+1)\Delta\lambda_{\nu+1}=\nu\Delta^2\lambda_\nu-\Delta\lambda_{\nu+1}.$$

Using (vi), conclusions (1) and (3) of Lemma 1.2, and the fact that $\{X_n\}$ is nondecreasing,

$$I_2=O(1)\sum_{\nu=1}^{m-1}\nu|\Delta^2\lambda_\nu|X_\nu+O(1)\sum_{\nu=1}^{m-1}|\Delta\lambda_{\nu+1}|X_{\nu+1}+O(1)m|\Delta\lambda_m|X_m=O(1).$$

Using Hölder's inequality, (4), (vii), (iii), summation by parts, (v), and conclusions (1) and (3) of Lemma 1.2,

$$\sum_{n=2}^{m+1}n^{k-1}|T_{n3}|^k=\sum_{n=2}^{m+1}n^{k-1}\left|\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}\lambda_{\nu+1}\frac{1}{\nu}t_\nu\right|^k$$

$$
\begin{aligned}
&\leq \sum_{n=2}^{m+1} n^{k-1} \left[\sum_{\nu=1}^{n-1} |\lambda_{\nu+1}| \frac{\hat{a}_{n,\nu+1}}{\nu} |t_\nu|\right]^k \\
&= O(1) \sum_{n=2}^{m+1} n^{k-1} \left[\sum_{\nu=1}^{n-1} |\lambda_{\nu+1}| \hat{a}_{n,\nu+1} |t_\nu| a_{\nu\nu}\right]^k \\
&= O(1) \sum_{n=2}^{m+1} n^{k-1} \left[\sum_{\nu=1}^{n-1} |\lambda_{\nu+1}|^k a_{\nu\nu} |t_\nu|^k \hat{a}_{n,\nu+1}\right] \left[\sum_{\nu=1}^{n-1} a_{\nu\nu} \hat{a}_{n,\nu+1}\right]^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} (n a_{nn})^{k-1} \sum_{\nu=1}^{n-1} |\lambda_{\nu+1}|^{k-1} |\lambda_{\nu+1}| a_{\nu\nu} |t_\nu|^k \hat{a}_{n,\nu+1} \\
&= O(1) \sum_{\nu=1}^{m} |\lambda_{\nu+1}| a_{\nu\nu} |t_\nu|^k \sum_{n=\nu+1}^{m+1} \hat{a}_{n,\nu+1} \\
&= O(1) \sum_{\nu=1}^{m} |\lambda_{\nu+1}| a_{\nu\nu} |t_\nu|^k \\
&= O(1) \sum_{\nu=1}^{m} |\lambda_{\nu+1}| \left[\sum_{r=1}^{\nu} a_{rr} |t_r|^k - \sum_{r=1}^{\nu-1} a_{rr} |t_r|^k\right] \\
&= O(1) \left[\sum_{\nu=1}^{m} |\lambda_{\nu+1}| \sum_{r=1}^{\nu} a_{rr} |t_r|^k - \sum_{\nu=0}^{m-1} |\lambda_{\nu+2}| \sum_{r=1}^{\nu} a_{rr} |t_r|^k\right] \\
&= O(1) \left[\sum_{\nu=1}^{m-1} |\Delta\lambda_{\nu+1}| \sum_{r=1}^{\nu} a_{rr} |t_r|^k + |\lambda_{m+1}| \sum_{r=1}^{m} a_{rr} |t_r|^k\right] \\
&= O(1) \left[\sum_{\nu=1}^{m-1} |\Delta\lambda_{\nu+1}| \sum_{r=1}^{\nu} \frac{1}{r} |t_r|^k + |\lambda_{m+1}| \sum_{r=1}^{m} \frac{1}{r} |t_r|^k\right] \\
&= O(1) \sum_{\nu=1}^{m-1} |\Delta\lambda_{\nu+1}| X_m + O(1) |\lambda_{m+1}| X_m = O(1).
\end{aligned}
$$

Finally using (iii) and (vii),

$$
\begin{aligned}
\sum_{n=1}^{m} n^{k-1} |T_{n4}|^k &= \sum_{n=1}^{m} n^{k-1} \left| \frac{(n+1) a_{nn} \lambda_n t_n}{n} \right|^k \\
&= O(1) \sum_{n=1}^{m} n^{k-1} |a_{nn}|^k |\lambda_n|^k |t_n|^k
\end{aligned}
$$

$$= O(1)\sum_{n=1}^{m}(na_{nn})^{k-1}a_{nn}|\lambda_n|^{k-1}|\lambda_n||t_n|^k$$
$$= O(1)\sum_{n=1}^{m}a_{nn}|\lambda_n||t_n|^k$$
$$= O(1)$$

as in I_1. ■

Corollary 1.3. Let $\{p_n\}$ be a positive sequence such that

$$P_n := \sum_{k=0}^{n} p_k \to \infty$$

and satisfies

(i) $np_n \asymp O(P_n)$.

If $\{X_n\}$ is a positive nondecreasing sequence such that

(ii) $\sum_{n=1}^{m}\frac{1}{n}|t_n|^k = O(X_m)$,

(iii) $\sum_{n=1}^{m} nX_n|\Delta^2\lambda_n| = O(1)$, and

(iv) $\lambda_n \to 0$ as $n \to \infty$,

then the series $\sum a_n\lambda_n$ is $|\overline{N}, p_n|_k$ summable, $k \in \mathbb{N}$.

Proof. Conditions (ii), (iii) and (iv) of Corollary 1.3 are, respectively, conditions (v), (vi) and (vii) of Theorem 1.1. Conditions (i), (ii) and (iv) of Theorem 1.1 are automatically satisfied for any weighted mean method. Condition (iii) of Theorem 1.1 becomes condition (i) of Corollary 1.3. ■

Corollary 1.4. [2] If

(i) $\sum_{n=1}^{m}\frac{1}{n}|t_k|^k = O(\log m)$,

(ii) $\sum_{n=1}^{m} n\log n|\Delta^2\lambda_n| = O(1)$, and

(iii) $\lambda_n \to 0$ as $n \to \infty$, then the series $\sum a_n \lambda_n$ is $|C, 1|_k$ summable, $k \in \mathbb{N}$.

Proof. Conditions (i) and (ii) of Corollary 1.4 are obtained by setting $X_m = \log m$ in conditions (ii) and (iii) of Corollary 1.3, respectively. Condition (iii) of Corollary 1.4 is condition (iv) of Corollary 1.3. The Cesáro matrix of order 1 is a weighted mean matrix with each $p_n = 1$. It then follows that $P_n = n + 1$, and condition (i) of Corollary 1.3 is automatically satisfied. The conclusion now follows from Corollary 1.3. ■

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