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Implicit Riccati Equations and Quadratic Functionals for Discrete Symplectic Systems*

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Abstract

In this paper we study discrete (implicit) Riccati matrix equations associated with discrete symplectic systems and related quadratic functionals $\mathcal{F}$ with variable endpoints. We derive these Riccati equations for nonnegative functionals $\mathcal{F}$ with separable and jointly varying endpoints. The result for jointly varying endpoints is in terms of the nonaugmented Riccati operator. The method also allows to simplify implicit Riccati equations known for the positivity of $\mathcal{F}$. Finally, we establish a comparison result (Riccati inequality) for solutions of Riccati equations associated with two discrete symplectic systems.

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1. Introduction and Motivation

In this paper we study *discrete symplectic systems*

$$X_{k+1} = \mathcal{A}_k X_k + \mathcal{B}_k U_k, \quad U_{k+1} = \mathcal{C}_k X_k + \mathcal{D}_k U_k, \tag{S}$$

where $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$, X_k, and U_k are real $n \times n$ matrices, and related discrete quadratic functionals and Riccati matrix equations. We assume throughout that system (S) has symplectic structure, i.e., the transition matrix $\mathcal{S}_k := \left(\begin{smallmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{smallmatrix}\right)$ is symplectic. This property is defined as $\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}$, where $\mathcal{J} := \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ is the $2n \times 2n$ skew-symmetric matrix with $n \times n$ block entries.

Discrete symplectic systems were introduced in [1] and they cover a large number of linear difference equations. More specifically, to mention a few, linear Hamiltonian difference systems

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \quad \Delta U_k = C_k X_{k+1} - A_k^T U_k \tag{H}$$

with $I - A_k$ invertible and symmetric B_k and C_k, discrete trigonometric systems, and higher order Sturm–Liouville difference equations are all special cases of system (S). Qualitative theory of Hamiltonian systems (H) with singular B_k was initiated in [4] and the results of that paper were generalized to discrete symplectic systems (in the directions discussed in this paper) in [6, 7, 9, 14].

With system (S) we consider the *discrete quadratic functional*

$$\mathcal{F}_0(x,u) := \sum_{k=0}^{N} \left\{ x_k^T \mathcal{C}_k^T \mathcal{A}_k x_k + 2\, x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{D}_k^T \mathcal{B}_k u_k \right\} \tag{1.1}$$

for admissible pairs (x, u), i.e., $x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k$, and the *Riccati operator*

$$R[Q]_k := Q_{k+1}(\mathcal{A}_k + \mathcal{B}_k Q_k) - (\mathcal{C}_k + \mathcal{D}_k Q_k),$$

where $x = \{x_k\}_{k=0}^{N+1}$ and $u = \{u_k\}_{k=0}^{N}$ are sequences of real n-vectors and Q_k are symmetric $n \times n$ matrices. From [14, Theorem 4] it is known that system (S) is the Jacobi system for the quadratic functional $\mathcal{F}_0$. Characterizations of the positivity ($\mathcal{F} > 0$) and nonnegativity ($\mathcal{F} \geq 0$) of discrete quadratic functionals $\mathcal{F}$ involving $\mathcal{F}_0$ were derived in [6, 7, 9, 11, 13, 14]. These results are in terms of certain solutions (called conjoined bases) of (S), generalized zeros and conjugate intervals of vector solutions of (S), and/or in terms of explicit and implicit Riccati matrix equations. The word "explicit" means the equation $R[Q]_k = 0$, while "implicit" means an equation involving the Riccati operator $R[Q]_k$ and some other matrices.

In this work we study in detail the latter condition. In particular, in [14, Theorem 5] we can find a characterization of a *positive definite* quadratic functional $\mathcal{F}$ with *separable endpoints* in terms of an *implicit Riccati equation* (see Proposition 3.1). In the same paper, a similar characterization of $\mathcal{F} > 0$ with *jointly varying endpoints* is given via an *augmented implicit Riccati equation* involving a certain Riccati operator $R^*[Q^*]_k$ in dimension $2n$ (see Proposition 3.3). The corresponding results regarding the *nonnegativity* of $\mathcal{F}$ are unknown.

In this paper we fill this gap and provide two additional results. In Section 3 we collect several results from the literature which we either directly need in the proofs or we wish to compare them with the results of this paper. In Section 4 we derive an implicit Riccati equation for a *nonnegative* quadratic functional $\mathcal{F}$ with separable endpoints. The main tool is utilizing the "image condition" $x_k \in \operatorname{Im} X_k$ from [7, 9]. In Section 5 we generalize the above result to a nonnegative quadratic functional $\mathcal{F}$ with jointly varying endpoints. We use a new technique which simplifies the augmented Riccati operator $R^*[Q^*]_k$, which is traditionally used for problems with general endpoints, see e.g., [4, 14]. In this way we obtain an implicit Riccati equation which involves the original Riccati operator $R[Q]_k$ in dimension n only. This technique naturally applies to a positive definite quadratic functional $\mathcal{F}$ with jointly varying endpoints. Hence, we derive in Section 6 a nonaugmented implicit Riccati equation for such positive definite quadratic functionals. Thus, the above mentioned results clarify and simplify the structure of the Riccati equation conditions used in characterizations of the definiteness of $\mathcal{F}$. At the same time, they complete the theory of discrete quadratic functionals. Our results in Sections 4–6 are new even for the special case of Hamiltonian system (H). Finally, in Section 7 we generalize a Riccati inequality result, known in [8, Theorem 1] for Hamiltonian system (H), to discrete symplectic system (S).

At the end of this section we mention that the results of this paper are valid also for complex-valued solutions and coefficients, provided that a transpose of a matrix is replaced by the conjugate transpose and the word "symmetric" is changed to "Hermitian".

2. Notation and Terminology

In this paper we denote by $[a, b]$ the discrete interval $\{a, a+1, \dots, b\}$. In particular, we shall use the intervals $[0, N]$ and $[0, N+1]$. Since any symplectic matrix is invertible (it has the determinant equal to 1), solutions of system (S) are uniquely determined by their initial values. From the formula for the inverse $\mathcal{S}_k^{-1} = \begin{pmatrix} \mathcal{D}_k^T & -\mathcal{B}_k^T \\ -\mathcal{C}_k^T & \mathcal{A}_k^T \end{pmatrix}$ of a symplectic matrix it follows that the time-reversed system takes the form

$$X_k = \mathcal{D}_k^T X_{k+1} - \mathcal{B}_k^T U_{k+1}, \quad U_k = -\mathcal{C}_k^T X_{k+1} + \mathcal{A}_k^T U_{k+1}. \tag{2.1}$$

A *conjoined basis* of (S) is a solution (X, U) such that $X_k^T U_k$ is symmetric as well as rank $\binom{X_k}{U_k} = n$ at some (and hence at any) index $k \in [0, N+1]$. The *principal solution* of (S) is the conjoined basis $(\hat{X}, \hat{U})$ starting with the initial values $\hat{X}_0 = 0$ and $\hat{U}_0 = I$. According to [6, Definition 3], a conjoined basis (X, U) of (S) has *no focal points in* $(k, k+1]$ if the two conditions

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k, \quad P_k := X_k X_{k+1}^{\dagger} \mathcal{B}_k \geq 0, \tag{2.2}$$

are satisfied, where † stands for the Moore–Penrose generalized inverse of the given matrix, see e.g., [2]. In [6, Lemma 3] it is shown that if the kernel condition in (2.2) holds, then the matrix P_k is symmetric. Two conjoined bases $(\bar{X}, \bar{U})$ and (X, U) of (S) are *normalized* if their (constant) Wronskian matrix is the identity matrix, that is, if $\bar{X}_k^T U_k - \bar{U}_k^T X_k = I$ for some (and hence for all) $k \in [0, N+1]$. For any conjoined basis (X, U) there always exists another conjoined basis $(\bar{X}, \bar{U})$, determined for example by the initial conditions $(\bar{X}_0, \bar{U}_0)$, such that $(\bar{X}, \bar{U})$ and (X, U) are normalized, and then these conjoined bases satisfy the identities

$$\bar{X}_k^T U_k - \bar{U}_k^T X_k = \bar{X}_k U_k^T - X_k \bar{U}_k^T = I, \quad \bar{X}_k^T \bar{U}_k,\ X_k^T U_k,\ \bar{X}_k X_k^T,\ \bar{U}_k U_k^T \text{ symmetric}. \tag{2.3}$$

In particular, we shall often use the normalized conjoined bases $(\tilde{X}, \tilde{U})$ and $(\hat{X}, \hat{U})$, where $(\hat{X}, \hat{U})$ is the principal solution, which are given by the initial conditions

$$(\hat{X}_0, \hat{U}_0) = (0, I), \quad (\tilde{X}_0, \tilde{U}_0) = (I, 0). \tag{2.4}$$

In this paper, the boundary conditions for quadratic functionals are given in terms of projections. For the case of *jointly varying endpoints*, we consider $2n \times 2n$ symmetric matrices $\mathcal{M}$ and Γ such that $\mathcal{M}$ is a projection. The corresponding boundary conditions are then

$$\mathcal{M} \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix} = 0 \tag{2.5}$$

and the associated quadratic functional is

$$\mathcal{F}(x, u) := \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix}^T \Gamma \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix} + \mathcal{F}_0(x, u). \tag{2.6}$$

Note that if a $2n$-vector α satisfies $\mathcal{M}\alpha = 0$, then $\alpha^T \Gamma \alpha = \alpha^T (I - \mathcal{M})\, \Gamma\, (I - \mathcal{M})\, \alpha$. Hence, we may assume without loss of generality that the matrices $\mathcal{M}$ and Γ satisfy the identity $\Gamma = (I - \mathcal{M})\, \Gamma\, (I - \mathcal{M})$. The quadratic functional $\mathcal{F}$ is *nonnegative*, we write $\mathcal{F} \geq 0$, if it takes nonnegative values on all admissible pairs (x, u) satisfying boundary conditions (2.5), while $\mathcal{F}$ is *positive* (*definite*), we write $\mathcal{F} > 0$, if it takes positive values on all such admissible pairs (x, u) with $x \not\equiv 0$.

In the case of *separated endpoints*, the matrices $\mathcal{M}$ and Γ are block diagonal, i.e., $\mathcal{M} = \text{diag}\{\mathcal{M}_0, \mathcal{M}_1\}$ and $\Gamma = \text{diag}\{\Gamma_0, \Gamma_1\}$ with $n \times n$ symmetric block entries such that $\mathcal{M}_0$ and $\mathcal{M}_1$ are projections and $\Gamma_i = (I - \mathcal{M}_i)\, \Gamma_i\, (I - \mathcal{M}_i)$, $i = 0, 1$. Condition (2.5) then reduces to the boundary conditions

$$\mathcal{M}_0 x_0 = 0, \quad \mathcal{M}_1 x_{N+1} = 0 \tag{2.7}$$

and the quadratic functional $\mathcal{F}$ takes the form

$$\mathcal{F}(x, u) = x_0^T \Gamma_0\, x_0 + x_{N+1}^T \Gamma_1\, x_{N+1} + \mathcal{F}_0(x, u). \tag{2.8}$$

The initial boundary condition in (2.7) gives rise to a specific conjoined basis of (S), called the *natural conjoined basis*, which is defined as the solution of (S) satisfying

$X_0 = I - \mathcal{M}_0$ and $U_0 = \Gamma_0 + \mathcal{M}_0$. Observe that when the initial endpoint is zero, i.e., when $\mathcal{M}_0 = I$, then the natural conjoined basis (X, U) reduces to the principal solution $(\hat{X}, \hat{U})$.

Admissible pairs (x, u) are generated by the $n \times n$ transition matrices $\Phi_{k,k} := I$ and

$$\Phi_{k,j} := \mathcal{A}_{k-1}\mathcal{A}_{k-2} \dots \mathcal{A}_j \quad \text{for } k > j,$$

and by the controllability matrices $G_0 := 0$ and

$$G_k := \begin{pmatrix} \Phi_{k,1}\mathcal{B}_0 & \Phi_{k,2}\mathcal{B}_1 & \dots & \Phi_{k,k-1}\mathcal{B}_{k-2} & \mathcal{B}_{k-1} \end{pmatrix} \in \mathbb{R}^{n\times(nk)}.$$

In particular, a pair (x, u) is admissible if and only if

$$x_k = \Phi_{k,0}x_0 + G_k\mathcal{V}_k\underline{u} \quad \text{for all } k \in [0, N+1], \tag{2.9}$$

where $\mathcal{V}_k$ is the restriction operator $\mathcal{V}_k : \mathbb{R}^{(N+1)n} \to \mathbb{R}^{kn}$ defined by

$$\mathcal{V}_k\,\underline{u} := \begin{pmatrix} u_0^T & \dots & u_{k-1}^T \end{pmatrix}^T \quad \text{with } \underline{u} := \begin{pmatrix} u_0^T & \dots & u_N^T \end{pmatrix}^T. \tag{2.10}$$

In fact, $\mathcal{V}_k$ is the $kn \times (N+1)\,n$ matrix $\mathcal{V}_k = \begin{pmatrix} I_{kn\times kn} & 0 \end{pmatrix}$. Observe that $\mathcal{V}_{N+1} = I$.

Since a discrete symplectic system can have the matrices $\mathcal{A}_k$ in general singular (as opposed to the Hamiltonian system (H) for which $\mathcal{A}_k = (I - A_k)^{-1}$ is always invertible), $\Phi_{k,j}$ may also be in general singular.

3. Results from Literature

In this section we collect known characterizations of the positivity and nonnegativity of $\mathcal{F}$ with variable endpoints which we wish to compare with the results of this paper. The following result regarding the quadratic functional $\mathcal{F}$ with separated endpoints is from [14, Theorem 5].

Proposition 3.1. The quadratic functional $\mathcal{F}$ in (2.8) is positive definite if and only if the implicit Riccati equation

$$R[Q]_k \begin{pmatrix} \Phi_{k,0}(I - \mathcal{M}_0) & G_k\mathcal{V}_k \end{pmatrix} = 0 \quad \text{on } \operatorname{Ker}\mathcal{M}_1 \begin{pmatrix} \Phi_{N+1,0}(I - \mathcal{M}_0) & G_{N+1} \end{pmatrix}, \quad k \in [0, N], \tag{3.1}$$

has a symmetric solution Q_k on $[0, N+1]$ such that $Q_0 = \Gamma_0$,

$$\mathcal{P}_k := (\mathcal{D}_k^T - \mathcal{B}_k^T Q_{k+1})\,\mathcal{B}_k \geq 0 \quad \text{for all } k \in [0, N], \tag{3.2}$$

and satisfying the final endpoint inequality

$$Q_{N+1} + \Gamma_1 > 0 \quad \text{on } \operatorname{Ker}\mathcal{M}_1 \cap \operatorname{Im} X_{N+1},$$

where (X, U) is the natural conjoined basis of (S).

When both endpoints are zero, we obtain from Proposition 3.1 the following, see also [6, Theorem 1].

Corollary 3.2. The quadratic functional $\mathcal{F}_0$ in (1.1) is positive definite over $x_0 = 0 = x_{N+1}$ if and only if the implicit Riccati equation

$$R[Q]_k G_k \mathcal{V}_k = 0 \quad \text{on Ker}\, G_{N+1}, \quad k \in [0, N], \tag{3.3}$$

has a symmetric solution Q_k on $[0, N+1]$ such that $Q_0 = 0$ and $\mathcal{P}$-condition (3.2) holds.

Next, we define the $2n \times 2n$ matrices $\mathcal{A}_k^* :=$ diag$\{I, \mathcal{A}_k\}$, $\mathcal{B}_k^* :=$ diag$\{0, \mathcal{B}_k\}$, $\mathcal{C}_k^* :=$ diag$\{0, \mathcal{C}_k\}$, and $\mathcal{D}_k^* :=$ diag$\{I, \mathcal{D}_k\}$, and the associated augmented Riccati operator

$$R^*[Q^*]_k := Q_{k+1}^*(\mathcal{A}_k^* + \mathcal{B}_k^* Q_k^*) - (\mathcal{C}_k^* + \mathcal{D}_k^* Q_k^*)$$

acting on a symmetric $2n \times 2n$ matrix Q_k^*. In fact, the above coefficients $\mathcal{A}_k^*$, $\mathcal{B}_k^*$, $\mathcal{C}_k^*$, $\mathcal{D}_k^*$ define an augmented discrete symplectic system

$$X_{k+1}^* = \mathcal{A}_k^* X_k^* + \mathcal{B}_k^* U_k^*, \quad U_{k+1}^* = \mathcal{C}_k^* X_k^* + \mathcal{D}_k^* U_k^*. \tag{S*}$$

In this paper we shall use a specific solution of (S*), namely the conjoined basis

$$\hat{X}_k^* = \begin{pmatrix} 0 & I \\ \hat{X}_k & \tilde{X}_k \end{pmatrix}, \quad \hat{U}_k^* = \begin{pmatrix} -I & 0 \\ \hat{U}_k & \tilde{U}_k \end{pmatrix}, \tag{3.4}$$

where $(\hat{X}, \hat{U})$ and $(\tilde{X}, \tilde{U})$ are the conjoined bases of (S) satisfying initial conditions (2.4).

The following result regarding the quadratic functional $\mathcal{F}$ with jointly varying endpoints is from [14, Theorem 10].

Proposition 3.3. The following conditions are equivalent.

(i) The quadratic functional $\mathcal{F}$ in (2.6) is positive definite.

(ii) The principal solution $(\hat{X}, \hat{U})$ has no focal points in $(0, N+1]$, and the matrices $\hat{X}_k^*$, $\hat{U}_k^*$ defined in (3.4) satisfy the final endpoint conditions

$$(\hat{X}_{N+1}^*)^T (\Gamma \hat{X}_{N+1}^* + \hat{U}_{N+1}^*) \geq 0 \quad \text{on Ker}\, \mathcal{M} \hat{X}_{N+1}^*, \tag{3.5}$$

$$\text{Ker}(I - \mathcal{M})\, (\Gamma \hat{X}_{N+1}^* + \hat{U}_{N+1}^*) \cap \text{Ker}\, \mathcal{M} \hat{X}_{N+1}^* \subseteq \text{Ker}\, \hat{X}_{N+1}^*. \tag{3.6}$$

(iii) The augmented implicit Riccati equation

$$R^*[Q^*]_k \begin{pmatrix} I & 0 \\ \Phi_{k,0} & G_k \mathcal{V}_k \end{pmatrix} = 0 \quad \text{on Ker}\, \mathcal{M} \begin{pmatrix} I & 0 \\ \Phi_{N+1,0} & G_{N+1} \end{pmatrix}, \quad k \in [0, N], \tag{3.7}$$

has a symmetric solution $Q_k^* = \begin{pmatrix} \star & \star \\ \star & Q_k \end{pmatrix}$ on $[0, N+1]$ such that $Q_0^* = 0$, $\mathcal{P}$-condition (3.2) holds, and satisfying the final endpoint inequality

$$Q_{N+1}^* + \Gamma > 0 \quad \text{on } \operatorname{Ker} \mathcal{M} \cap \operatorname{Im} \hat{X}_{N+1}^*,$$

where $\hat{X}_k^*$ is given in (3.4).

Remark 3.4. (i) The restriction of implicit Riccati equations (3.1), (3.3), and (3.7) to the kernel of the matrices $\mathcal{M}_1 \left(\Phi_{N+1,0}(I - \mathcal{M}_0) \quad G_{N+1} \right)$, G_{N+1}, and $\mathcal{M} \begin{pmatrix} I & 0 \\ \Phi_{N+1,0} & G_{N+1} \end{pmatrix}$, respectively, can be eliminated in the above characterizations of the positivity of $\mathcal{F}$. This follows from a close examination of the proofs of these statements in [14, 15]. In this way we obtain implicit Riccati equations of the same nature as those in [4, Theorems 2 and 3].

(ii) When the quadratic functional $\mathcal{F}$ is the second variation of a (nonlinear) discrete optimization problem, its positivity is a second order sufficient optimality condition. Keeping this in mind, we are interested in (at least formally) the *weakest* conditions, which imply $\mathcal{F} > 0$. Therefore, with respect to this fact, we formulate the implicit Riccati equations for $\mathcal{F} > 0$ over the appropriate kernel.

Following [16], for any conjoined basis (X, U) of (S) we define the $n \times n$ matrices

$$M_k := (I - X_{k+1} X_{k+1}^\dagger)\, \mathcal{B}_k, \quad T_k := I - M_k^\dagger M_k. \tag{3.8}$$

It is known from [16] that $M_k = 0$ if and only if the kernel condition in (2.2) holds, and that this kernel condition is not necessary for the nonnegativity of $\mathcal{F}$, see [9]. On the other hand, it is proven in the same paper and in [7] that, instead of the above mentioned kernel condition, the image condition $x_k \in \operatorname{Im} X_k$ should be used in the characterization of the nonnegativity of $\mathcal{F}$. Moreover, the matrix $T_k P_k T_k = T_k \mathcal{P}_k T_k$ is always symmetric, where P_k and $\mathcal{P}_k$ are defined in (2.2) and (3.2), respectively.

The following result regarding the quadratic functional $\mathcal{F}$ with separable endpoints is from [7, Theorem 2].

Proposition 3.5. The quadratic functional $\mathcal{F}$ in (2.8) is nonnegative if and only if the natural conjoined basis (X, U) satisfies the P-condition $T_k P_k T_k \geq 0$ for all $k \in [0, N]$, the image condition

$$x_k \in \operatorname{Im} X_k \quad \text{for all } k \in [0, N+1] \text{ and all admissible } (x, u) \text{ satisfying (2.7)}, \tag{3.9}$$

and the final endpoint inequality

$$X_{N+1}^T (\Gamma_1 X_{N+1} + U_{N+1}) \geq 0 \quad \text{on } \operatorname{Ker} \mathcal{M}_1 X_{N+1}. \tag{3.10}$$

The next result regarding the quadratic functional $\mathcal{F}$ with jointly varying endpoints is from [13, Theorem 2].

Proposition 3.6. Let $(\hat{X}, \hat{U})$ and $(\tilde{X}, \tilde{U})$ be the conjoined bases of (S) satisfying initial conditions (2.4). The quadratic functional $\mathcal{F}$ in (2.6) is nonnegative if and only if the principal solution $(\hat{X}, \hat{U})$ satisfies the P-condition $T_k P_k T_k \geq 0$ for all $k \in [0, N]$, the image condition

$$x_k - \tilde{X}_k x_0 \in \operatorname{Im} \hat{X}_k \quad \text{for all } k \in [0, N+1] \text{ and all admissible } (x, u) \text{ satisfying (2.5),} \tag{3.11}$$

and the final endpoint inequality (3.5) holds.

4. Nonnegativity for Separable Endpoints

In this section we derive a characterization of the *nonnegativity* of $\mathcal{F}$ with separable endpoints in terms of an *implicit Riccati equation*. This is a parallel result to Proposition 3.1 in which $\mathcal{F}$ is positive definite. Recall that the matrices $\mathcal{P}_k$, M_k, T_k are defined by (3.2) and (3.8).

Theorem 4.1. Let (X, U) be the natural conjoined basis of (S). The quadratic functional $\mathcal{F}$ in (2.8) is nonnegative if and only if the implicit Riccati equation

$$\begin{pmatrix} (I - \mathcal{M}_0)\, \Phi_{k+1,0}^T \\ \mathcal{V}_{k+1}^T G_{k+1}^T \end{pmatrix} R[Q]_k \left(\Phi_{k,0}(I - \mathcal{M}_0) \quad G_k \mathcal{V}_k \right) = 0 \tag{4.1}$$
$$\text{on } \operatorname{Ker} \mathcal{M}_1 \left(\Phi_{N+1,0}(I - \mathcal{M}_0) \quad G_{N+1} \right), \quad k \in [0, N],$$

has a symmetric solution Q_k on $[0, N+1]$ such that the $\mathcal{P}$-condition

$$T_k \mathcal{P}_k T_k \geq 0 \quad \text{for all } k \in [0, N], \tag{4.2}$$

holds, $Q_k X_k = U_k X_k^{\dagger} X_k$ on $[0, N+1]$, and satisfying the final endpoint inequality

$$Q_{N+1} + \Gamma_1 \geq 0 \quad \text{on } \operatorname{Ker} \mathcal{M}_1 \cap \operatorname{Im} X_{N+1}. \tag{4.3}$$

Remark 4.2. In contrast to Remark 3.4(i) in which $\mathcal{F}$ is positive definite, the restriction of implicit Riccati equation (4.1) to $\operatorname{Ker} \mathcal{M}_1 \left(\Phi_{N+1,0}(I - \mathcal{M}_0) \quad G_{N+1} \right)$ cannot be eliminated when characterizing the nonnegativity of $\mathcal{F}$. The same applies to implicit Riccati equation (5.1) in Theorem 5.1 in Section 5.

The proof of the above theorem is based on the following auxiliary lemmas. The first one follows from (2.9) and it describes the set of admissible pairs (x, u) satisfying the initial boundary condition.

Lemma 4.3. A pair (x, u) is admissible and $\mathcal{M}_0 x_0 = 0$ if and only if

$$x_k = \Phi_{k,0}(I - \mathcal{M}_0)\, x_0 + G_k \mathcal{V}_k \underline{u} \quad \text{for all } k \in [0, N+1], \tag{4.4}$$

where the operator $\mathcal{V}_k$ and $\underline{u}$ are defined in (2.10).

Lemma 4.4. For any solution (X, U) of (S) and any symmetric matrix Q_k with $Q_k X_k = U_k X_k^\dagger X_k$ and $X_k^T U_k$ symmetric on $[0, N+1]$ we have for $k \in [0, N]$

$$R[Q]_k X_k = -U_{k+1}(I - X_{k+1}^\dagger X_{k+1})\, X_k^\dagger X_k, \quad X_{k+1}^T R[Q]_k X_k = 0.$$

Proof. The first identity follows by a direct calculation, see e.g., the proof of [13, Lemma 1(ii)]. The second identity is a consequence of the first one. ■

Proof of Theorem 4.1. Assume $\mathcal{F}(x, u) \geq 0$ over admissible (x, u) with $\mathcal{M}_0 x_0 = 0$ and $\mathcal{M}_1 x_{N+1} = 0$. Let (X, U) be the natural conjoined basis of (S) and define Q_k by

$$Q_k := U_k X_k^\dagger - (U_k X_k^\dagger \bar{X}_k - \bar{U}_k)\,(I - X_k^\dagger X_k)\, U_k^T, \tag{4.5}$$

where $(\bar{X}, \bar{U})$ and (X, U) are normalized. Then Q_k is symmetric and $Q_k X_k = U_k X_k^\dagger X_k$ on $[0, N+1]$, $T_k \mathcal{P}_k T_k = T_k P_k T_k \geq 0$ on $[0, N]$ by Proposition 3.5, and final endpoint inequality (4.3) follows from condition (3.10). It remains to show that implicit Riccati equation (4.1) is satisfied. Let $\binom{\alpha}{\underline{u}} \in \operatorname{Ker} \mathcal{M}_1 \big(\Phi_{N+1,0}(I - \mathcal{M}_0) \quad G_{N+1}\big)$ be arbitrary. Define the sequence $x = \{x_k\}_{k=0}^{N+1}$ by equation (4.4) with $x_0 := \alpha$. Then (x, u) is admissible, $\mathcal{M}_0 x_0 = 0$, $\mathcal{M}_1 x_{N+1} = 0$, and hence, by (3.9) in Proposition 3.5, we have $x_k = X_k c_k \in \operatorname{Im} X_k$ for all $k \in [0, N+1]$ and some $c_k \in \mathbb{R}^n$. Consequently,

$$\begin{aligned}\begin{pmatrix}\alpha \\ \underline{u}\end{pmatrix}^T \begin{pmatrix}(I - \mathcal{M}_0)\,\Phi_{k+1,0}^T \\ \mathcal{V}_{k+1}^T G_{k+1}^T\end{pmatrix} R[Q]_k \left(\Phi_{k,0}(I - \mathcal{M}_0) \quad G_k \mathcal{V}_k\right) \begin{pmatrix}\alpha \\ \underline{u}\end{pmatrix} \\ = x_{k+1}^T R[Q]_k x_k = c_{k+1}^T X_{k+1}^T R[Q]_k X_k c_k = 0,\end{aligned}$$

where we used $X_{k+1}^T R[Q]_k X_k = 0$ from Lemma 4.4.

Conversely, assume that Q_k satisfies the conditions in Theorem 4.1. We must show that the image condition (3.9) holds, since then Proposition 3.5 yields the nonnegativity of $\mathcal{F}$. Therefore, let (x, u) be admissible with $\mathcal{M}_0 x_0 = 0$ and $\mathcal{M}_1 x_{N+1} = 0$. Assume that

$$x_k \in \operatorname{Im} X_k \quad \text{for all } k \in [0, m] \tag{4.6}$$

with some $m \in [0, N]$. This is always true for $m = 0$. If we show that $x_{m+1} \in \operatorname{Im} X_{m+1}$, then the proof is complete. Let $d \in \mathbb{R}^n$ be arbitrary and define

$$(\tilde{x}_k, \tilde{u}_k) := \begin{cases} (X_k, U_k)\,(I - X_{k+1}^\dagger X_{k+1})\, d, & \text{for } k \in [0, m], \\ (0, 0), & \text{for } k \in [m+1, N+1]. \end{cases}$$

Then $(\tilde{x}, \tilde{u})$ is admissible, $\mathcal{M}_0 \tilde{x}_0 = 0$ and $\tilde{x}_{N+1} = 0$. Consequently, the pair $(\bar{x}, \bar{u}) := (x, u) + (\tilde{x}, \tilde{u})$ is also admissible, $\mathcal{M}_0 \bar{x}_0 = 0$, and $\mathcal{M}_1 \bar{x}_{N+1} = 0$. By Lemma 4.3 it follows that

$$\bar{x}_k = \Phi_{k,0}(I - \mathcal{M}_0)\,\bar{x}_0 + G_k \mathcal{V}_k \bar{u} \quad \text{for all } k \in [0, N+1],$$

where $\underline{\bar{u}} := \begin{pmatrix} \bar{u}_0^T & \dots & \bar{u}_N^T \end{pmatrix}^T$, and where $\left(\begin{smallmatrix} \bar{x}_0 \\ \underline{\bar{u}} \end{smallmatrix}\right) \in \operatorname{Ker} \mathcal{M}_1 \begin{pmatrix} \Phi_{N+1,0}(I - \mathcal{M}_0) & G_{N+1} \end{pmatrix}$. Hence, Riccati equation (4.1) yields that

$$\bar{x}_{k+1}^T R[Q]_k \bar{x}_k = 0 \quad \text{for } k \in [0, N]. \tag{4.7}$$

The same argument implies that $x_{k+1}^T R[Q]_k x_k = 0$ and $\tilde{x}_{k+1}^T R[Q]_k \tilde{x}_k = 0$ for all $k \in [0, N]$. Hence, from (4.7) at $k = m$ and from $\tilde{x}_{m+1} = 0$ we get

$$\begin{aligned} 0 = x_{m+1}^T R[Q]_m \tilde{x}_m &= x_{m+1}^T R[Q]_m X_m (I - X_{m+1}^\dagger X_{m+1})\, d \\ &= -x_{m+1}^T U_{m+1} (I - X_{m+1}^\dagger X_{m+1})\, X_m^\dagger X_m (I - X_{m+1}^\dagger X_{m+1})\, d, \end{aligned}$$

where the last equality follows from Lemma 4.4. Now the choice of $d := U_{m+1}^T x_{m+1}$ yields that $X_m^\dagger X_m (I - X_{m+1}^\dagger X_{m+1})\, U_{m+1}^T x_{m+1} = 0$ and hence,

$$X_m (I - X_{m+1}^\dagger X_{m+1})\, U_{m+1}^T x_{m+1} = 0. \tag{4.8}$$

Since, by (4.6), $x_m = X_m c \in \operatorname{Im} X_m$ with some $c \in \mathbb{R}^n$, we have

$$\begin{aligned} U_{m+1}^T x_{m+1} &= U_{m+1}^T (\mathcal{A}_m X_m c + \mathcal{B}_m u_m) = U_{m+1}^T [X_{m+1} c + \mathcal{B}_m (u_m - U_m c)] \\ &\overset{(2.1)}{=} X_{m+1}^T [U_{m+1} c + \mathcal{D}_m (u_m - U_m c)] - X_m^T (u_m - U_m c). \end{aligned} \tag{4.9}$$

We now insert the above formula into (4.8) and get

$$X_m (I - X_{m+1}^\dagger X_{m+1})\, X_m^T (u_m - U_m c) = 0,$$

which, upon multiplying by $(u_m - U_m c)^T$ from the left, implies

$$(I - X_{m+1}^\dagger X_{m+1})\, X_m^T (u_m - U_m c) = 0. \tag{4.10}$$

Then, we use the formula for $X_m^T (u_m - U_m c)$ from (4.9) in equation (4.10) and obtain

$$(I - X_{m+1}^\dagger X_{m+1})\, U_{m+1}^T x_{m+1} = 0. \tag{4.11}$$

Let $(\bar{X}, \bar{U})$ be a conjoined basis of (S) such that $(\bar{X}, \bar{U})$ and (X, U) are normalized. If we multiply equation (4.11) by $\bar{X}_{m+1}$ and use properties (2.3) of normalized conjoined bases, then

$$\begin{aligned} 0 &= \bar{X}_{m+1} (I - X_{m+1}^\dagger X_{m+1})\, U_{m+1}^T x_{m+1} \\ &\overset{(2.3)}{=} x_{m+1} + X_{m+1} [\bar{U}_{m+1}^T - \bar{X}_{m+1}^T (X_{m+1}^\dagger)^T U_{m+1}^T]\, x_{m+1}. \end{aligned}$$

Hence, $x_{m+1} \in \operatorname{Im} X_{m+1}$ and the proof is complete. ■

For completeness and comparison with Corollary 3.2 we present the corresponding zero endpoints result.

Corollary 4.5. Let $(\hat{X}, \hat{U})$ be the principal solution of (S). The quadratic functional $\mathcal{F}_0$ in (1.1) is nonnegative over $x_0 = 0 = x_{N+1}$ if and only if the implicit Riccati equation

$$\mathcal{V}_{k+1}^T G_{k+1}^T R[Q]_k G_k \mathcal{V}_k = 0 \quad \text{on } \operatorname{Ker} G_{N+1}, \quad k \in [0, N], \tag{4.12}$$

has a symmetric solution Q_k on $[0, N+1]$ such that $Q_k \hat{X}_k = \hat{U}_k \hat{X}_k^\dagger \hat{X}_k$ on $[0, N+1]$ and $\mathcal{P}$-condition (4.2) holds, where T_k is defined in (3.8) through the principal solution $(\hat{X}, \hat{U})$.

The condition that $Q_k X_k = U_k X_k^\dagger X_k$ cannot be removed from Theorem 4.1 (or from Corollary 4.5, or Theorem 5.1 in the next section) as it is shown in the following example, where we have $\mathcal{F}_0 \not\geq 0$ and there is a symmetric solution Q_k on $[0, N+1]$ of equation (4.12) satisfying all the conditions in Corollary 4.5 except of $Q_k \hat{X}_k = \hat{U}_k \hat{X}_k^\dagger \hat{X}_k$ on $[0, N+1]$.

Example 4.6. Let $n = 1$, $N = 3$, and $\mathcal{S}_k \equiv \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ for $k \in [0, 3]$, i.e., $\mathcal{A}_k \equiv 0$ and $\mathcal{B}_k = \mathcal{D}_k = -\mathcal{C}_k \equiv 1$. Assume that both endpoints are zero. Then the functional $\mathcal{F}_0$ in (1.1) takes the form $\mathcal{F}_0(x, u) = \sum_{k=0}^{3} \{u_k^2 - 2\, x_k u_k\}$ over pairs (x, u) satisfying $x_{k+1} = u_k$ for $k \in [0, 3]$ and $x_0 = 0 = x_4$. The principal solution $(\hat{X}, \hat{U})$ of (S) is in this case $\hat{X} = \{0, 1, 1, 0, -1\} = \hat{X}^\dagger$ and $\hat{U} = \{1, 1, 0, -1, -1\}$. Then $\mathcal{F}_0 \not\geq 0$, since $\mathcal{F}(\hat{x}, \hat{u}) = -1$ for the admissible pair $(\hat{x}, \hat{u})$ defined by $\hat{x} := \{0, 1, 1, 1, 0\}$ and $\hat{u} := \{1, 1, 1, 0\}$. Note that $\hat{x}_3 \notin \operatorname{Im} \hat{X}_3$, i.e., image condition (3.9) is violated.

Define $Q := \left\{2, \frac{1}{2}, -1, 2, 0\right\}$. Then Q_k satisfies all the conditions in Corollary 4.5 except of $Q_k \hat{X}_k = \hat{U}_k \hat{X}_k^\dagger \hat{X}_k$ for $k \in [0, 4]$. This can be verified when we calculate the sequences $R[Q] = \{0, 0, 0, -1\}$, $\mathcal{P} = \left\{\frac{1}{2}, 2, -1, 1\right\}$, $M = \{0, 0, 1, 0\} = M^\dagger$, $T = \{1, 1, 0, 1\}$, $T\mathcal{P}T = \left\{\frac{1}{2}, 2, 0, 1\right\} \geq 0$, $G_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$, and $G_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$.

5. Nonnegativity for Jointly Varying Endpoints

In this section we establish the following characterization of the nonnegativity of $\mathcal{F}$ with jointly varying endpoints in terms of an *implicit Riccati equation*. In contrast to Proposition 3.3, this implicit Riccati equation uses the original Riccati operator $R[Q]_k$ instead of the augmented one $R^*[Q^*]_k$.

Theorem 5.1. Let $(\hat{X}, \hat{U})$ and $(\tilde{X}, \tilde{U})$ be the conjoined bases of (S) satisfying initial conditions (2.4). The quadratic functional $\mathcal{F}$ in (2.6) is nonnegative if and only if the implicit Riccati equation

$$\begin{pmatrix} \Phi^T_{k+1,0} - \tilde{X}^T_{k+1} \\ \mathcal{V}^T_{k+1} G^T_{k+1} \end{pmatrix} R[Q]_k \begin{pmatrix} \Phi_{k,0} - \tilde{X}_k & G_k \mathcal{V}_k \end{pmatrix} = 0 \qquad (5.1)$$

$$\text{on } \operatorname{Ker} \mathcal{M} \begin{pmatrix} I & 0 \\ \Phi_{N+1,0} & G_{N+1} \end{pmatrix}, \quad k \in [0, N],$$

has a symmetric solution Q_k on $[0, N+1]$ such that $Q_k \hat{X}_k = \hat{U}_k \hat{X}^\dagger_k \hat{X}_k$ on $[0, N+1]$, $\mathcal{P}$-condition (4.2) holds, where T_k is defined in (3.8) through the principal solution $(\hat{X}, \hat{U})$, and satisfying the final endpoint inequality

$$\hat{Q}^*_{N+1} + \Gamma \geq 0 \quad \text{on } \operatorname{Ker} \mathcal{M} \cap \operatorname{Im} \hat{X}^*_{N+1},$$

where $\hat{X}^*_k$ is defined in (3.4) and

$$\hat{Q}^*_k := \begin{pmatrix} \tilde{X}^T_k Q_k \tilde{X}_k - \tilde{X}^T_k \tilde{U}_k & \tilde{U}^T_k - \tilde{X}^T_k Q_k \\ \tilde{U}_k - Q_k \tilde{X}_k & Q_k \end{pmatrix}. \qquad (5.2)$$

Remark 5.2. When both endpoints are zero, i.e., when $\mathcal{M} = I$ and $\Gamma = 0$, then Theorem 5.1 reduces to Corollary 4.5.

The proof of Theorem 5.1 is displayed below after some preparatory results, in which we use arbitrary normalized conjoined bases $(\bar{X}, \bar{U})$ and (X, U) of (S) and a symmetric Q_k. Hence, define the $2n \times 2n$ matrices

$$X^*_k = \begin{pmatrix} 0 & I \\ X_k & \bar{X}_k \end{pmatrix}, \; U^*_k = \begin{pmatrix} -I & 0 \\ U_k & \bar{U}_k \end{pmatrix}, \; Q^*_k := \begin{pmatrix} \bar{X}^T_k Q_k \bar{X}_k - \bar{X}^T_k \bar{U}_k & \bar{U}^T_k - \bar{X}^T_k Q_k \\ \bar{U}_k - Q_k \bar{X}_k & Q_k \end{pmatrix}. \qquad (5.3)$$

First we show some properties of the matrix Q^*_k.

Lemma 5.3. Let Q^*_k, X^*_k, and U^*_k be defined by (5.3) with a symmetric matrix Q_k. Then

(i) $(X^*_k)^T Q^*_k X^*_k = (X^*_k)^T U^*_k$ if and only if $X^T_k Q_k X_k = X^T_k U_k$,

(ii) $Q^*_k X^*_k = U^*_k (X^*_k)^\dagger X^*_k$ if and only if $Q_k X_k = U_k X^\dagger_k X_k$.

Proof. Part (i) follows by a direct calculation. The second assertion is a consequence of the formula for the pseudoinverse $(X^*_k)^\dagger$, see [4, Remark 8(i)] adopted to the present setting. In particular, identity $(X^*_k)^\dagger X^*_k = \operatorname{diag}\{X^\dagger_k X_k, I\}$ holds. ■

The form of the matrix Q^*_k in (5.3) enables to simplify the augmented Riccati operator $R^*[Q^*]_k$ in terms of the original Riccati operator $R[Q]_k$. This is a new feature which was not utilized by the previously considered solutions Q^*_k, such as those in [14, Theorem 10] or [4, Theorem 3].

Lemma 5.4. Let Q_k^* and X_k^* be defined in (5.3) with a symmetric matrix Q_k. Then the augmented Riccati operator $R^*[Q^*]_k$ has the form

$$R^*[Q^*]_k = \begin{pmatrix} \bar{X}_{k+1}^T R[Q]_k \bar{X}_k & -\bar{X}_{k+1}^T R[Q]_k \\ -R[Q]_k \bar{X}_k & R[Q]_k \end{pmatrix} = \begin{pmatrix} -\bar{X}_{k+1}^T \\ I \end{pmatrix} R[Q]_k \begin{pmatrix} -\bar{X}_k & I \end{pmatrix}. \quad (5.4)$$

Consequently, we have the identities

$$R^*[Q^*]_k X_k^* = \begin{pmatrix} -\bar{X}_{k+1}^T R[Q]_k X_k & 0 \\ R[Q]_k X_k & 0 \end{pmatrix} = \begin{pmatrix} -\bar{X}_{k+1}^T \\ I \end{pmatrix} R[Q]_k \begin{pmatrix} X_k & 0 \end{pmatrix},$$

$$(X_{k+1}^*)^T R^*[Q^*]_k X_k^* = \begin{pmatrix} X_{k+1}^T R[Q]_k X_k & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{k+1}^T \\ 0 \end{pmatrix} R[Q]_k \begin{pmatrix} X_k & 0 \end{pmatrix}.$$

Proof. All identities are shown by direct calculations. ■

Proof of Theorem 5.1. It is similar to the proof of Theorem 4.1 in which we replace the natural conjoined basis (X, U) by the principal solution $(\hat{X}, \hat{U})$, and instead of image condition (3.9) we use image condition (3.11). The symmetric matrix Q_k satisfying the conditions in Theorem 5.1 is defined by equation (4.5) where $(X, U) := (\hat{X}, \hat{U})$ and $(\bar{X}, \bar{U}) := (\tilde{X}, \tilde{U})$. Then we can apply Lemmas 5.3, 5.4 to $(\hat{X}^*, \hat{U}^*)$ and $\hat{Q}_k^*$, and Lemma 4.4 to the principal solution $(\hat{X}, \hat{U})$. The sufficiency of the implicit Riccati equation (5.1) is proven similarly as in calculations (4.6)–(4.11), but with the image condition $x_k - \tilde{X}_k x_0 \in \operatorname{Im} \hat{X}_k$ for all $k \in [0, m]$, and with the vector $d := \hat{U}_{m+1}^T (x_{m+1} - \tilde{X}_{m+1} x_0)$. ■

6. Positivity for Jointly Varying Endpoints

In this section we establish a characterization of the positivity of $\mathcal{F}$ with jointly varying endpoints in terms of an *implicit Riccati equation*, which is simpler than the one in Proposition 5.1. We use the simplified form of the augmented Riccati operator $R^*[Q^*]_k$ from Lemma 5.4.

Theorem 6.1. Let $(\hat{X}, \hat{U})$ and $(\tilde{X}, \tilde{U})$ be the conjoined bases of (S) satisfying initial conditions (2.4). The quadratic functional $\mathcal{F}$ in (2.6) is positive definite if and only if the implicit Riccati equation

$$R[Q]_k \begin{pmatrix} \Phi_{k,0} - \tilde{X}_k & G_k \mathcal{V}_k \end{pmatrix} = 0 \quad \text{on } \operatorname{Ker} \mathcal{M} \begin{pmatrix} I & 0 \\ \Phi_{N+1,0} & G_{N+1} \end{pmatrix}, \quad k \in [0, N], \quad (6.1)$$

has a symmetric solution Q_k on $[0, N+1]$ such that $Q_0 = 0$, $\mathcal{P}$-condition (3.2) holds, and satisfying the final endpoint inequality

$$\hat{Q}_{N+1}^* + \Gamma > 0 \quad \text{on } \operatorname{Ker} \mathcal{M} \cap \operatorname{Im} \hat{X}_{N+1}^*, \quad (6.2)$$

where $\hat{Q}_k^*$ and $\hat{X}_k^*$ are defined in (5.2) and (3.4), respectively.

In the proof we need the following result from [6, Lemma 3], which can also be deduced directly from Lemma 4.4.

Lemma 6.2. For any solution (X, U) of (S) such that the kernel condition

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k \quad \text{for all } k \in [0, N] \tag{6.3}$$

holds, and for any symmetric matrix Q_k with $Q_k X_k = U_k X_k^\dagger X_k$ and $X_k^T U_k$ symmetric on $[0, N+1]$ we have $R[Q]_k X_k = 0$ for $k \in [0, N]$.

Proof of Theorem 6.1. The positivity of $\mathcal{F}$ implies, by Proposition 3.3 (ii), that the principal solution $(\hat{X}, \hat{U})$ has no focal points, i.e., it satisfies kernel condition (6.3) and $\mathcal{P}_k = P_k \geq 0$ on $[0, N]$. Let Q_k be defined by formula (4.5) with $(X, U) := (\hat{X}, \hat{U})$ and $(\bar{X}, \bar{U}) := (\tilde{X}, \tilde{U})$, and let $\hat{Q}_k^*$ be as in (5.2). Then Lemma 6.2 implies that $R[Q]_k \hat{X}_k = 0$ on $[0, N]$. Take any $\left(\begin{smallmatrix} \alpha \\ \underline{u} \end{smallmatrix}\right) \in \operatorname{Ker} \mathcal{M} \left(\begin{smallmatrix} I & 0 \\ \Phi_{N+1,0} & G_{N+1} \end{smallmatrix}\right)$ with $\underline{u}$ as in (2.10), put $x_0 := \alpha$, and define x_k by equation (2.9) for $k \in [1, N+1]$. Then (x, u) is admissible, $x_0 - \tilde{X}_0 \alpha = 0 \in \operatorname{Im} \hat{X}_0$, so that kernel condition (6.3) implies that $x_k - \tilde{X}_k \alpha = \hat{X}_k c_k \in \operatorname{Im} \hat{X}_k$ for some $c_k \in \mathbb{R}^n$ and all $k \in [0, N+1]$. Thus, we get

$$R[Q]_k \begin{pmatrix} \Phi_{k,0} - \tilde{X}_k & G_k \mathcal{V}_k \end{pmatrix} \begin{pmatrix} \alpha \\ \underline{u} \end{pmatrix} = R[Q]_k (x_k - \tilde{X}_k \alpha) = R[Q]_k \hat{X}_k c_k = 0.$$

Final endpoint inequality (6.2) follows from conditions (3.5) and (3.6). Conversely, suppose that a symmetric matrix Q_k on $[0, N+1]$ satisfies all the conditions in Theorem 6.1. Then, upon taking Lemma 5.3 (ii) and Lemma 5.4 into account, the matrix $\hat{Q}_k^*$ in (5.2) satisfies the conditions in Proposition 3.3 (iii), that is, the functional $\mathcal{F}$ is positive definite. ■

Alternatively, the sufficiency of Riccati equation (6.1) for $\mathcal{F} > 0$ in Theorem 6.1 can be directly proven similarly to the implication (iv) ⇒ (i) in [15, Theorem 1] by means of the next two lemmas. The first one is the Picone identity, which is a generalization of [6, Lemma 2], see also [4, Proposition 4] adopted to the setting of this paper. The second lemma shows, similarly as in [15, Lemma 2], that the conditions on Q_k in Theorem 6.1 imply that the principal solution $(\hat{X}, \hat{U})$ satisfies kernel condition (6.3). However, these two results have their own importance for future reference.

Lemma 6.3. Let $(\bar{X}, \bar{U})$ and (X, U) be any normalized conjoined bases of (S). For any admissible (x, u) and symmetric Q_k on $[0, N+1]$ and for any $\alpha \in \mathbb{R}^n$ we have

$$\begin{aligned} \Delta \left\{ \begin{pmatrix} \alpha \\ x_k \end{pmatrix}^T Q_k^* \begin{pmatrix} \alpha \\ x_k \end{pmatrix} \right\} - \{ x_k^T \mathcal{C}_k^T \mathcal{A}_k x_k + 2\, x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{D}_k^T \mathcal{B}_k u_k \} + w_k^T \mathcal{P}_k w_k \\ = 2\, (u_k - \bar{U}_k \alpha)^T \mathcal{B}_k^T R[Q]_k (x_k - \bar{X}_k \alpha) \\ + (x_k - \bar{X}_k \alpha)^T \big\{ R^T[Q]_k \mathcal{A}_k - Q_k \mathcal{B}_k^T R[Q]_k \big\} (x_k - \bar{X}_k \alpha) \end{aligned}$$

for all $k \in [0, N]$, and the identity

$$\begin{aligned}(\mathcal{D}_k^T - \mathcal{B}_k^T Q_{k+1})\, x_{k+1} = x_k + \mathcal{P}_k w_k + \mathcal{B}_k^T (\bar{U}_{k+1} - Q_{k+1}\bar{X}_{k+1})\alpha \\ -\mathcal{B}_k^T R[Q]_k (x_k - \bar{X}_k \alpha)\end{aligned} \tag{6.4}$$

holds, where $\mathcal{P}_k$ and Q_k^* are defined in (3.2) and (5.3), and where $w_k := u_k - Q_k x_k - (\bar{U}_k - Q_k \bar{X}_k)\, \alpha$ on $[0, N]$.

Proof. Let (x, u) be admissible. Then (x^*, u^*), where $x_k^* := \left(\begin{smallmatrix} \alpha \\ x_k \end{smallmatrix}\right)$ and $u_k^* := \left(\begin{smallmatrix} 0 \\ u_k \end{smallmatrix}\right)$, is admissible and Q_k^* is symmetric. The desired identity follows from [6, Lemma 2] applied to the given augmented quantities (x^*, u^*), $R^*[Q^*]_k$, and X_k^* from (5.3), where we used simplified form (5.4) of $R^*[Q^*]_k$ in Lemma 5.4. ■

Lemma 6.4. Assume that a symmetric Q_k on $[0, N+1]$ solves implicit Riccati equation (6.1) and $\mathcal{P}$-condition (3.2) holds. Then the principal solution $(\hat{X}, \hat{U})$ satisfies kernel condition (6.3).

Proof. Suppose that $\hat{X}_{m+1} d = 0$ and $\hat{X}_m d \neq 0$ for some vector $d \in \mathbb{R}^n$ and index $m \in [0, N]$. Then the pair $(\tilde{x}, \tilde{u})$ defined as $(\hat{X}, \hat{U})d$ on $[0, m]$ and $(0, 0)$ on $[m+1, N+1]$ is admissible and $\tilde{x}_0 = 0 = \tilde{x}_{N+1}$. Hence, $\mathcal{F}(\tilde{x}, \tilde{u}) = \tilde{x}_k^T \tilde{u}_k \big|_0^{m+1} = 0$. Since equation (2.9) implies that $\tilde{x}_k = G_k \begin{pmatrix} \tilde{u}_0^T & \dots & \tilde{u}_{k-1}^T \end{pmatrix}^T$ on $[0, N+1]$, Riccati equation (6.1) yields $R[Q]_k \tilde{x}_k = 0$ on $[0, N]$. Now we apply Lemma 6.3 with $\alpha := 0$ and obtain $\mathcal{F}(\tilde{x}, \tilde{u}) = \sum_{k=0}^{N} w_k^T \mathcal{P}_k w_k$. Since $\mathcal{P}_k \geq 0$, we get $\mathcal{P}_k w_k = 0$ on $[0, N]$, which in turn implies $\tilde{x}_m = (\mathcal{D}_m^T - \mathcal{B}_m^T Q_{m+1})\, \tilde{x}_{m+1} = 0$, by (6.4) and the definition of $\tilde{x}_{m+1}$. This contradicts the fact that $\tilde{x}_m = \hat{X}_m d \neq 0$. ■

Alternative proof of Theorem 6.1. For sufficiency part, assume that a symmetric matrix Q_k on $[0, N+1]$ satisfies the conditions in Theorem 6.1 and let (x, u) be admissible and $\mathcal{M}\left(\begin{smallmatrix} x_0 \\ x_{N+1} \end{smallmatrix}\right) = 0$. By Lemma 6.4, we get that $(\hat{X}, \hat{U})$ satisfies kernel condition (6.3). In turn, since $x_0 - \tilde{X}_0 x_0 = 0 \in \operatorname{Im} \hat{X}_0$, we obtain that $x_k - \tilde{X}_k x_0 \in \operatorname{Im} \hat{X}_k$ on $[0, N+1]$. The positivity of $\mathcal{F}$ then follows from Lemma 6.2 and from the Picone identity in Lemma 6.3 with $\alpha := x_0$. ■

The final result of this section says that, with respect to Remark 3.4, unrestricted implicit Riccati equations (6.1) and (3.3) are equivalent.

Lemma 6.5. Assume that Q_k are symmetric for $k \in [0, N+1]$. Then $R[Q]_k G_k \mathcal{V}_k = 0$ for all $k \in [0, N]$ if and only if $R[Q]_k \begin{pmatrix} \Phi_{k,0} - \tilde{X}_k & G_k \mathcal{V}_k \end{pmatrix} = 0$ for all $k \in [0, N]$.

Proof. Assume that $R[Q]_k G_k \mathcal{V}_k = 0$ for $k \in [0, N]$ and let $\left(\begin{smallmatrix} \alpha \\ \underline{u} \end{smallmatrix}\right) \in \mathbb{R}^{(N+2)\,n}$ be arbitrary with $\underline{u}$ as in (2.10). Define

$$\begin{aligned} \tilde{x}_k &:= (\Phi_{k,0} - \tilde{X}_k)\, \alpha + G_k \mathcal{V}_k \underline{u}, \qquad && k \in [0, N+1], \\ \tilde{u}_k &:= u_k - \tilde{U}_k \alpha, && k \in [0, N]. \end{aligned}$$

Then $(\tilde{x}, \tilde{u})$ is admissible, because

$$\mathcal{A}_k \tilde{x}_k + \mathcal{B}_k \tilde{u}_k = \Phi_{k+1,0}\alpha - (\mathcal{A}_k \tilde{X}_k + \mathcal{B}_k \tilde{U}_k)\,\alpha + \mathcal{A}_k G_k \mathcal{V}_k \underline{u} + \mathcal{B}_k u_k = \tilde{x}_{k+1}.$$

Furthermore, $\tilde{x}_0 = 0$ and (2.9) implies that $\tilde{x}_k = G_k \mathcal{V}_k \underline{\tilde{u}}$, where $\underline{\tilde{u}} = \begin{pmatrix} \tilde{u}_0^T & \dots & \tilde{u}_N^T \end{pmatrix}^T$. Consequently,

$$R[Q]_k \begin{pmatrix} \Phi_{k,0} - \tilde{X}_k & G_k \mathcal{V}_k \end{pmatrix} \begin{pmatrix} \alpha \\ \underline{u} \end{pmatrix} = R[Q]_k \tilde{x}_k = R[Q]_k G_k \mathcal{V}_k \underline{\tilde{u}} = 0.$$

The converse implication is satisfied trivially. ■

Remark 6.6. Similarly as in Lemma 6.5, one can show that unrestricted implicit Riccati equations (5.1) and (4.12) are equivalent. More precisely,

$$\begin{pmatrix} \Phi_{k+1,0}^T - \tilde{X}_{k+1}^T \\ \mathcal{V}_{k+1}^T G_{k+1}^T \end{pmatrix} R[Q]_k \begin{pmatrix} \Phi_{k,0} - \tilde{X}_k & G_k \mathcal{V}_k \end{pmatrix} = 0 \text{ for all } k \in [0, N]$$

iff $\mathcal{V}_{k+1}^T G_{k+1}^T R[Q]_k G_k \mathcal{V}_k = 0$ for all $k \in [0, N]$. Further details on this topic will be included in the Ph.D. dissertation of the second author.

7. Riccati Inequality

This section contains a comparison result for solutions of two discrete symplectic systems and the corresponding Riccati-type matrices. It is a generalization of the corresponding result known in [8, Theorem 1] for linear Hamiltonian systems to discrete symplectic systems.

If (x, u) is admissible, then

$$x_k^T \mathcal{C}_k^T \mathcal{A}_k x_k + 2\, x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{D}_k^T \mathcal{B}_k u_k = \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}^T \mathcal{G}_k \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}, \qquad (7.1)$$

where $\mathcal{G}_k$ is the symmetric $2n \times 2n$ matrix

$$\mathcal{G}_k := \begin{pmatrix} \mathcal{A}_k^T \mathcal{E}_k \mathcal{A}_k - \mathcal{A}_k^T \mathcal{C}_k & \mathcal{C}_k^T - \mathcal{A}_k^T \mathcal{E}_k \\ \mathcal{C}_k - \mathcal{E}_k \mathcal{A}_k & \mathcal{E}_k \end{pmatrix}$$

and $\mathcal{E}_k$ is any symmetric $n \times n$ matrix satisfying $\mathcal{D}_k^T \mathcal{B}_k = \mathcal{B}_k^T \mathcal{E}_k \mathcal{B}_k$, for example $\mathcal{E}_k = \mathcal{B}_k \mathcal{B}_k^\dagger \mathcal{D}_k \mathcal{B}_k^\dagger$, see e.g., [12].

With system (S) we consider another discrete symplectic system, denoted by ($\underline{\text{S}}$), with coefficient matrices $\underline{\mathcal{A}}_k$, $\underline{\mathcal{B}}_k$, $\underline{\mathcal{C}}_k$, $\underline{\mathcal{D}}_k$. Let $\underline{\mathcal{F}}_0$ be the discrete quadratic functional corresponding to symplectic system ($\underline{\text{S}}$) with associated symmetric matrices $\underline{\mathcal{E}}_k$ and $\underline{\mathcal{G}}_k$, which are defined in a parallel way as the matrices $\mathcal{E}_k$ and $\mathcal{G}_k$ above. Admissible pairs $(\underline{x}, \underline{u})$ are now defined by the equation $\underline{x}_{k+1} = \underline{\mathcal{A}}_k \underline{x}_k + \underline{\mathcal{B}}_k \underline{u}_k$, $k \in [0, N]$, and we shall emphasize this fact by saying that $(\underline{x}, \underline{u})$ is *admissible with respect to* $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$.

Theorem 7.1. Let (X, U) and $(\underline{X}, \underline{U})$ be any conjoined bases of (S) and (S̲), respectively. Furthermore, let Q_k and $\underline{Q}_k$ be symmetric matrices such that $X_k^T Q_k X_k = X_k^T U_k$ and $\underline{X}_k^T \underline{Q}_k \underline{X}_k = \underline{X}_k^T \underline{U}_k$ on $[0, N+1]$, and conditions

$$\text{Im}\begin{pmatrix}\underline{\mathcal{A}}_k - \mathcal{A}_k & \underline{\mathcal{B}}_k\end{pmatrix} \subseteq \text{Im}\,\mathcal{B}_k \quad \text{for all } k \in [0, N], \tag{7.2}$$

$$\text{Im}\,\underline{X}_0 \subseteq \text{Im}\,X_0, \quad \underline{X}_0^T (\underline{Q}_0 - Q_0)\, \underline{X}_0 \geq 0, \tag{7.3}$$

$$\mathcal{G}_k \leq \underline{\mathcal{G}}_k \quad \text{for all } k \in [0, N], \tag{7.4}$$

hold. If (X, U) has no focal points in $(0, N+1]$, then $(\underline{X}, \underline{U})$ has no focal points in $(0, N+1]$ either, and

$$\underline{X}_k^T (\underline{Q}_k - Q_k)\, \underline{X}_k \geq 0 \quad \text{for all } k \in [0, N+1]. \tag{7.5}$$

Before presenting the proof of Theorem 7.1 we need some preparatory considerations. If (X, U) is a conjoined basis of (S) satisfying kernel condition (6.3), then for any admissible (x, u) with $x_0 \in \text{Im}\,X_0$ we have $x_k \in \text{Im}\,X_k$ for all $k \in [0, N+1]$ and, by Lemma 6.3 with $\alpha = 0$,

$$x_k^T \mathcal{C}_k^T \mathcal{A}_k x_k + 2\, x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{D}_k^T \mathcal{B}_k u_k = \Delta(x_k^T \bar{Q}_k x_k) + \bar{w}_k^T \mathcal{P}_k \bar{w}_k, \tag{7.6}$$

where the symmetric matrix $\bar{Q}_k$ is given by the right-hand side of equation (4.5) and $\bar{w}_k = u_k - \bar{Q}_k x_k$, compare also with [6, Lemma 3]. It is interesting to observe that once formula (7.6) is established, then it is satisfied with *any* symmetric Q_k such that $X_k^T Q_k X_k = X_k^T \bar{Q}_k X_k$. In order to see this, we let $x_k = X_k c_k$ for $k \in [0, N+1]$ and some $c_k \in \mathbb{R}^n$, and then

$$x_k^T \bar{Q}_k x_k = c_k^T X_k^T \bar{Q}_k X_k c_k = c_k^T X_k^T Q_k X_k c_k = x_k^T Q_k x_k. \tag{7.7}$$

The following auxiliary result is a "symplectic" version of [5, Proposition 1], see also a more general result [10, Theorem 10.45].

Lemma 7.2. Let (X, U) be any conjoined basis of (S). Then (X, U) has no focal points in $(0, N+1]$ if and only if $\mathcal{F}_0(x, u) + d^T X_0^T U_0 d > 0$ for all admissible (x, u) with $x_0 = X_0 d$, $x_{N+1} = 0$, and $x \not\equiv 0$.

Proof of Theorem 7.1. Let (X, U) be a conjoined basis of (S) with no focal points in $(0, N+1]$. We proceed in the proof by showing the following steps.

Claim 1. For all $k \in [0, N]$ we have the inclusions

$$\text{Im}(\underline{X}_{k+1} - \mathcal{A}_k \underline{X}_k) \subseteq \text{Im}\,\mathcal{B}_k \subseteq \text{Im}\,X_{k+1}.$$

The first inclusion follows from the identity $\underline{X}_{k+1} - \mathcal{A}_k \underline{X}_k = (\underline{\mathcal{A}}_k - \mathcal{A}_k)\, \underline{X}_k + \underline{\mathcal{B}}_k \underline{U}_k$ and from assumption (7.2). The second inclusion is equivalent to $\text{Ker}\,X_{k+1} \subseteq \text{Ker}\,X_k$, by [16, Lemma 1(ii)], hence it is satisfied.

Claim 2. For all $k \in [0, N+1]$ we have the inclusion

$$\operatorname{Im} \underline{X}_k \subseteq \operatorname{Im} X_k. \tag{7.8}$$

We shall prove it by induction. By assumption (7.3), the statement holds for $k = 0$. Suppose that condition (7.8) holds for some $k \in [0, N]$. Then $\underline{X}_k = X_k X_k^\dagger \underline{X}_k$, see e.g., [3, Lemma A5], and then we have

$$\begin{aligned}\underline{X}_{k+1} &= \underline{X}_{k+1} - \mathcal{A}_k \underline{X}_k + \mathcal{A}_k X_k X_k^\dagger \underline{X}_k \\ &= X_{k+1} X_k^\dagger \underline{X}_k + (\underline{X}_{k+1} - \mathcal{A}_k \underline{X}_k) - (X_{k+1} - \mathcal{A}_k X_k) X_k^\dagger \underline{X}_k \\ &= X_{k+1} X_k^\dagger \underline{X}_k + (\underline{X}_{k+1} - \mathcal{A}_k \underline{X}_k) - \mathcal{B}_k U_k X_k^\dagger \underline{X}_k. \end{aligned} \tag{7.9}$$

The inclusion $\operatorname{Im} \underline{X}_{k+1} \subseteq \operatorname{Im} X_{k+1}$ then follows from Claim 1, since each of the three terms in (7.9) above is contained in $\operatorname{Im} X_{k+1}$.

Claim 3. If $(x, \underline{u})$ is admissible with respect to $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$, then $x_{k+1} - \mathcal{A}_k x_k \in \operatorname{Im} \mathcal{B}_k$ for all $k \in [0, N]$, i.e., there exist vectors $\{u_k\}_{k=0}^N$ such that (x, u) is admissible with respect to $(\mathcal{A}, \mathcal{B})$.

This is a consequence of assumption (7.2), since

$$x_{k+1} - \mathcal{A}_k x_k = (\underline{\mathcal{A}}_k - \mathcal{A}_k)\, x_k + \underline{\mathcal{B}}_k \underline{u}_k \in \operatorname{Im} \mathcal{B}_k.$$

Claim 4. If $(\underline{X}, \underline{U})$ is a solution of (S̲), then

$$\underline{X}_k^T \underline{\mathcal{C}}_k^T \underline{\mathcal{A}}_k \underline{X}_k + 2\, \underline{X}_k^T \underline{\mathcal{C}}_k^T \underline{\mathcal{B}}_k \underline{U}_k + \underline{U}_k^T \underline{\mathcal{D}}_k^T \underline{\mathcal{B}}_k \underline{U}_k = \Delta(\underline{X}_k^T \underline{U}_k) = \Delta(\underline{X}_k^T \underline{Q}_k \underline{X}_k). \tag{7.10}$$

This follows by a direct calculation.

Claim 5. For all $k \in [0, N]$ we have $\underline{X}_k^T (\underline{Q}_k - Q_k)\, \underline{X}_k \geq 0$, i.e., condition (7.5) holds.

We will show that $\Delta[\underline{X}_k^T (\underline{Q}_k - Q_k)\, \underline{X}_k] \geq 0$ for $k \in [0, N]$, which together with initial condition (7.3) imply the statement. Let $c \in \mathbb{R}^n$ be arbitrary and put $x_k := \underline{X}_k c$ and $\underline{u}_k := \underline{U}_k c$ on $[0, N+1]$. Then $(x, \underline{u})$ is admissible with respect to $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$ and

$$\begin{aligned} c^T \Delta[\underline{X}_k^T (\underline{Q}_k - Q_k)\, \underline{X}_k]\, c &= c^T \Delta(\underline{X}_k^T \underline{U}_k)\, c - c^T \Delta(\underline{X}_k^T Q_k \underline{X}_k)\, c \\ &\overset{(7.10),\,(7.1)}{=} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}^T \underline{\mathcal{G}}_k \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix} - \Delta(x_k^T Q_k x_k). \end{aligned} \tag{7.11}$$

By Claim 3, there exists $u = \{u_k\}_{k=0}^N$ such that (x, u) is admissible with respect to $(\mathcal{A}, \mathcal{B})$, while Claim 2 yields that $x_k \in \operatorname{Im} X_k$ for all $k \in [0, N+1]$. Hence, we get

$$\Delta(x_k^T Q_k x_k) \overset{(7.7)}{=} \Delta(x_k^T \bar{Q}_k x_k) \overset{(7.6),\,(7.1)}{=} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}^T \mathcal{G}_k \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix} - \bar{w}_k^T \mathcal{P}_k \bar{w}_k.$$

Using this identity in formula (7.11) and using assumption (7.4) we obtain

$$c^T \Delta[\underline{X}_k^T (\underline{Q}_k - Q_k)\, \underline{X}_k]\, c = \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}^T (\underline{\mathcal{G}}_k - \mathcal{G}_k) \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix} + \bar{w}_k^T \mathcal{P}_k \bar{w}_k \geq 0,$$

where we used the fact that $\mathcal{P}_k = P_k = X_k X_{k+1}^\dagger \mathcal{B}_k \geq 0$, since (X, U) is assumed to have no focal points in $(0, N+1]$.

Claim 6. The conjoined basis $(\underline{X}, \underline{U})$ has no focal points in $(0, N+1]$.

We shall prove this via Lemma 7.2. Let $(x, \underline{u})$ be admissible with respect to $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$ with $x_0 = \underline{X}_0 \underline{d}$ for some $\underline{d} \in \mathbb{R}^n$, $x_{N+1} = 0$, and $x \not\equiv 0$. Then, by Claim 3, there is $u = \{u_k\}_{k=0}^N$ such that (x, u) is admissible with respect to $(\mathcal{A}, \mathcal{B})$ and, by assumption (7.3), $x_0 = X_0 d$ for some $d \in \mathbb{R}^n$. Hence, we have

$$\begin{aligned}
\underline{\mathcal{F}}_0(x, \underline{u}) + \underline{d}^T \underline{X}_0^T \underline{U}_0 \underline{d} &\overset{(7.1)}{=} \sum_{k=0}^{N} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}^T \underline{\mathcal{G}}_k \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix} + \underline{d}^T \underline{X}_0^T \underline{Q}_0 \underline{X}_0 \underline{d} \\
&\overset{(7.4),\,(7.3)}{\geq} \sum_{k=0}^{N} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}^T \mathcal{G}_k \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix} + \underline{d}^T \underline{X}_0^T Q_0 \underline{X}_0 \underline{d} \\
&\overset{(7.1)}{=} \mathcal{F}_0(x, u) + d^T X_0^T U_0 d > 0,
\end{aligned}$$

since (X, U) is assumed to have no focal points in $(0, N+1]$. Thus, by Lemma 7.2, $(\underline{X}, \underline{U})$ has no focal points in $(0, N+1]$. This theorem is now proven. ■

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