On Global Asymptotic Stability of Nonlinear Stochastic Difference Equations with Delays

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Abstract

We consider the stochastic difference equation

\[ X_{n+1} = aF \left( X_n - \sum_{l=1}^{k} b_l X_{n-l} \right) + g(n, X_n, X_{n-1}, \ldots, X_{n-k}) \xi_{n+1}, \quad n \in \mathbb{N}_0 \]  

(0.1)

with arbitrary initial conditions \( X_0, X_{-1}, \ldots, X_{-k} \in \mathbb{R} \), non-linear continuous functions \( F \) and \( g \), and independent zero mean random variables \( \xi_n \). Equation (0.1) describes the dynamics of a neural network under stochastic perturbations.

We prove results on global a.s. asymptotic stability of the trivial solution \( X_n \) of equation (0.1). We show that (0.1) is a good discrete model for a corresponding stochastic continuous Itô equation, since under the same conditions on the functions \( F \) and \( g \), their solutions have similar asymptotic behavior.

Keywords: Stochastic difference and differential equations, a.s. asymptotic stability, martingale convergence theorems, Lyapunov–Krasovkii functionals, neural networks.

1. Introduction and Discussion of Result

To describe the dynamics of a single isolated neuron, Gopalsamy and Leung in [4] proposed the following delay differential equation

\[ \frac{dx(t)}{dt} = -x(t) + a \tanh [x(t) - bx(t - \tau)], \quad t \geq 0. \]  

(1.1)

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Here \( x(t) \) denotes the activation level of a neuron at time \( t \) and \( a \) is a real constant which is related to the range of the continuous variable \( x(\cdot) \), while \( b \) denotes a measure of the inhibitory influence of the past history, \( a > 0, \ b \geq 0 \), and \( \tau \) is the time delay, \( \tau \in [0, \infty) \) (for details see \([4, 17]\)). Accordingly, the equation

\[
\frac{dx(t)}{dt} = \left[ -x(t) + a \tanh \left( x(t) - \sum_{i=1}^{k} b_i x(t - \tau_i) \right) \right] \ dt, \quad t > 0, \quad (1.2)
\]
makes a neural network (see \([3]\) and references therein).

Hamaya and Sato in \([5]\) investigated the global attractivity of the equilibrium point of the difference equation

\[
x_{n+1} = a \tanh(x_n - bx_{n-k}), \quad n \in \mathbb{N}_0, \quad (1.3)
\]
where \( k \) is a nonnegative integer and \( a, b \in \mathbb{R} \). They proved that when

\[
a(1 - b) < 1, \quad a(1 + b) < 1, \quad (1.4)
\]

\[
\lim_{n \to \infty} x_n = 0, \text{ where } (x_n) \text{ solves (1.3) with arbitrary initial values } x_0, x_{-1}, \ldots, x_{-k} \in \mathbb{R}.
\]

They obtained a similar result for equations with several delays

\[
x_{n+1} = a \tanh \left( x_n - \sum_{i=1}^{m} b_i x_{n-k_i} \right), \quad n \in \mathbb{N}_0, \quad (1.5)
\]
under the assumptions that

\[
a \left( 1 - \sum_{i=1}^{m} b_i \right) < 1, \quad a \left( 1 + \sum_{i=1}^{m} b_i \right) < 1, \quad a > 0, \quad b_i \geq 0, \quad i = 1, \ldots, m. \quad (1.6)
\]

We note that (1.3) and (1.5) can be treated as Euler discretizations (with mesh size \( h = 1 \)) of equations (1.1) and (1.2) respectively.

In this paper we prove a stability result for the stochastic difference equation

\[
X_{n+1} = (1 - h)X_n + ahF \left( X_n - \sum_{l=1}^{k} b_l X_{n-l} \right) + \sqrt{h}g(n, X_n, X_{n-1}, \ldots, X_{n-k})\xi_{n+1}, \quad n \in \mathbb{N}_0, \quad (1.7)
\]
with arbitrary nonrandom initial values \( X_0, X_{-1}, \ldots, X_{-k} \in \mathbb{R} \). Here \( h \in (0, 1], \ F: \mathbb{R} \to \mathbb{R} \) is a continuous function and \( \xi_n \) are independent random variables, with zero mean \( \mathbb{E}\xi_n = 0 \) and unit variance \( \mathbb{E}\xi_n^2 = 1 \). Furthermore we attempt to connect our result for the difference equation (1.7) with the corresponding differential equation. That is
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(1.7) can be viewed as the Euler–Maruyama discretization of the stochastic differential Itô equation

\[
\begin{align*}
dX(t) &= \left[-X(t) + aF\left(X(t) - \sum_{i=1}^{r} \beta_i X(t - \tau_i)\right)\right] dt \\
&\quad + g(t, X(t), X(t - \tau_1), \ldots, X(t - \tau_r))dW_t, \quad t > 0,
\end{align*}
\]

with the one-dimensional Wiener process \( W = \{W(t)\}_{t \geq 0} \). In doing so, we show that the Euler–Maruyama method is dynamically consistent with the stochastic differential equation (1.8). More precisely, for a step size \( h \leq 1 \), solutions of the difference equation (1.7) converge almost surely under conditions which imply the almost sure convergence of solutions of the stochastic differential equation (1.8).

Equation (1.8) can be used to model the dynamics of a neural network under the influence of random perturbations.

We note that in this paper, we use the notation \( X(t) \) for the solution of (1.8) at time \( t \), although the notation \( X_t \) is commonly accepted for this purpose in the literature on stochastic processes. This enables one to avoid double indices, when we consider \( X(t - \tau_i) \).

The main result of the paper states that if for all \( u, u_0, u_1, \ldots, u_k \in \mathbb{R}, n \in \mathbb{N}, \)

\[
|F(u)| \leq |u|, \quad (1.9)
\]

\[
|g(n, u_0, u_1, \ldots, u_k)|^2 \leq \sum_{l=0}^{k} c_l |u_l|^2 + \gamma_n^2, \quad g(n, 0, \ldots, 0) = 0, \quad (1.10)
\]

\[
\sum_{i=1}^{\infty} \gamma_i^2 < \infty, \quad (1.11)
\]

and

\[
a^2 \left(1 + \sum_{l=1}^{k} |b_l| \right)^2 + \sum_{j=0}^{k} c_j < 1, \quad (1.12)
\]

then \( \lim_{n \to \infty} X_n = 0 \) a.s., where \( (X_n) \) is a solution of equation (1.7). We note that conditions (1.10) and (1.11) on the noise intensity \( g \) comprise of both additive and multiplicative types of noises.

Equation (1.5) can be obtained from (1.7) when \( h = 1 \), noise intensity \( g \equiv 0 \) and function \( F(u) \) coincides with \( \tanh(u) \). However condition (1.12) is more general than (1.6) since in (1.12) we do not assume positivity of the coefficients \( a \) and \( b_i, i = 1, \ldots, k \). We also note that in order that condition (1.12) is fulfilled it is necessary that

\[
|a| \left(1 + \sum_{l=1}^{k} |b_l| \right) < 1. \quad (1.13)
\]

On the other hand, if (1.13) holds, then (1.12) also holds for small enough coefficients \( c_i \) (i.e., for a small enough noise level).
Global asymptotic behaviour of the nonlinear stochastic difference equations was investigated in a series of papers e.g., [1, 18–23], where, among other approaches, the Liapunov–Krasovkii functional technique (see e.g., [12, 13]) and the semimartingale convergence theorems (see e.g., [14, 25]) play a central role. In this paper to prove our results we also construct special Liapunov–Krasovkii functionals and use the convergence theorem for semimartingale inequalities.

The structure of the paper is the following. To make our approach more transparent in Section 2 we prove a stability result for deterministic equations with one delay. Section 3 starts with necessary definitions and facts from the Theory of Stochastic Processes (Subsection 3.1) and gives the proof of the main result on a.s. global asymptotic stability of solution of (1.7) (Subsection 3.2). In Subsection 3.3 we show that (1.7) can be viewed as a model for an explicit Euler–Maruyama discretization of the continuous Itô equation (1.8) and demonstrate that sufficient conditions for the almost sure asymptotic stability of (1.8) also suffice to give the global asymptotic stability of the difference scheme (1.7). In Subsection 3.4 we present a proof of the a.s. asymptotic stability result for the Itô equation (1.8).

2. Deterministic One-Delay Difference Equation

In this section, we show how to improve the stability result in [5] by applying the appropriate Liapunov functional. To make our method more clear we consider equations with single delay

$$x_{n+1} = (1 - h)x_n + ahF(x_n - bx_{n-k}), \quad n \in \mathbb{N}_0,$$

with arbitrary initial conditions $x_0, x_{-1}, \ldots, x_{-k} \in \mathbb{R}$ and $a, b \in \mathbb{R}$. We suppose that the mesh size $h \in (0, 1]$. We also suppose that $F : \mathbb{R} \to \mathbb{R}$ is a nonrandom continuous function,

$$|F(u)| \leq |u| \quad \forall u \in \mathbb{R}, \quad (2.2)$$

and

$$|a|(1 + |b|) < 1. \quad (2.3)$$

**Theorem 2.1.** Let conditions (2.2) and (2.3) hold. Then $\lim_{n \to \infty} x_n = 0$, where $(x_n)$ is a solution to equation (2.1) with arbitrary $h \in (0, 1]$.

**Proof.** From the Hölder inequality we easily obtain that for all $m \in \mathbb{N}$ and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \ldots, m$,

$$\sum_{i=1}^{m} \alpha_i \beta_i \leq \left( \sum_{i=1}^{m} |\alpha_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{m} |\beta_i|^2 \right)^{\frac{1}{2}} \leq \sqrt{\sum_{i=1}^{m} |\alpha_i|} \sqrt{\sum_{i=1}^{m} |\beta_i|^2}. \quad (2.4)$$

We estimate

$$|(1 - h)x_n + ahF(x_n - bx_{n-k})| \leq (1 - h)|x_n| + h|a|(|x_n| + |b||x_{n-k}|). \quad (2.5)$$
Letting $m = 3, \quad \alpha_1 = 1 - h, \quad \beta_1 = |x_n|, \quad \alpha_2 = h|a|, \quad \beta_2 = |x_n|, \\
\alpha_3 = h|b||a|, \quad \beta_3 = |x_{n-k}|$

we get from (2.4) and (2.5)

$$x_{n+1}^2 \leq \left(1 - h + h|a|(1 + |b|)\right)^2 \left(1 - h + h|a|\right)x_n^2 + h|a|b|x_{n-k}^2. \quad (2.6)$$

We define

$$\alpha_0 = h|a||b| \left[1 - h + h|a|(1 + |b|)\right], \quad (2.7)$$

$$V_n^{(2)} = \alpha_0 \sum_{l=n-k}^{n-1} x_l^2, \quad (2.8)$$

$$V_n = x_n^2 + V_n^{(2)} . \quad (2.9)$$

Then

$$\Delta V_n^{(2)} = V_{n+1}^{(2)} - V_n^{(2)} = \alpha_0 \sum_{l=n+1-k}^{n} x_l^2 - \alpha_0 \sum_{l=n-k}^{n-1} x_l^2 = \alpha_0 x_n^2 - \alpha_0 x_{n-k}^2 . \quad (2.10)$$

Applying (2.6) and (2.10) we arrive at

$$\Delta V_n = x_{n+1}^2 - x_n^2 + V_{n+1}^{(2)} - V_n^{(2)} \leq - \left(1 - \left[1 - h + h|a|(1 + |b|)\right]^2\right) x_n^2 . \quad (2.11)$$

Condition (2.3) implies that for all $h \in (0, 1]$

$$\alpha^* = 1 - \left[1 - h + h|a|(1 + |b|)\right]^2 = h \left(1 - |a|(1 + |b|)\right) \left(2 - h + h|a|(1 + |b|)\right) > 0 .$$

By summation in (2.11) we get

$$V_{n+1} \leq V_0 - \sum_{i=0}^{n} \alpha^* x_i^2 . \quad (2.12)$$

To prove that $\lim_{n \to \infty} x_n = 0$ we suppose the opposite: $\limsup_{n \to \infty} x_n > 0$. Then for some $\varepsilon > 0$ there is a sequence $\{n_l\}_{l \in \mathbb{N}}$ such that $x_{n_l}^2 > \varepsilon$ for all $l \in \mathbb{N}$. We define

$$K(n) = \text{number of members of sequence } \{n_l\} \leq n,$$

and note that $K(n) \to \infty$ when $n \to \infty$. Remembering that $V_n \geq 0$ and applying (2.12) we arrive at a contradiction:

$$V_{n+1} \leq V_0 - \sum_{i=0}^{n} \alpha^* x_i^2 \leq V_0 - \sum_{l: n_l \leq n} \alpha^* x_{n_l}^2 \leq V_0 - \varepsilon \alpha^* K(n) \to -\infty ,$$

as $n \to \infty$. This concludes the proof. \[\blacksquare\]
3. Stochastic Equations

3.1. Auxiliary Definitions and Facts

In this section we state a number of necessary definitions and lemmas that we will use to prove our results. A detailed exposition of the definitions and facts of the theory of random processes can be found in, for example, [10, 14, 25].

Let $\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \in \mathbb{N}}, \mathbb{P}$ be a complete filtered probability space. Let $\{\xi_n\}_{n \in \mathbb{N}}$ be a sequence of independent random variables with $\mathbb{E}\xi_n = 0$ and $\mathbb{E}\xi_n^2 = 1$. We assume that the filtration $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ is naturally generated: $\mathcal{F}_{n+1} = \sigma\{\xi_i : i = 0, 1, \ldots, n\}$.

We use the standard abbreviation “a.s.” for the wordings “almost sure” or “almost surely” with respect to the fixed probability measure $\mathbb{P}$ throughout the text.

Among all sequences $\{X_n\}_{n \in \mathbb{N}}$ of random variables we distinguish those for which $X_n$ is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$.

A stochastic sequence $\{X_n\}_{n \in \mathbb{N}}$ is said to be an $\mathcal{F}_n$-martingale, if $\mathbb{E}|X_n| < \infty$ and $\mathbb{E}(X_n | \mathcal{F}_{n-1}) = X_{n-1}$ a.s. for all $n \in \mathbb{N}$. A stochastic sequence $\{\xi_n\}_{n \in \mathbb{N}}$ is said to be an $\mathcal{F}_n$-martingale-difference, if $\mathbb{E} |\xi_n| < \infty$ and $\mathbb{E}(\xi_n | \mathcal{F}_{n-1}) = 0$ a.s. for all $n \in \mathbb{N}$.

The following Lemma 3.1 gives a simple, but important type of martingale which appears in this paper.

**Lemma 3.1.** Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of independent $\mathcal{F}_n$-measurable random variables, $\mathbb{E}|x_n| = 0$, $\mathbb{E}|x_n| < \infty$. Let also $\{y_n\}_{n \in \mathbb{N}}$ be a sequence of $\mathcal{F}_n$-measurable random variables such that $\mathbb{E}|y_{n-1}x_n| < \infty$ for all $n \geq 1$. Then $\{Z_n\}_{n \in \mathbb{N}}$, $Z_n = \sum_{i=1}^{n} y_{i-1}x_i$ for all $n \in \mathbb{N}$, is an $\mathcal{F}_n$-martingale and $\{y_{n-1}x_n\}_{n \in \mathbb{N}}$ is an $\mathcal{F}_n$-martingale-difference.

The next lemma can be easily deduced from Doob’s decomposition theorem (cf., e.g., [25]). For the proof see [19].

**Lemma 3.2.** Let $\{\xi_n\}_{n \in \mathbb{N}}$ be an $\mathcal{F}_n$-martingale-difference. Then there exist an $\mathcal{F}_n$-martingale-difference $\{\mu_n\}_{n \in \mathbb{N}}$ as well as a positive $\mathcal{F}_{n-1}$-measurable stochastic sequence $\{\eta_n\}_{n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$,

$$\xi_n^2 = \mu_n + \eta_n, \quad \text{a.s.} \tag{3.1}$$

If $\xi_n$ are independent for all $n \geq 0$, then

$$\eta_n = \mathbb{E}\left(\xi_n^2\right), \quad \mu_n = \xi_n^2 - \mathbb{E}\left(\xi_n^2\right). \tag{3.2}$$

Below we present a version of the convergence theorem for semimartingale inequalities. This result is convenient for establishing the stability results in this paper. For the proof see [1].

**Lemma 3.3.** Let $\{Z_n\}_{n \in \mathbb{N}}$ be a non-negative $\mathcal{F}_n$-measurable process, $\mathbb{E}|Z_n| < \infty \ \forall n \in \mathbb{N}$, and

$$Z_{n+1} \leq Z_n + u_n - v_n + \nu_{n+1}, \quad n \in \mathbb{N}_0, \tag{3.3}$$
where \( \{\nu_n\}_{n \in \mathbb{N}} \) is an \( \mathcal{F}_n \)-martingale-difference, \( \{u_n\}_{n \in \mathbb{N}}, \{v_n\}_{n \in \mathbb{N}} \) are nonnegative \( \mathcal{F}_n \)-measurable processes, \( \mathbb{E}|u_n|, \mathbb{E}|v_n| < \infty \) \( \forall n \in \mathbb{N} \). Then

\[
\left\{ \omega : \sum_{n=1}^{\infty} u_n < \infty \right\} \subseteq \left\{ \omega : \sum_{n=1}^{\infty} v_n < \infty \right\} \bigcap \{Z_n \to \}.
\]

By \( \{Z_n \to \} \) we denote the set of all \( \omega \in \Omega \) for which \( \lim_{n \to \infty} Z_n(\omega) \) exists and is finite.

### 3.2. Stochastic Multi-Delay Difference Equation

We consider the equation

\[
X_{n+1} = (1 - h)X_n + ahF \left( X_n - \sum_{l=1}^{k} b_l X_{n-l} \right) + \sqrt{h}g(n, X_n, X_{n-1}, \ldots, X_{n-k}) \xi_{n+1}, \quad n \in \mathbb{N}_0,
\]

(3.4)

with arbitrary nonrandom initial values \( X_0, X_{-1}, \ldots, X_{-k} \in \mathbb{R} \), \( h \in (0, 1] \), continuous function \( F : \mathbb{R} \to \mathbb{R} \), satisfying (2.2), continuous function \( g : \mathbb{R}^{k+2} \to \mathbb{R} \), and independent random variables \( \xi_n, \mathbb{E}\xi_n = 0, \mathbb{E}\xi_n^2 = 1 \).

Suppose that the function \( g \) is non-random and there exist \( c_l \geq 0, \gamma_n \in \mathbb{R}, l = 0, 1, \ldots, k, n \in \mathbb{N} \), such that for all \( u_l \in \mathbb{R}, l = 0, 1, \ldots, k \), satisfies

\[
|g(n, u_0, u_1, \ldots, u_k)|^2 \leq \sum_{l=0}^{k} c_l |u_l|^2 + \gamma_n^2, \quad g(n, 0, \ldots, 0) = 0,
\]

(3.5)

\[
\sum_{i=1}^{\infty} \gamma_i^2 < \infty.
\]

(3.6)

We also suppose that

\[
a^2 \left( 1 + \sum_{l=1}^{k} |b_l| \right)^2 + \sum_{j=0}^{k} c_j < 1.
\]

(3.7)

**Theorem 3.4.** Let condition (2.2), (3.5), (3.6) and (3.7) be fulfilled. Then \( \lim_{n \to \infty} X_n = 0 \) a.s. where \( (X_n) \) is a solution to equation (3.4) with arbitrary \( h \in (0, 1] \).
Proof. We square equation (3.4) and obtain

\[
X_{n+1}^2 = \left[ (1 - h)X_n + ahF \left( X_n - \sum_{l=1}^{k} b_lX_{n-l} \right) \right]^2 \\
+ 2 \left[ (1 - h)X_n + ahF \left( X_n - \sum_{l=1}^{k} b_lX_{n-l} \right) \right] \sqrt{h}g(n, X_{n-k}, \ldots, X_n)\xi_{n+1} \\
+ hg^2(n, X_{n-k}, \ldots, X_n)\xi_{n+1}^2 \\
= \left[ (1 - h)X_n + ahF \left( X_n - \sum_{l=1}^{k} b_lX_{n-l} \right) \right]^2 + hg^2(n, X_{n-k}, \ldots, X_n) \\
+ \rho_{n+1}. 
\]

(3.8)

Here \( \{\rho_n\}_{n \in \mathbb{N}} \) is an \( \mathcal{F}_n \)-martingale-difference, defined by

\[
\rho_{n+1} = 2 \left[ (1 - h)X_n + ahF \left( X_n - \sum_{l=1}^{k} b_lX_{n-l} \right) \right] \sqrt{h}g(n, X_{n-k}, \ldots, X_n)\xi_{n+1} \\
+ hg^2(n, X_{n-k}, \ldots, X_n)[\xi_{n+1}^2 - 1]. 
\]

(3.9)

Indeed, by assumption, we have \( E\xi_{n+1}^2 = 1 \), and therefore \( \{\xi_{n+1}^2 - 1\}_{n \in \mathbb{N}} \) is \( \mathcal{F}_{n+1} \)-martingale-difference. Then, from Lemma 3.1, we conclude that both terms on the right-hand side of (3.9) are \( \mathcal{F}_{n+1} \)-martingale-differences.

By (2.2) we estimate

\[
\left| (1 - h)X_n + ahF \left( X_n - \sum_{l=1}^{k} b_lX_{n-l} \right) \right| \\
\leq |(1 - h)X_n| + |a|h \left| F \left( X_n - \sum_{l=1}^{k} b_lX_{n-l} \right) \right| \\
\leq |(1 - h)X_n| + |a|h \left| X_n - \sum_{l=1}^{k} b_lX_{n-l} \right| \\
\leq (1 - h)|X_n| + |a|h \left| X_n \right| + \sum_{l=1}^{k} |b_l||X_{n-l}| \\
= (1 - h + |a|h)|X_n| + |a|h \sum_{l=1}^{k} |b_l||X_{n-l}|. 
\]
Letting
\[ m = k + 1, \quad \alpha_1 = 1 - h + |a|h, \quad \beta_1 = |X_n|, \quad \alpha_i = |a|h|b_{i-1}|, \quad \beta_i = |X_{n+1-i}|, \]
i = 2, \ldots, k + 1, and applying inequality (2.4) we arrive at
\[
(1 - h + |a|h)X_n + |a|h \sum_{l=1}^{k} |b_l||X_{n-l}| \geq 0
\]
\[ \leq \left[ 1 - h + |a|h + |a|h \sum_{l=1}^{k} |b_l| \right] \left[ (1 - h + |a|h)X_n^2 + |a|h \sum_{l=1}^{k} |b_l|X_{n-l}^2 \right].
\] (3.10)

Taking into consideration (3.10) and (3.5) we estimate the right-hand side of (3.8)
\[
X_{n+1}^2 \leq \left[ 1 - h + |a|h + |a|h \sum_{l=1}^{k} |b_l| \right] \left[ (1 - h + |a|h)X_n^2 + |a|h \sum_{l=1}^{k} |b_l|X_{n-l}^2 \right] \]
\[ + h \sum_{l=0}^{k} c_l X_{n-l}^2 + h\gamma_n^2 + \rho_{n+1}
\]
\[ = \left[ \left( 1 - h + |a|h + |a|h \sum_{l=1}^{k} |b_l| \right) (1 - h + |a|h) + hc_0 \right] X_n^2
\]
\[ + h \sum_{l=1}^{k} \left[ |a| \left( 1 - h + |a|h + |a|h \sum_{j=1}^{k} |b_j| \right) |b_l| + c_l \right] X_{n-l}^2 + h\gamma_n^2 + \rho_{n+1}.
\] (3.11)

Let \( \tilde{c}_l = 0 \) for \( l = -k, -k + 1, \ldots, 0, k + 1, k + 2, \ldots \) and let
\[ \tilde{c}_l = |a| \left( 1 - h + |a|h + |a|h \sum_{j=1}^{k} |b_j| \right) |b_l| + c_l,
\]
for \( l = 1, 2, \ldots, k \). For \( n \in \mathbb{N} \) we define
\[ V_n^{(2)} = h \sum_{l=-k}^{n-1} X_{l}^2 \sum_{j=n-l}^{\infty} \tilde{c}_j, \quad V_n = x_n^2 + V_n^{(2)}.
\]

To find the increment of \( V_n^{(2)} \) we perform the following calculations:
\[
\Delta V_n^{(2)} = h \sum_{l=-k}^{n} X_{l}^2 \sum_{j=n+1-l}^{\infty} \tilde{c}_j - h \sum_{l=-k}^{n-1} X_{l}^2 \sum_{j=n-l}^{\infty} \tilde{c}_j
\]
\[ = h \sum_{l=-k}^{n-1} X_{l}^2 \sum_{j=n+1-l}^{\infty} \tilde{c}_j + hX_n^2 \sum_{j=1}^{\infty} \tilde{c}_j - h \sum_{l=-k}^{n-1} X_{l}^2 \sum_{j=n-l}^{\infty} \tilde{c}_j
\]
\[= h \sum_{l=-k}^{n-1} X_l^2 \sum_{j=-1}^{\infty} \tilde{c}_j - h \sum_{l=-k}^{n-1} X_l^2 \tilde{c}_{n-l} + h X_n^2 \sum_{j=1}^{\infty} \tilde{c}_j - h \sum_{l=-k}^{n-1} X_l^2 \sum_{j=-1}^{\infty} \tilde{c}_j \]

\[= -h \sum_{l=-k}^{n-1} X_l^2 \tilde{c}_{n-l} + h X_n^2 \sum_{j=1}^{\infty} \tilde{c}_j.\]

Since \( \tilde{c}_l = 0 \) for \( l \geq k + 1 \), the term \( h \sum_{l=-k}^{n-1} X_l^2 \tilde{c}_{n-l} \) can be written as

\[h \sum_{l=-k}^{n-1} X_l^2 \tilde{c}_{n-l} = h \sum_{l=1}^{k} X_{n-l}^2 \tilde{c}_l.\]

Therefore

\[\Delta V_{(2)}^{(2)} = -h \sum_{l=1}^{k} X_{n-l}^2 \tilde{c}_l + h X_n^2 \sum_{j=1}^{\infty} \tilde{c}_j. \tag{3.12}\]

Applying (3.11) and (3.12) we arrive at the following bounds:

\[\Delta V_n = X_{n+1}^2 - X_n^2 + V_{n+1}^{(2)} - V_n^{(2)} \leq \left\{ \left( 1 - h + |a|h + |a|h \sum_{l=1}^{k} |b_l| \right) \right\} \left( 1 - h + |a|h + h c_0 \right.\]

\[+ h \sum_{l=1}^{k} \left. \left| a \right| \left( 1 - h + |a|h + |a|h \sum_{j=1}^{k} |b_j| \right) |b_l| + c_l \right\}, \]

\[\times X_n^2 + h \gamma_n^2 + \rho_{n+1} - X_n^2 \]

\[= \left\{ \left( 1 - h + |a|h + |a|h \sum_{l=1}^{k} |b_l| \right) \right\} \left( 1 - h + |a|h + |a|h \sum_{j=1}^{k} |b_j| \right)

\[+ h \sum_{j=0}^{k} c_j \right\} X_n^2 + h \gamma_n^2 + \rho_{n+1} - X_n^2 \]

\[= \left\{ \left( 1 - h \left[ 1 - |a| \left( 1 + \sum_{l=1}^{k} |b_l| \right) \right] \right) \right\} \left( 1 - h + h \sum_{j=0}^{k} c_j \right\} X_n^2 + h \gamma_n^2 + \rho_{n+1}. \tag{3.13}\]
From (3.7) we obtain that for all $h \in (0, 1]$

$$0 < 1 - |a| \left(1 + \sum_{l=1}^{k} |b_l| \right) < 1, \quad 0 < h \left[1 - |a| \left(1 + \sum_{l=1}^{k} |b_l| \right) \right] < 1,$$

$$0 < 1 - h \left[1 - |a| \left(1 + \sum_{l=1}^{k} |b_l| \right) \right] \leq |a| \left(1 + \sum_{l=1}^{k} |b_l| \right),$$

and, therefore,

$$\left(1 - h \left[1 - |a| \left(1 + \sum_{l=1}^{k} |b_l| \right) \right] \right)^2 + h \sum_{j=0}^{k} c_j \leq a^2 \left(1 + \sum_{l=1}^{k} |b_l| \right)^2 + \sum_{j=0}^{k} c_j < 1.$$ 

We denote

$$\varepsilon = 1 - \left(1 - h \left[1 - |a| \left(1 + \sum_{l=1}^{k} |b_l| \right) \right] \right)^2 - h \sum_{j=0}^{k} c_j.$$

Then (3.13) takes the form

$$V_{n+1} \leq V_n - \varepsilon X_n^2 + h \gamma_n^2 + \rho_{n+1}. \quad (3.14)$$

We let $Z_n = V_n$, $u_n = h \gamma_n^2$, $v_n = \varepsilon X_n^2$ and $\nu_{n+1} = \rho_{n+1}$, and apply Lemma 3.3 to inequality (3.14). Since $\mathbb{P} \left\{ \omega : \sum_{n=1}^{\infty} u_n < \infty \right\} = \mathbb{P} \left\{ \omega : \sum_{n=1}^{\infty} \gamma_n < \infty \right\} = 1$ by condition (3.6), both the limits $\lim_{n \to \infty} Z_n = \lim_{n \to \infty} V_n$ and $\lim_{n \to \infty} \sum_{i=0}^{n} v_i = \varepsilon \lim_{n \to \infty} \sum_{i=1}^{n} X_i^2$ exist and are a.s. finite. Suppose that $\lim_{n \to \infty} X_n \neq 0$ with non zero probability. Then there is a set $\Omega_1$ such that $\mathbb{P} \Omega_1 > 0$, an a.s. finite random variable $\zeta(\omega) > 0$ and a sequence $\{n_l(\omega)\}_{l \in \mathbb{N}}$ such that $X_{n_l}^2(\omega) \geq \zeta(\omega), l \in \mathbb{N}, \omega \in \Omega_1$. For all $n \in \mathbb{N}$ and $\omega \in \Omega_1$, we define

$$K(n, \omega) = \text{number of members of sequence } \{n_k(\omega)\} \leq n,$$

and note that $K(n, \omega) \to \infty$ when $n \to \infty$. Then we arrive at the contradiction: for $\omega \in \Omega_1$

$$\infty > \sum_{n=1}^{\infty} v_n(\omega) \geq \varepsilon \sum_{i=0}^{n} X_i^2(\omega) \geq \varepsilon \sum_{l=0}^{n} X_{n_l}^2(\omega) \geq \zeta(\omega) \varepsilon K(n, \omega) \to \infty,$$

as $n \to \infty$. This concludes the proof.
3.3. Remarks on Numerical Methods and Stochastically Perturbed Equation of Neural Network

In this section, we compare the asymptotic behaviour of the solution of the stochastic difference equation (3.4) with the solution of the stochastic differential Itô equation.

\[
dX(t) = \left[ -X(t) + aF[X(t)] - \sum_{i=1}^{r} \beta_i X(t - \tau_i) \right] dt + g(t, X(t), X(t - \tau_1), \ldots, X(t - \tau_r))dW_t, \quad t > 0. \tag{3.15}
\]

Here \( W = \{W(t)\}_{t \geq 0} \) is a one-dimensional Wiener process, \( \tau_i, i = 1, \ldots, r, \) are non-random numbers, \( 0 < \tau_1 < \cdots < \tau_r = \tau. \) We suppose that an initial condition has the following form

\[
X_t(\omega) = \varphi(t), \quad t \in [-\tau, 0], \quad \omega \in \Omega, \tag{3.16}
\]

where function \( \varphi : [-\tau, 0] \rightarrow \mathbb{R} \) is nonrandom and continuous.

In the case when \( F(u) \equiv \tanh(u) \) and \( g(t, u_0, u_1, \ldots, u_r) \equiv 0, \) equation (3.15) coincides with (1.2). Therefore we can treat (3.15) as an equation describing the dynamics of a neural network under the influence of random perturbations.

Let \( h > 0 \) be the constant step size and \( \Delta W_{n+1} = W((n + 1)h) - W(nh). \) We approximate \( t \approx nh, \tau_i \approx m_i h, X_n(h) \approx X(nh), X_{n-m_i}(h) \approx X(h(n - m_i)), m_1 < m_2 < \cdots < m_r. \) Here \( X_n(h) \) are computed by the Euler–Maruyama numerical method for equation (3.15) by

\[
X_{n+1}(h) = (1 - h)X_n(h) - haF\left(X_n(h) - \sum_{i=1}^{r} \beta_i X_{n-m_i}(h)\right) \tag{3.17}
\]

\[
+ \sqrt{h}g(nh, X_n(h), X_{n-m_1}(h), \ldots, X_{n-m_r}(h))\Delta W_{n+1}.
\]

We set

\[
\xi_{n+1} = \frac{W((n + 1)h) - W(nh)}{\sqrt{h}}. \tag{3.18}
\]

Then \( \{\xi_n\}_{n \geq 0} \) is a sequence of standardized normally distributed random variables. We put \( k = m_r \) and, for all \( j = 1, 2, \ldots, k = m_r, \) we define

\[
b_j = \beta_i, \quad \text{if} \quad j = m_i \quad \text{for some} \quad i = 1, \ldots, r, \quad \text{otherwise} \quad b_i = 0.
\]

We also define

\[
\tilde{g}(t, a_0, a_1, a_2, \ldots, a_k) \equiv g(t, a_0, a_{m_1}, \ldots, a_{m_r}).
\]

Then equation (3.17) takes the form

\[
X_{n+1}(h) = (1 - h)X_n(h) - haF\left(X_n(h) - \sum_{j=1}^{k} b_j X_{n-j}(h)\right) + \sqrt{h}\tilde{g}(nh, X_n(h), X_{n-1}(h), \ldots, X_{n-k}(h))\Delta W_{n+1}. \tag{3.19}
\]
When $\xi_n$ are defined by (3.18), the equation (3.19) coincides with (3.4).

There is an extensive literature on the finite-time strong convergence of the Euler–Maruyama approximation (3.19) to the true solution of (3.15), for example, [7, 11, 15, 16]. However, we are not going to show the finite-time strong convergence of $X_n$ to $X(nh)$, we only analyze the stability of the numerical method: we find a step size $h$ such that the numerical method reproduces the characteristics of the test equation. In the case of linear stochastic differential equations the mean-square asymptotic stability of the numerical methods has been studied by several authors e.g., [6, 24]. The exponential mean-square stability of numerical methods for general nonlinear $n$-dimensional stochastic equation was investigated in [8]. But the almost sure asymptotic stability of numerical methods has been less studied. Only recently [24] discussed the almost sure asymptotic stability of the weak Euler–Maruyama method to a linear scalar stochastic differential equation.

In this paper we consider the almost sure asymptotic stability of the strong Euler–Maruyama method (3.17) to the nonlinear scalar stochastic differential equation (3.15). It transpires that (3.19) is a good discrete model of (3.15), since under the conditions stipulated in Theorem 3.4, solutions of the continuous problem have the same asymptotic behavior. Below we formulate an analog of Theorem 3.4 for equation (3.15). The proof of Theorem 3.5 is deferred to Subsection 3.4.

**Theorem 3.5.** Let condition (2.2), (3.5), (3.6) and (3.7) be fulfilled. Also suppose that the function $F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz-continuous and the function $g : [0, \infty) \times \mathbb{R}^{k+1} \to \mathbb{R}$ is Borel-measurable with respect to the first variable and locally Lipschitz-continuous with respect to last $k+1$ variables. Then $\lim_{t \to \infty} X(t) = 0$ a.s. where $X(t)$ is a solution of equation (3.15) with arbitrary initial conditions given in (3.16).

### 3.4. Proof of Asymptotic Stability for Stochastically Perturbed Equations for Neural Networks

For simplicity we give a proof of Theorem 3.5 for a one delay stochastic Itô equation:

$$dX(t) = [-X(t) + aF[X(t) - \beta X(t - \tau)]] dt + g(t, X(t), X(t - \tau))dW_t, \quad t > 0. \tag{3.20}$$

Multi-delay case can be treated in similar way.

In this subsection we will apply the following variant of the Itô formula (see [9, 10, 14, 15]).

**Lemma 3.6.** Let $\{X_t\}_{t \geq 0}$ be a stochastic process with differential $dX_t = c(t, X_t)dt + g(t, X_t)dW_t$. Let $V(t, x)$ be a differentiable function with respect to the first argument
and a twice differentiable function with respect to the second argument. Then

\[
V(t, X_t) = V(0, X_0) + \int_0^t \frac{\partial V(s, X_s)}{\partial x} c(s, X_s) ds + \int_0^t \frac{\partial V(s, X_s)}{\partial s} ds + \frac{1}{2} \int_0^t \frac{\partial^2 V(s, X_s)}{\partial x^2} g^2(s, X_s) ds + \int_0^t \frac{\partial V(s, X_s)}{\partial x} g(s, X_s) dW_s. \tag{3.21}
\]

To establish asymptotic stability of the solution to (3.20) we will use a continuous analogue of Lemma 3.3.

**Lemma 3.7.** Suppose that \( \{Z_t\}_{t \geq 0} \) is a nonnegative continuous \( \mathcal{F}_t \)-semimartingale and its stochastic differential \( dZ_t \) is estimated by the inequality

\[
dZ_t \leq dA_t^{(1)} - dA_t^{(2)} + dM_t, \quad t \geq 0, \tag{3.22}
\]

where \( \{A_t^{(i)}\}_{t \geq 0}, i = 1, 2 \), are continuous nondecreasing \( \mathcal{F}_t \)-measurable processes with \( A_0^{(i)} = 0 \) and \( \{M_t\}_{t \geq 0} \) is a continuous \( \mathcal{F}_t \)-martingale started at \( M_0 = 0 \). Then

\[
\{\omega \in \Omega : A^{(1)}_\infty(\omega) < \infty\} \subseteq \{\omega \in \Omega : Z_t(\omega) \to \} \cap \{\omega \in \Omega : A^{(2)}_\infty(\omega) < \infty\} \quad \text{a.s.}
\]

**Proof of Theorem 3.5.** Since the coefficients of equation (3.15) are locally Lipschitz-continuous in the space variables and Borel-measurable with respect to time \( t \), there exists local unique, a.s. continuous, strong solution \( X(t) \) of (3.15), (3.16) on the interval \([0, \tau_n \wedge T]\), where \( \tau_n = \inf\{t : |X(t)| \geq n\}, \tau_n \wedge T = \min\{\tau_n, T\} \) (see [9, Theorem 2.5 on page 287] or [10]).

Applying the Itô formula to (3.20), and then Hölder inequality and inequality (2.4), we obtain for \( t \leq \tau_n \wedge T \)

\[
dX^2(t) = 2X(t) \left(-X(t) + aF[X(t) - bX(t - \tau)] + g^2[t, X(t), X(t - \tau)]\right) dt
+ 2X(t)g(t, X(t), X(t - \tau))dW_t
\leq \left(-2X^2(t) + a^2F^2[X(t) - bX(t - \tau)] + X(t)^2 + c_1X^2(t)
+ c_2X^2(t - \tau) + \gamma_2^2\right) dt2X(t)g(t, X(t), X(t - \tau))dW_t
\leq \left(-(1 - c_1)X^2(t) + a^2[X(t) + bX(t - \tau)]^2 + c_2X^2(t - \tau) + \gamma_2^2\right) dt
+ 2X(t)g(t, X(t), X(t - \tau))dW_t
\leq \left(-(1 - c_1)X^2(t) + a^2(1 + |b|)[X^2(t) + bX^2(t - \tau)]
+ c_2X^2(t - \tau) + \gamma_2^2\right) dt2X(t)g(t, X(t), X(t - \tau))dW_t
\leq \left(-(1 - c_1 - a^2(1 + |b|))X^2(t) + (c_2 + a^2(1 + |b|)|b|)X^2(t - \tau) + \gamma_2^2\right) dt
+ 2X(t)g(t, X(t), X(t - \tau))dW_t. \tag{3.23}
\]

We define

\[
\tilde{c} := c_2 + a^2(1 + |b|)|b|, \quad \tilde{\epsilon} := 1 - (c_1 + c_2) - a^2(1 + |b|)^2,
\]
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and

\[ V = X^2(t) + V^{(2)}(t), \quad V^{(2)}(t) = \tilde{c} \int_{t-\tau}^t X(s)^2 ds. \]  (3.24)

It is easy to see that

\[ \frac{\partial V^{(2)}(t)}{\partial t} = \tilde{c}X^2(t) - \tilde{c}X^2(t - \tau). \]  (3.25)

Applying the Itô formula once again we get for \( t \leq \tau_n \wedge T \)

\[ dV(t) \leq (- (1 - c_1 - a^2(1 + |b|))X^2(t) + (c_2 + a^2(1 + |b|)|b|)X^2(t) + \gamma^2) dt \]
\[ + 2X(t)g(t, X(t), X(t - \tau))dW_t \]
\[ = - \tilde{c}X^2(t)dt + \tilde{c}\gamma^2 dt + 2X(t)g(t, X(t), X(t - \tau))dW_t. \]  (3.26)

Set \( t_n = t \wedge \tau_n \) for \( n \in \mathbb{N} \). We note that \( \tau_n \) is a Markov time (stopping time), hence \( t_n \) is a finite Markov time (finite stopping time). After the integration of inequality (3.26), we arrive at

\[ V(t_n) \leq V(0) - \tilde{c} \int_0^{t_n} X^2(s)ds + \int_0^{t_n} \gamma^2 ds + 2 \int_0^{t_n} X(s)g(s, X(s), X(s - \tau))dW_s \]
\[ \leq V(0) + \int_0^{t_n} \gamma^2 ds + 2 \int_0^{t_n} X(s)g(s, X(s), X(s - \tau))dW_s. \]  (3.27)

To prove the theorem we need to prove that:

(a) for any initial condition (3.16) solution \( X(t) \) of equation (3.20) exists on \([0, \infty)\),

(b) \( \lim_{t \to \infty} X(t) = 0 \) a.s.

(a) Global existence of solutions. We take expectations on both sides of the integral inequality (3.27) and obtain

\[ \mathbb{E}V(t_n) \leq V_0 + \int_0^{t_n} \gamma^2 ds \leq K(t). \]  (3.28)

Then we may conclude that, for arbitrary \( t \in (0, \infty) \), the estimate

\[ 0 \leq \mathbb{P}\{t > \tau_n\} \leq \frac{\mathbb{E}V(t_n)}{n} \leq \frac{K(t)}{n} \to 0 \]

holds as \( n \to \infty \). Since \( t > 0 \) is arbitrary this means that \( \tau_n \to \infty \) a.s. as \( n \to \infty \), i.e., the first explosion time \( \tau \) of solution \( X(t) \) is equal to \( \infty \) a.s. Hence the solution \( X(t) \) of (3.20), (3.16) exists and is unique on \([-\tau, \infty)\).
A.s. asymptotic stability of a trivial solutions. Notice that \( V = \{ V(t) \}_{t \geq 0} \) is a nonnegative continuous \( \mathcal{F}_t \)-semimartingale. Now, since condition (3.6) is fulfilled, we may apply Lemma 3.7 with
\[
A_t^{(1)} = \int_0^t \gamma_s^2 ds,
\]
\[
A_t^{(2)} = \tilde{\varepsilon} \int_0^t X^2(s) ds,
\]
\[
M_t = 2 \int_0^t X(s) g(s, X(s), X(s - \tau)) dW_s,
\]
and conclude that both, \( \lim_{t \to \infty} V(t) \) and
\[
\lim_{t \to \infty} A_t^{(2)} = \tilde{\varepsilon} \int_0^\infty X^2(s) ds,
\]
exist and are a.s. finite. By (3.24) and by convergence of \( V(t) \), as \( t \to \infty \), we can conclude that \( X(t) \) is a.s. bounded on \([0, \infty)\), and, therefore, there exists a.s. finite random variable \( K = K(\omega) \) such that \( \mathbb{P} \{ \sup_{t > 0} |X(t)| < K \} = 1 \).

It remains to show that \( X(t) \) converges to 0 a.s., as \( t \to \infty \). To do this we note that a.s.
\[
|V^{(2)}(t_1) - V^{(2)}(t_2)| = \tilde{c} \left| \int_{t_2}^{t_1} X^2(\tau) d\tau - \int_{t_1}^{t_2} X^2(\tau) d\tau \right| \leq 2\tilde{c}K^2|t_2 - t_1|.
\]
Suppose now that \( \mathbb{P} \left\{ \lim_{t \to \infty} \sup_{t \geq 0} X^2(t) = \zeta_0(\omega) > 0 \right\} = p_0 > 0 \). Then there exists a sequence \( \{t_k(\omega)\}_{k \in \mathbb{N}} \) such that \( \mathbb{P}(\Omega_1) = p_0 \) with \( \Omega_1 = \{ \omega: X^2(t_k(\omega), \omega) > \zeta_0(\omega)/2 > 0 \} \).

From definition (3.24) of \( V \), we have \( X^2(t) = V(t) - V^{(2)}(t) \), and then
\[
|X^2(t) - X^2(s)| \leq |V(t) - V(s)| + |V^{(2)}(t) - V^{(2)}(s)|.
\]
Due to convergence of \( V(t) \), there exists \( T = T(\omega) \) such that
\[
|V(t) - V(s)| \leq \zeta_0(\omega)/8
\]
for \( t, s \geq T \). From (3.30) we can find \( \delta = \delta(\zeta_0(\omega)) \) such that \( |V^{(2)}(t) - V^{(2)}(s)| \leq \zeta_0(\omega)/8 \), when \( |t - s| \leq \delta \). Then
\[
|X^2(t_k) - X^2(s)| \leq \zeta_0(\omega)/4
\]
for \( \omega \in \Omega_1, |s - t_k| \leq \delta, t_k > T \). Thus we obtain for \( \omega \in \Omega_1, |s - t_k| \leq \delta, t_k > T, \)
\[
|X^2(s)| \geq |X^2(t_k)| - |X^2(t_k) - X^2(s)| \geq \zeta_0(\omega)/2 - \zeta_0(\omega)/4 = \zeta_0(\omega)/4.
\]
(3.31)
Let \( k(n) = k(n, \omega) \) be a number of elements of the sequence \( \{t_k\} \) in the interval \([T, n]\). We note that \( k(n) \to \infty \) as \( n \to \infty \). Applying inequality \((3.31)\) we obtain for \( \omega \in \Omega_1, n > T, \)

\[
\int_0^n X^2(s)ds \geq \int_T^n X^2(s)ds \geq \sum_{k:T \leq t_k+\delta \leq n} \int_{t_k-\delta}^{t_k+\delta} X^2(s)ds \\
\geq \frac{\zeta_0(\omega)}{4} \sum_{k:T \leq t_k+\delta \leq n} \int_{t_k-\delta}^{t_k+\delta} ds = \frac{2\delta \zeta_0^2(\omega)}{4} \sum_{k \leq k(n)} 1 = \frac{k(n)\delta \zeta_0^2(\omega)}{2} \to \infty
\]
as \( n \to \infty \). Hence \( \mathbb{P}\{A^2(\infty) = \infty\} \geq p_0 > 0 \). This contradicts \((3.29)\) proving the result. ■

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**References**


