International Journal of Difference Equations.
ISSN 0973-6069 Volume 1 Number 1 (2006), pp. 101–118

http://www.ripublication.com/ijde.htm

On Global Asymptotic Stability of Nonlinear Stochastic Difference Equations with Delays

Yoshihiro Hamaya* **and Alexandra Rodkina**†
**Department of Information Science, Okayama University of Science, Japan*
†Department of Math/CSci, University of the West Indies, Jamaica

Abstract

We consider the stochastic difference equation

$$X_{n+1} = aF\left(X_n - \sum_{l=1}^{k} b_l X_{n-l}\right) + g(n, X_n, X_{n-1}, \dots, X_{n-k})\xi_{n+1}, \quad n \in \mathbb{N}_0 \tag{0.1}$$

with arbitrary initial conditions $X_0, X_{-1}, \dots, X_{-k} \in \mathbb{R}$, non-linear continuous functions F and g, and independent zero mean random variables ξ_n. Equation (0.1) describes the dynamics of a neural network under stochastic perturbations.

We prove results on global a.s. asymptotic stability of the trivial solution X_n of equation (0.1). We show that (0.1) is a good discrete model for a corresponding stochastic continuous Itô equation, since under the same conditions on the functions F and g, their solutions have similar asymptotic behavior.

AMS subject classification: 39A11, 34K20, 34F05, 93E15, 60H10, 92B20.
Keywords: Stochastic difference and differential equations, a.s. asymptotic stability, martingale convergence theorems, Lyapunov–Krasovkii functionals, neural networks.

1. Introduction and Discussion of Result

To describe the dynamics of a single isolated neuron, Gopalsamy and Leung in [4] proposed the following delay differential equation

$$\frac{dx(t)}{dt} = -x(t) + a\tanh\left[x(t) - bx(t-\tau)\right], \qquad t \geq 0. \tag{1.1}$$

Received March 12, 2006; Accepted May 10, 2006

Here $x(t)$ denotes the activation level of a neuron at time t and a is a real constant which is related to the range of the continuous variable $x(\cdot)$, while b denotes a measure of the inhibitory influence of the past history, $a > 0,\ \ b \geq 0$, and τ is the time delay, $\tau \in [0, \infty)$ (for details see [4, 17]). Accordingly, the equation

$$dx(t) = \left[-x(t) + a \tanh \left(x(t) - \sum_{i=1}^{k} b_i x(t - \tau_i) \right) \right] dt, \quad t > 0, \tag{1.2}$$

models a neural network (see [3] and references therein).

Hamaya and Sato in [5] investigated the global attractivity of the equilibrium point of the difference equation

$$x_{n+1} = a \tanh(x_n - b x_{n-k}), \qquad n \in \mathbb{N}_0, \tag{1.3}$$

where k is a nonnegative integer and $a, b \in \mathbb{R}$. They proved that when

$$a(1 - b) < 1, \quad a(1 + b) < 1, \tag{1.4}$$

$\lim\limits_{n\to\infty} x_n = 0$, where (x_n) solves (1.3) with arbitrary initial values $x_0, x_{-1}, \ldots, x_{-k} \in \mathbb{R}$. They obtained a similar result for equations with several delays

$$x_{n+1} = a \tanh \left(x_n - \sum_{i=1}^{m} b_i x_{n-k_i} \right), \qquad n \in \mathbb{N}_0, \tag{1.5}$$

under the assumptions that

$$a \left(1 - \sum_{i=1}^{m} b_i \right) < 1, \quad a \left(1 + \sum_{i=1}^{m} b_i \right) < 1, \quad a > 0,\ b_i \geq 0, \quad i = 1, \ldots, m. \tag{1.6}$$

We note that (1.3) and (1.5) can be treated as Euler discretizations (with mesh size $h = 1$) of equations (1.1) and (1.2) respectively.

In this paper we prove a stability result for the stochastic difference equation

$$\begin{aligned} X_{n+1} = (1 - h) X_n + ahF \left(X_n - \sum_{l=1}^{k} b_l X_{n-l} \right) \\ + \sqrt{h} g(n, X_n, X_{n-1}, \ldots, X_{n-k}) \xi_{n+1}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{1.7}$$

with arbitrary nonrandom initial values $X_0, X_{-1}, \ldots, X_{-k} \in \mathbb{R}$. Here $h \in (0, 1]$, $F : \mathbb{R} \to \mathbb{R}$ is a continuous function and ξ_n are independent random variables, with zero mean $\mathbb{E}\xi_n = 0$ and unit variance $\mathbb{E}\xi_n^2 = 1$. Furthermore we attempt to connect our result for the difference equation (1.7) with the corresponding differential equation. That is

(1.7) can be viewed as the Euler–Maruyama discretization of the stochastic differential Itô equation

$$\begin{aligned} dX(t) &= \left[-X(t) + aF\left(X(t) - \sum_{i=1}^{r} \beta_i X(t-\tau_i)\right)\right] dt \\ &\quad + g(t, X(t), X(t-\tau_1), \dots, X(t-\tau_r)) dW_t, \quad t > 0, \end{aligned} \tag{1.8}$$

with the one-dimensional Wiener process $W = \{W(t)\}_{t\geq 0}$. In doing so, we show that the Euler–Maruyama method is dynamically consistent with the stochastic differential equation (1.8). More precisely, for a step size $h \leq 1$, solutions of the difference equation (1.7) converge almost surely under conditions which imply the almost sure convergence of solutions of the stochastic differential equation (1.8).

Equation (1.8) can be used to model the dynamics of a neural network under the influence of random perturbations.

We note that in this paper, we use the notation $X(t)$ for the solution of (1.8) at time t, although the notation X_t is commonly accepted for this purpose in the literature on stochastic processes. This enables one to avoid double indices, when we consider $X(t-\tau_i)$.

The main result of the paper states that if for all $u, u_0, u_1, \dots, u_k \in \mathbb{R}$, $n \in \mathbb{N}$,

$$|F(u)| \leq |u|, \tag{1.9}$$

$$|g(n, u_0, u_1, \dots u_k)|^2 \leq \sum_{l=0}^{k} c_l |u_l|^2 + \gamma_n^2, \quad g(n, 0, \dots, 0) = 0, \tag{1.10}$$

$$\sum_{i=1}^{\infty} \gamma_i^2 < \infty, \tag{1.11}$$

and

$$a^2 \left(1 + \sum_{l=1}^{k} |b_l|\right)^2 + \sum_{j=0}^{k} c_j < 1, \tag{1.12}$$

then $\lim_{n\to\infty} X_n = 0$ a.s., where (X_n) is a solution of equation (1.7). We note that conditions (1.10) and (1.11) on the noise intensity g comprise of both additive and multiplicative types of noises.

Equation (1.5) can be obtained from (1.7) when $h = 1$, noise intensity $g \equiv 0$ and function $F(u)$ coincides with $\tanh(u)$. However condition (1.12) is more general than (1.6) since in (1.12) we do not assume positivity of the coefficients a and b_i, $i = 1, \dots, k$. We also note that in order that condition (1.12) is fulfilled it is necessary that

$$|a| \left(1 + \sum_{l=1}^{k} |b_l|\right) < 1. \tag{1.13}$$

On the other hand, if (1.13) holds, then (1.12) also holds for small enough coefficients c_i (i.e., for a small enough noise level).

Global asymptotic behaviour of the nonlinear stochastic difference equations was investigated in a series of papers e.g., [1, 18–23], where, among other approaches, the Liapunov–Krasovkii functional technique (see e.g., [12, 13]) and the semimartingale convergence theorems (see e.g., [14, 25]) play a central role. In this paper to prove our results we also construct special Liapunov–Krasovkii functionals and use the convergence theorem for semimartingale inequalities.

The structure of the paper is the following. To make our approach more transparent in Section 2 we prove a stability result for deterministic equations with one delay. Section 3 starts with necessary definitions and facts from the Theory of Stochastic Processes (Subsection 3.1) and gives the proof of the main result on a.s. global asymptotic stability of solution of (1.7) (Subsection 3.2). In Subsection 3.3 we show that (1.7) can be viewed as a model for an explicit Euler–Maruyama discretization of the continuous Itô equation (1.8) and demonstrate that sufficient conditions for the almost sure asymptotic stability of (1.8) also suffice to give the global asymptotic stability of the difference scheme (1.7). In Subsection 3.4 we present a proof of the a.s. asymptotic stability result for the Itô equation (1.8).

2. Deterministic One-Delay Difference Equation

In this section, we show how to improve the stability result in [5] by applying the appropriate Liapunov functional. To make our method more clear we consider equations with single delay

$$x_{n+1} = (1-h)x_n + ahF(x_n - bx_{n-k}), \quad n \in \mathbb{N}_0, \tag{2.1}$$

with arbitrary initial conditions $x_0, x_{-1}, \dots, x_{-k} \in \mathbb{R}^1$ and $a, b \in \mathbb{R}$. We suppose that the mesh size $h \in (0,1]$. We also suppose that $F : \mathbb{R} \to \mathbb{R}$ is a nonrandom continuous function,

$$|F(u)| \le |u| \quad \forall u \in \mathbb{R}, \tag{2.2}$$

and

$$|a|(1+|b|) < 1. \tag{2.3}$$

Theorem 2.1. Let conditions (2.2) and (2.3) hold. Then $\lim\limits_{n\to\infty} x_n = 0$, where (x_n) is a solution to equation (2.1) with arbitrary $h \in (0,1]$.

Proof. From the Hölder inequality we easily obtain that for all $m \in \mathbb{N}$ and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \dots, m$,

$$\sum_{i=1}^m \alpha_i\beta_i \le \sum_{i=1}^m |\alpha_i|^{\frac12}\big(|\alpha_i|^{\frac12}|\beta_i|\big) \le \sqrt{\sum_{i=1}^m |\alpha_i|}\,\sqrt{\sum_{i=1}^m |\alpha_i||\beta_i|^2}. \tag{2.4}$$

We estimate

$$|(1-h)x_n + ahF(x_n - bx_{n-k})| \le (1-h)|x_n| + h|a|\big(|x_n| + |b||x_{n-k}|\big). \tag{2.5}$$

Letting

$$m = 3, \quad \alpha_1 = 1 - h, \quad \beta_1 = |x_n|, \quad \alpha_2 = h|a|, \quad \beta_2 = |x_n|,$$
$$\alpha_3 = h|b||a|, \qquad \beta_3 = |x_{n-k}|,$$

we get from (2.4) and (2.5)

$$\begin{aligned} x_{n+1}^2 &\le \left[(1-h)|x_n| + h|a|\big(|x_n| + |b||x_{n-k}|\big)\right]^2 \\ &\le \left[1 - h + h|a|\big(1+|b|\big)\right]\left[(1-h+h|a|)x_n^2 + h|a||b|x_{n-k}^2\right]. \end{aligned} \tag{2.6}$$

We define

$$\alpha_0 = h|a||b|\left[1 - h + h|a|\big(1+|b|\big)\right], \tag{2.7}$$

$$V_n^{(2)} = \alpha_0 \sum_{l=n-k}^{n-1} x_l^2, \tag{2.8}$$

$$V_n = x_n^2 + V_n^{(2)}. \tag{2.9}$$

Then

$$\Delta V_n^{(2)} = V_{n+1}^{(2)} - V_n^{(2)} = \alpha_0 \sum_{l=n+1-k}^{n} x_l^2 - \alpha_0 \sum_{l=n-k}^{n-1} x_l^2 = \alpha_0 x_n^2 - \alpha_0 x_{n-k}^2. \tag{2.10}$$

Applying (2.6) and (2.10) we arrive at

$$\Delta V_n = x_{n+1}^2 - x_n^2 + V_{n+1}^{(2)} - V_n^{(2)} \le -\left(1 - \left[1 - h + h|a|\big(1+|b|\big)\right]^2\right) x_n^2. \tag{2.11}$$

Condition (2.3) implies that for all $h \in (0,1]$

$$\alpha^* = 1 - \left[1 - h + h|a|\big(1+|b|\big)\right]^2 = h\left(1 - |a|\big(1+|b|\big)\right)\left(2 - h + h|a|\big(1+|b|\big)\right) > 0.$$

By summation in (2.11) we get

$$V_{n+1} \le V_0 - \sum_{i=0}^{n} \alpha^* x_i^2. \tag{2.12}$$

To prove that $\lim_{n\to\infty} x_n = 0$ we suppose the opposite: $\limsup_{n\to\infty} x_n > 0$. Then for some $\varepsilon > 0$ there is a sequence $\{n_l\}_{l\in\mathbb{N}}$ such that $x_{n_l}^2 > \varepsilon$ for all $l \in \mathbb{N}$. We define

$$K(n) = \text{number of members of sequence } \{n_l\} \le n,$$

and note that $K(n) \to \infty$ when $n \to \infty$. Remembering that $V_n \ge 0$ and applying (2.12) we arrive at a contradiction:

$$V_{n+1} \le V_0 - \sum_{i=0}^{n} \alpha^* x_i^2 \le V_0 - \sum_{l: i_l \le n} \alpha^* x_{i_l}^2 \le V_0 - \varepsilon\alpha^* K(n) \to -\infty,$$

as $n \to \infty$. This concludes the proof. ■

3. Stochastic Equations

3.1. Auxiliary Definitions and Facts

In this section we state a number of necessary definitions and lemmas that we will use to prove our results. A detailed exposition of the definitions and facts of the theory of random processes can be found in, for example, [10, 14, 25].

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a complete filtered probability space. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of independent random variables with $\mathbb{E}\xi_n = 0$ and $\mathbb{E}\xi_n^2 = 1$. We assume that the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is naturally generated: $\mathcal{F}_{n+1} = \sigma\{\xi_{i+1} : i = 0, 1, \dots, n\}$.

We use the standard abbreviation "a.s." for the wordings "almost sure" or "almost surely" with respect to the fixed probability measure $\mathbb{P}$ throughout the text.

Among all sequences $\{X_n\}_{n\in\mathbb{N}}$ of random variables we distinguish those for which X_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$.

A stochastic sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be an $\mathcal{F}_n$*-martingale*, if $\mathbb{E}|X_n| < \infty$ and $\mathbb{E}\big(X_n \big| \mathcal{F}_{n-1}\big) = X_{n-1}$ a.s. for all $n \in \mathbb{N}$. A stochastic sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is said to be an $\mathcal{F}_n$*-martingale-difference*, if $\mathbb{E}|\xi_n| < \infty$ and $\mathbb{E}\big(\xi_n \big| \mathcal{F}_{n-1}\big) = 0$ a.s. for all $n \in \mathbb{N}$.

The following Lemma 3.1 gives a simple, but important type of martingale which appears in this paper.

Lemma 3.1. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of independent $\mathcal{F}_n$-measurable random variables, $\mathbb{E}[x_n] = 0$, $\mathbb{E}|x_n| < \infty$. Let also $\{y_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}_n$-measurable random variables such that $\mathbb{E}|y_{n-1}x_n| < \infty$ for all $n \geq 1$. Then $\{Z_n\}_{n\in\mathbb{N}}$, $Z_n = \sum_{i=1}^{n} y_{i-1}x_i$ for all $n \in \mathbb{N}$, is an $\mathcal{F}_n$-martingale and $\{y_{n-1}x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{F}_n$-martingale-difference.

The next lemma can be easily deduced from Doob's decomposition theorem (cf., e.g., [25]). For the proof see [19].

Lemma 3.2. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be an $\mathcal{F}_n$-martingale-difference. Then there exist an $\mathcal{F}_n$-martingale-difference $\{\mu_n\}_{n\in\mathbb{N}}$ as well as a positive $\mathcal{F}_{n-1}$-measurable stochastic sequence $\{\eta_n\}_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$,

$$\xi_n^2 = \mu_n + \eta_n, \quad \text{a.s.} \tag{3.1}$$

If ξ_n are independent for all $n \geq 0$, then

$$\eta_n = \mathbb{E}\left(\xi_n^2\right), \quad \mu_n = \xi_n^2 - \mathbb{E}\left(\xi_n^2\right). \tag{3.2}$$

Below we present a version of the convergence theorem for semimartingale inequalities. This result is convenient for establishing the stability results in this paper. For the proof see [1].

Lemma 3.3. Let $\{Z_n\}_{n\in\mathbb{N}}$ be a non-negative $\mathcal{F}_n$-measurable process, $\mathbb{E}|Z_n| < \infty\ \forall n \in \mathbb{N}$, and

$$Z_{n+1} \leq Z_n + u_n - v_n + \nu_{n+1}, \quad n \in \mathbb{N}_0, \tag{3.3}$$

where $\{\nu_n\}_{n\in\mathbb{N}}$ is an $\mathcal{F}_n$-martingale-difference, $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$ are nonnegative $\mathcal{F}_n$-measurable processes, $\mathbb{E}|u_n|, \mathbb{E}|v_n| < \infty\ \forall n \in \mathbb{N}$. Then

$$\left\{\omega : \sum_{n=1}^{\infty} u_n < \infty\right\} \subseteq \left\{\omega : \sum_{n=1}^{\infty} v_n < \infty\right\} \bigcap \{Z_n \to\}.$$

By $\{Z_n \to\}$ we denote the set of all $\omega \in \Omega$ for which $\lim_{n\to\infty} Z_n(\omega)$ exists and is finite.

3.2. Stochastic Multi-Delay Difference Equation

We consider the equation

$$\begin{aligned} X_{n+1} = (1-h)X_n + ahF\left(X_n - \sum_{l=1}^{k} b_l X_{n-l}\right) \\ + \sqrt{h} g(n, X_n, X_{n-1}, \ldots, X_{n-k})\xi_{n+1}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{3.4}$$

with arbitrary nonrandom initial values $X_0, X_{-1}, \ldots, X_{-k} \in \mathbb{R}$, $h \in (0, 1]$, continuous function $F : \mathbb{R} \to \mathbb{R}$, satisfying (2.2), continuous function $g : \mathbb{R}^{k+2} \to \mathbb{R}$, and independent random variables ξ_n, $\mathbb{E}\xi_n = 0$, $\mathbb{E}\xi_n^2 = 1$.

Suppose that the function g is non-random and there exist $c_l \geq 0$, $\gamma_n \in \mathbb{R}$, $l = 0, 1, \ldots, k$, $n \in \mathbb{N}$, such that for all $u_l \in \mathbb{R}$, $l = 0, 1, \ldots, k$, satisfies

$$|g(n, u_0, u_1, \ldots u_k)|^2 \leq \sum_{l=0}^{k} c_l |u_l|^2 + \gamma_n^2, \quad g(n, 0, \ldots, 0) = 0, \tag{3.5}$$

$$\sum_{i=1}^{\infty} \gamma_i^2 < \infty. \tag{3.6}$$

We also suppose that

$$a^2 \left(1 + \sum_{l=1}^{k} |b_l|\right)^2 + \sum_{j=0}^{k} c_j < 1. \tag{3.7}$$

Theorem 3.4. Let condition (2.2), (3.5), (3.6) and (3.7) be fulfilled. Then $\lim_{n\to\infty} X_n = 0$ a.s. where (X_n) is a solution to equation (3.4) with arbitrary $h \in (0, 1]$.

Proof. We square equation (3.4) and obtain

$$\begin{aligned}
X_{n+1}^2 = &\left[(1-h)X_n + ahF\left(X_n - \sum_{l=1}^{k} b_l X_{n-l}\right)\right]^2 \\
&+ 2\left[(1-h)X_n + ahF\left(X_n - \sum_{l=1}^{k} b_l X_{n-l}\right)\right]\sqrt{h}g(n, X_{n-k}, \dots, X_n)\xi_{n+1} \\
&+ hg^2(n, X_{n-k}, \dots, X_n)\xi_{n+1}^2 \\
= &\left[(1-h)X_n + ahF\left(X_n - \sum_{l=1}^{k} b_l X_{n-l}\right)\right]^2 + hg^2(n, X_{n-k}, \dots, X_n) \\
&+ \rho_{n+1}.
\end{aligned} \tag{3.8}$$

Here $\{\rho_n\}_{n\in\mathbb{N}}$ is an $\mathcal{F}_n$-martingale-difference, defined by

$$\begin{aligned}
\rho_{n+1} = 2&\left[(1-h)X_n + ahF\left(X_n - \sum_{l=1}^{k} b_l X_{n-l}\right)\right]\sqrt{h}g(n, X_{n-k}, \dots, X_n)\xi_{n+1} \\
&+ hg^2(n, X_{n-k}, \dots, X_n)[\xi_{n+1}^2 - 1].
\end{aligned} \tag{3.9}$$

Indeed, by assumption, we have $\mathbb{E}\xi_{n+1}^2 = 1$, and therefore $\{\xi_{n+1}^2 - 1\}_{n\in\mathbb{N}}$ is $\mathcal{F}_{n+1}$-martingale-difference. Then, from Lemma 3.1, we conclude that both terms on the right-hand side of (3.9) are $\mathcal{F}_{n+1}$-martingale-differences.

By (2.2) we estimate

$$\begin{aligned}
&\left|(1-h)X_n + ahF\left(X_n - \sum_{l=1}^{k} b_l X_{n-l}\right)\right| \\
\le &|(1-h)X_n| + |a|h\left|F\left(X_n - \sum_{l=1}^{k} b_l X_{n-l}\right)\right| \\
\le &|(1-h)X_n| + |a|h\left|X_n - \sum_{l=1}^{k} b_l X_{n-l}\right| \\
\le &(1-h)|X_n| + |a|h\left[|X_n| + \sum_{l=1}^{k} |b_l||X_{n-l}|\right] \\
= &(1-h+|a|h)|X_n| + |a|h\sum_{l=1}^{k} |b_l||X_{n-l}|.
\end{aligned}$$

Letting

$$m = k + 1, \quad \alpha_1 = 1 - h + |a|h, \quad \beta_1 = |X_n|, \quad \alpha_i = |a|h|b_{i-1}|, \quad \beta_i = |X_{n+1-i}|,$$

$i = 2, \ldots, k+1$, and applying inequality (2.4) we arrive at

$$\begin{aligned} &\left[(1 - h + |a|h)|X_n| + |a|h \sum_{l=1}^{k} |b_l||X_{n-l}|\right]^2 \\ &\quad \leq \left[1 - h + |a|h + |a|h \sum_{l=1}^{k} |b_l|\right] \left[(1 - h + |a|h)X_n^2 + |a|h \sum_{l=1}^{k} |b_l|X_{n-l}^2\right]. \end{aligned} \tag{3.10}$$

Taking into consideration (3.10) and (3.5) we estimate the right-hand side of (3.8)

$$\begin{aligned} X_{n+1}^2 &\leq \left[1 - h + |a|h + |a|h \sum_{l=1}^{k} |b_l|\right] \left[(1 - h + |a|h)X_n^2 + |a|h \sum_{l=1}^{k} |b_l|X_{n-l}^2\right] \\ &\quad + h \sum_{l=0}^{k} c_l X_{n-l}^2 + h\gamma_n^2 + \rho_{n+1} \\ &= \left[\left(1 - h + |a|h + |a|h \sum_{l=1}^{k} |b_l|\right)(1 - h + |a|h) + hc_0\right] X_n^2 \\ &\quad + h \sum_{l=1}^{k} \left[|a| \left(1 - h + |a|h + |a|h \sum_{j=1}^{k} |b_j|\right)|b_l| + c_l\right] X_{n-l}^2 + h\gamma_n^2 + \rho_{n+1}. \end{aligned} \tag{3.11}$$

Let $\tilde{c}_l = 0$ for $l = -k, -k+1, \ldots, 0, k+1, k+2, \ldots$ and let

$$\tilde{c}_l = |a| \left(1 - h + |a|h + |a|h \sum_{j=1}^{k} |b_j|\right)|b_l| + c_l,$$

for $l = 1, 2, \ldots, k$. For $n \in \mathbb{N}$ we define

$$V_n^{(2)} = h \sum_{l=-k}^{n-1} X_l^2 \sum_{j=n-l}^{\infty} \tilde{c}_j, \quad V_n = x_n^2 + V_n^{(2)}.$$

To find the increment of $V_n^{(2)}$ we perform the following calculations:

$$\begin{aligned} \Delta V_n^{(2)} &= h \sum_{l=-k}^{n} X_l^2 \sum_{j=n+1-l}^{\infty} \tilde{c}_j - h \sum_{l=-k}^{n-1} X_l^2 \sum_{j=n-l}^{\infty} \tilde{c}_j \\ &= h \sum_{l=-k}^{n-1} X_l^2 \sum_{j=n+1-l}^{\infty} \tilde{c}_j + hX_n^2 \sum_{j=1}^{\infty} \tilde{c}_j - h \sum_{l=-k}^{n-1} X_l^2 \sum_{j=n-l}^{\infty} \tilde{c}_j \end{aligned}$$

$$
\begin{aligned}
&= h\sum_{l=-k}^{n-1} X_l^2 \sum_{j=n-l}^{\infty} \tilde{c}_j - h\sum_{l=-k}^{n-1} X_l^2\tilde{c}_{n-l} + hX_n^2\sum_{j=1}^{\infty}\tilde{c}_j - h\sum_{l=-k}^{n-1} X_l^2 \sum_{j=n-l}^{\infty}\tilde{c}_j \\
&= -h\sum_{l=-k}^{n-1} X_l^2\tilde{c}_{n-l} + hX_n^2\sum_{j=1}^{\infty}\tilde{c}_j.
\end{aligned}
$$

Since $\tilde{c}_l = 0$ for $l \geq k+1$, the term $h\sum\limits_{l=-k}^{n-1} X_l^2\tilde{c}_{n-l}$ can be written as

$$
h\sum_{l=-k}^{n-1} X_l^2\tilde{c}_{n-l} = h\sum_{l=1}^{k} X_{n-l}^2\tilde{c}_l.
$$

Therefore

$$
\Delta V_n^{(2)} = -h\sum_{l=1}^{k} X_{n-l}^2\tilde{c}_l + hX_n^2\sum_{j=1}^{\infty}\tilde{c}_j. \tag{3.12}
$$

Applying (3.11) and (3.12) we arrive at the following bounds:

$$
\begin{aligned}
\Delta V_n =& X_{n+1}^2 - X_n^2 + V_{n+1}^{(2)} - V_n^{(2)} \\
\leq& \Bigg\{ \left(1 - h + |a|h + |a|h\sum_{l=1}^{k}|b_l|\right)(1 - h + |a|h) + hc_0 \\
&+ h\sum_{l=1}^{k}\left[|a|\left(1 - h + |a|h + |a|h\sum_{j=1}^{k}|b_j|\right)|b_l| + c_l\right]\Bigg\} \\
&\times X_n^2 + h\gamma_n^2 + \rho_{n+1} - X_n^2 \\
=& \Bigg\{\left(1 - h + |a|h + |a|h\sum_{l=1}^{k}|b_l|\right)\left(1 - h + |a|h + |a|h\sum_{j=1}^{k}|b_j|\right) \\
&+ h\sum_{j=0}^{k} c_j\Bigg\} X_n^2 + h\gamma_n^2 + \rho_{n+1} - X_n^2 \\
=& \left\{\left(1 - h\left[1 - |a|\left(1 + \sum_{l=1}^{k}|b_l|\right)\right]\right)^2 - 1 + h\sum_{j=0}^{k} c_j\right\} X_n^2 + h\gamma_n^2 + \rho_{n+1}.
\end{aligned} \tag{3.13}
$$

From (3.7) we obtain that for all $h \in (0,1]$

$$0 < 1 - |a|\left(1 + \sum_{l=1}^{k} |b_l|\right) < 1, \quad 0 < h\left[1 - |a|\left(1 + \sum_{l=1}^{k} |b_l|\right)\right] < 1,$$
$$0 < 1 - h\left[1 - |a|\left(1 + \sum_{l=1}^{k} |b_l|\right)\right] \le |a|\left(1 + \sum_{l=1}^{k} |b_l|\right),$$

and, therefore,

$$\left(1 - h\left[1 - a\left(1 + \sum_{l=1}^{k} |b_l|\right)\right]\right)^2 + h\sum_{j=0}^{k} c_j \le a^2\left(1 + \sum_{l=1}^{k} |b_l|\right)^2 + \sum_{j=0}^{k} c_j < 1.$$

We denote

$$\varepsilon = 1 - \left(1 - h\left[1 - a\left(1 + \sum_{l=1}^{k} |b_l|\right)\right]\right)^2 - h\sum_{j=0}^{k} c_j.$$

Then (3.13) takes the form

$$V_{n+1} \le V_n - \varepsilon X_n^2 + h\gamma_n^2 + \rho_{n+1}. \tag{3.14}$$

We let $Z_n = V_n$, $u_n = h\gamma_n^2$, $v_n = \varepsilon X_n^2$ and $\nu_{n+1} = \rho_{n+1}$, and apply Lemma 3.3 to inequality (3.14). Since $\mathbb{P}\left\{\omega : \sum_{n=1}^{\infty} u_n < \infty\right\} = \mathbb{P}\left\{\omega : \sum_{n=1}^{\infty} \gamma_n < \infty\right\} = 1$ by condition (3.6), both the limits $\lim_{n\to\infty} Z_n = \lim_{n\to\infty} V_n$ and $\lim_{n\to\infty} \sum_{i=1}^{n} v_i = \varepsilon \lim_{n\to\infty} \sum_{i=1}^{n} X_i^2$ exist and are a.s. finite. Suppose that $\lim_{n\to\infty} X_n \ne 0$ with non zero probability. Then there is a set Ω_1 such that $\mathbb{P}\Omega_1 > 0$, an a.s. finite random variable $\zeta(\omega) > 0$ and a sequence $\{n_l(\omega)\}_{l\in\mathbb{N}}$ such that $X_{n_l}^2(\omega) > \zeta(\omega)$, $l \in \mathbb{N}$, $\omega \in \Omega_1$. For all $n \in \mathbb{N}$ and $\omega \in \Omega_1$, we define

$$K(n,\omega) = \text{number of members of sequence } \{n_k(\omega)\} \le n,$$

and note that $K(n,\omega) \to \infty$ when $n \to \infty$. Then we arrive at the contradiction: for $\omega \in \Omega_1$

$$\infty > \sum_{n=1}^{\infty} v_n(\omega) \ge \varepsilon \sum_{i=0}^{n} X_i^2(\omega) \ge \varepsilon \sum_{l:i_l \le n} X_{i_l}^2(\omega) \ge \zeta(\omega)\varepsilon K(n,\omega) \to \infty,$$

as $n \to \infty$. This concludes the proof. ∎

3.3. Remarks on Numerical Methods and Stochastically Perturbed Equation of Neural Network

In this section, we compare the asymptotic behaviour of the solution of the stochastic difference equation (3.4) with the solution of the stochastic differential Itô equation

$$\begin{aligned} dX(t) = &\left[-X(t) + aF[X(t) - \sum_{i=1}^{r} \beta_i X(t-\tau_i)]\right] dt \\ &+ g(t, X(t), X(t-\tau_1), \dots, X(t-\tau_r)) dW_t, \quad t > 0. \end{aligned} \tag{3.15}$$

Here $W = \{W(t)\}_{t\geq 0}$ is a one-dimensional Wiener process, τ_i, $i = 1, \dots, r$, are nonrandom numbers, $0 < \tau_1 < \cdots < \tau_r = \tau$. We suppose that an initial condition has the following form

$$X_t(\omega) = \varphi(t), \quad t \in [-\tau, 0], \quad \omega \in \Omega, \tag{3.16}$$

where function $\varphi : [-\tau, 0] \to \mathbb{R}$ is nonrandom and continuous.

In the case when $F(u) \equiv \tanh(u)$ and $g(t, u_0, u_1, \dots, u_r) \equiv 0$, equation (3.15) coincides with (1.2). Therefore we can treat (3.15) as an equation describing the dynamics of a neural network under the influence of random perturbations.

Let $h > 0$ be the constant step size and $\Delta W_{n+1} = W((n+1)h) - W(nh)$. We approximate $t \approx nh$, $\tau_i \approx m_i h$, $X_n(h) \approx X(nh)$, $X_{n-m_i}(h) \approx X(h(n-m_i))$, $m_1 < m_2 < \cdots < m_r$. Here $X_n(h)$ are computed by the Euler–Maruyama numerical method for equation (3.15) by

$$\begin{aligned} X_{n+1}(h) = &(1-h)X_n(h) - haF\left(X_n(h) - \sum_{i=1}^{r} \beta_i X_{n-m_i}(h)\right) \\ &+ \sqrt{h} g(nh, X_n(h), X_{n-m_1}(h), \dots, X_{n-m_r}(h)) \Delta W_{n+1}. \end{aligned} \tag{3.17}$$

We set

$$\xi_{n+1} = \frac{W((n+1)h) - W(nh)}{\sqrt{h}}. \tag{3.18}$$

Then $\{\xi_n\}_{n\geq 0}$ is a sequence of standardized normally distributed random variables. We put $k = m_r$ and, for all $j = 1, 2, \dots, k = m_r$, we define

$$b_j = \beta_i, \quad \text{if} \quad j = m_i \quad \text{for some} \quad i = 1, \dots, r, \quad \text{otherwise} \quad b_i = 0.$$

We also define

$$\tilde{g}(t, a_0, a_1, a_2, \dots, a_k) \equiv g(t, a_0, a_{m_1}, \dots, a_{m_r}).$$

Then equation (3.17) takes the form

$$\begin{aligned} X_{n+1}(h) = &(1-h)X_n(h) - haF\left(X_n(h) - \sum_{j=1}^{k} b_j X_{n-j}(h)\right) \\ &+ \sqrt{h} \tilde{g}(nh, X_n(h), X_{n-1}(h), \dots, X_{n-k}(h)) \Delta W_{n+1}. \end{aligned} \tag{3.19}$$

When ξ_n are defined by (3.18), the equation (3.19) coincides with (3.4).

There is an extensive literature on the finite-time strong convergence of the Euler–Maruyama approximation (3.19) to the true solution of (3.15), for example, [7, 11, 15, 16]. However, we are not going to show the finite-time strong convergence of X_n to $X(nh)$, we only analyze the stability of the numerical method: we find a step size h such that the numerical method reproduces the characteristics of the test equation. In the case of linear stochastic differential equations the mean-square asymptotic stability of the numerical methods has been studied by several authors e.g., [6, 24]. The exponential mean-square stability of numerical methods for general nonlinear n-dimensional stochastic equation was investigated in [8]. But the almost sure asymptotic stability of numerical methods has been less studied. Only recently [24] discussed the almost sure asymptotic stability of the weak Euler–Maruyama method to a linear scalar stochastic differential equation.

In this paper we consider the almost sure asymptotic stability of the strong Euler–Maruyama method (3.17) to the nonlinear scalar stochastic differential equation (3.15). It transpires that (3.19) is a good discrete model of (3.15), since under the conditions stipulated in Theorem 3.4, solutions of the continuous problem have the same asymptotic behavior. Below we formulate an analog of Theorem 3.4 for equation (3.15). The proof of Theorem 3.5 is deferred to Subsection 3.4.

Theorem 3.5. Let condition (2.2), (3.5), (3.6) and (3.7) be fulfilled. Also suppose that the function $F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz-continuous and the function $g : [0, \infty) \times \mathbb{R}^{k+1} \to \mathbb{R}$ is Borel-measurable with respect to the first variable and locally Lipschitz-continuous with respect to last $k+1$ variables. Then $\lim_{t\to\infty} X(t) = 0$ a.s. where $X(t)$ is a solution of equation (3.15) with arbitrary initial conditions given in (3.16).

3.4. Proof of Asymptotic Stability for Stochastically Perturbed Equations for Neural Networks

For simplicity we give a proof of Theorem 3.5 for a one delay stochastic Itô equation:

$$dX(t) = [-X(t) + aF[X(t) - \beta X(t-\tau)]]\, dt + g(t, X(t), X(t-\tau))dW_t, \quad t > 0. \tag{3.20}$$

Multi-delay case can be treated in similar way.

In this subsection we will apply the following variant of the Itô formula (see [9, 10, 14, 15]).

Lemma 3.6. Let $\{X_t\}_{t\geq 0}$ be a stochastic process with differential $dX_t = c(t, X_t)dt + g(t, X_t)dW_t$. Let $V(t, x)$ be a differentiable function with respect to the first argument and a twice differentiable function with respect to the second argument. Then

$$V(t,X_t) = V(0,X_0) + \int_0^t \frac{\partial V(s,X_s)}{\partial x} c(s,X_s)ds + \int_0^t \frac{\partial V(s,X_s)}{\partial s} ds + \frac{1}{2}\int_0^t \frac{\partial^2 V(s,X_s)}{\partial x^2} g^2(s,X_s)ds + \int_0^t \frac{\partial V(s,X_s)}{\partial x} g(s,X_s)dW_s. \quad (3.21)$$

To establish asymptotic stability of the solution to (3.20) we will use a continuous analogue of Lemma 3.3.

Lemma 3.7. Suppose that $\{Z_t\}_{t\geq 0}$, is a nonnegative continuous $\mathcal{F}_t$-semimartingale and its stochastic differential dZ_t is estimated by the inequality

$$dZ_t \leq dA_t^{(1)} - dA_t^{(2)} + dM_t, \quad t \geq 0, \quad (3.22)$$

where $\{A_t^{(i)}\}_{t\geq 0}$, $i = 1,2$, are continuous nondecreasing $\mathcal{F}_t$-measurable processes with $A_0^{(i)} = 0$ and $\{M_t\}_{t\geq 0}$ is a continuous $\mathcal{F}_t$-martingale started at $M_0 = 0$. Then

$$\{\omega \in \Omega : A_\infty^{(1)}(\omega) < \infty\} \subseteq \{\omega \in \Omega : Z_t(\omega) \to\} \cap \{\omega \in \Omega : A_\infty^{(2)}(\omega) < \infty\} \quad a.s.$$

Proof of Theorem 3.5. Since the coefficients of equation (3.15) are locally Lipschitz-continuous in the space variables and Borel-measurable with respect to time t, there exists local unique, a.s. continuous, strong solution $X(t)$ of (3.15), (3.16) on the interval $[0, \tau_n \wedge T)$, where $\tau_n = \inf\{t : |X(t)| \geq n\}$, $\tau_n \wedge T = \min\{\tau_n, T\}$ (see [9, Theorem 2.5 on page 287] or [10]).

Applying the Itô formula to (3.20), and then Hölder inequality and inequality (2.4), we obtain for $t \leq \tau_n \wedge T$

$$\begin{aligned} dX^2(t) &= 2X(t)\left(-X(t) + aF[X(t) - bX(t-\tau)] + g^2[t, X(t), X(t-\tau)]\right)dt \\ &\quad + 2X(t)g(t,X(t),X(t-\tau))dW_t \\ &\leq \left(-2X^2(t) + a^2F^2[X(t) - bX(t-\tau)] + X(t)^2 + c_1X^2(t)\right. \\ &\quad \left. + c_2X^2(t-\tau)) + \gamma_t^2\right) dt 2X(t)g(t,X(t),X(t-\tau))dW_t \\ &\leq \left(-(1-c_1)X^2(t) + a^2[|X(t)| + b|X(t-\tau)|]^2 + c_2X^2(t-\tau) + \gamma_t^2\right) dt \\ &\quad + 2X(t)g(t,X(t),X(t-\tau))dW_t \\ &\leq \left(-(1-c_1)X^2(t) + a^2(1+|b|)[X^2(t) + |b|X^2(t-\tau)]\right. \\ &\quad \left. + c_2X^2(t-\tau) + \gamma_t^2\right) dt + 2X(t)g(t,X(t),X(t-\tau))dW_t \\ &\leq \left(-(1-c_1-a^2(1+|b|))X^2(t) + (c_2 + a^2(1+|b|)|b|)X^2(t-\tau) + \gamma_t^2\right) dt \\ &\quad + 2X(t)g(t,X(t),X(t-\tau))dW_t. \end{aligned} \quad (3.23)$$

We define

$$\tilde{c} := c_2 + a^2(1+|b|)|b|, \quad \tilde{\varepsilon} := 1 - (c_1 + c_2) - a^2(1+|b|)^2,$$

and

$$V = X^2(t) + V^{(2)}(t), \quad V^{(2)}(t) = \tilde{c}\int_{t-\tau}^{t} X(s)^2 ds. \tag{3.24}$$

It is easy to see that

$$\frac{\partial V^{(2)}(t)}{\partial t} = \tilde{c}X^2(t) - \tilde{c}X^2(t-\tau). \tag{3.25}$$

Applying the Itô formula once again we get for $t \le \tau_n \wedge T$

$$\begin{aligned} dV(t) \le & \left(-(1-c_1-a^2(1+|b|)X^2(t) + (c_2+a^2(1+|b|)|b|)X^2(t) + \gamma_t^2\right) dt \\ & + 2X(t)g(t, X(t), X(t-\tau))dW_t \\ = & -\tilde{\varepsilon}X^2(t)dt + \tilde{c}\gamma_t^2 dt + 2X(t)g(t, X(t), X(t-\tau))dW_t. \end{aligned} \tag{3.26}$$

Set $t_n = t \wedge \tau_n$ for $n \in \mathbb{N}$. We note that τ_n is a Markov time (stopping time), hence t_n is a finite Markov time (finite stopping time). After the integration of inequality (3.26), we arrive at

$$\begin{aligned} V(t_n) &\le V(0) - \tilde{\varepsilon}\int_0^{t_n} X^2(s)ds + \int_0^{t_n}\gamma_s^2 ds + 2\int_0^{t_n} X(s)g(s, X(s), X(s-\tau))dW_s \\ &\le V(0) + \int_0^{t_n}\gamma_s^2 ds + 2\int_0^{t_n} X(s)g(s, X(s), X(s-\tau))dW_s. \end{aligned} \tag{3.27}$$

To prove the theorem we need to prove that:

(a) for any initial condition (3.16) solution $X(t)$ of equation (3.20) exists on $[0,\infty)$,

(b) $\lim_{t\to\infty} X(t) = 0$ a.s.

(a) *Global existence of solutions.* We take expectations on both sides of the integral inequality (3.27) and obtain

$$\mathbb{E}V(t_n) \le V_0 + \int_0^{t_n}\gamma_s^2 ds \le K(t). \tag{3.28}$$

Then we may conclude that, for arbitrary $t \in (0,\infty)$, the estimate

$$0 \ \le \ \mathbb{P}\{t > \tau_n\} \le \frac{\mathbb{E}V(t_n)}{n} \le \frac{K(t)}{n} \to 0$$

holds as $n \to \infty$. Since $t > 0$ is arbitrary this means that $\tau_n \to \infty$ a.s. as $n \to \infty$, i.e., the first explosion time τ of solution $X(t)$ is equal to ∞ a.s. Hence the solution $X(t)$ of (3.20), (3.16) exists and is unique on $[-\tau,\infty)$.

A.s. asymptotic stability of a trivial solutions. Notice that $V = \{V(t)\}_{t\geq 0}$ is a nonnegative continuous $\mathcal{F}_t$-semimartingale. Now, since condition (3.6) is fulfilled, we may apply Lemma 3.7 with

$$\begin{aligned} A_t^{(1)} &= \int_0^t \gamma_s^2 ds, \\ A_t^{(2)} &= \tilde{\varepsilon} \int_0^t X^2(s) ds, \\ M_t &= 2 \int_0^t X(s) g(s, X(s), X(s-\tau)) dW_s, \end{aligned}$$

and conclude that both, $\lim\limits_{t\to\infty} V(t)$ and

$$\lim_{t\to\infty} A_t^{(2)} = \tilde{\varepsilon} \int_0^\infty X^2(s) ds, \tag{3.29}$$

exist and are a.s. finite. By (3.24) and by convergence of $V(t)$, as $t \to \infty$, we can conclude that $X(t)$ is a.s. bounded on $[0, \infty)$, and, therefore, there exists a.s. finite random variable $K = K(\omega)$ such that $\mathbb{P}\{\sup\limits_{t>0} |X(t)| < K\} = 1$.

It remains to show that $X(t)$ converges to 0 a.s., as $t \to \infty$. To do this we note that a.s.

$$\left|V^{(2)}(t_1) - V^{(2)}(t_2)\right| = \tilde{c} \left| \int\limits_{t_2}^{t_1} X^2(\tau) d\tau - \int\limits_{t_2 - s}^{t_1 - s} X^2(\tau) d\tau \right| \leq 2\tilde{c} K^2 |t_2 - t_1|. \tag{3.30}$$

Suppose now that $\mathbb{P}\left\{\limsup\limits_{t\to\infty} X^2(t) = \zeta_0(\omega) > 0\right\} = p_0 > 0$. Then there exists a sequence $\{t_k(\omega)\}_{k\in\mathbb{N}}$ such that $\mathbb{P}(\Omega_1) = p_0$ with $\Omega_1 = \{\omega\colon X^2(t_k, \omega) > \zeta_0(\omega)/2 > 0\}$. From definition (3.24) of V, we have $X^2(t) = V(t) - V^{(2)}(t)$, and then

$$|X^2(t) - X^2(s)| \leq |V(t) - V(s)| + \left|V^{(2)}(t) - V^{(2)}(s)\right|.$$

Due to convergence of $V(t)$, there exists $T = T(\omega)$ such that

$$|V(t) - V(s)| \leq \zeta_0(\omega)/8$$

for $t, s \geq T$. From (3.30) we can find $\delta = \delta(\zeta_0(\omega))$ such that $\left|V^{(2)}(t) - V^{(2)}(s)\right| \leq \zeta_0(\omega)/8$, when $|t - s| \leq \delta$. Then

$$|X^2(t_k) - X^2(s)| \leq \zeta_0(\omega)/4$$

for $\omega \in \Omega_1$, $|s - t_k| \leq \delta$, $t_k > T$. Thus we obtain for $\omega \in \Omega_1$, $|s - t_k| \leq \delta$, $t_k > T$,

$$|X^2(s)| \geq |X^2(t_k)| - |X^2(t_k) - X^2(s)| \geq \zeta_0(\omega)/2 - \zeta_0(\omega)/4 = \zeta_0(\omega)/4. \tag{3.31}$$

Let $k(n) = k(n,\omega)$ be a number of elements of the sequence $\{t_k\}$ in the interval $[T,n]$. We note that $k(n) \to \infty$ as $n \to \infty$. Applying inequality (3.31) we obtain for $\omega \in \Omega_1$, $n > T$,

$$\int_0^n X^2(s)ds \geq \int_T^n X^2(s)ds \geq \sum_{k:\, T \leq t_k+\delta \leq n} \int_{t_k-\delta}^{t_k+\delta} X^2(s)ds$$
$$\geq \frac{\zeta_0(\omega)}{4} \sum_{k:\, T \leq t_k+\delta \leq n} \int_{t_k-\delta}^{t_k+\delta} ds = \frac{2\delta\zeta_0^2(\omega)}{4} \sum_{k \leq k(n)} 1 = \frac{k(n)\delta\zeta_0^2(\omega)}{2} \to \infty$$

as $n \to \infty$. Hence $\mathbb{P}\{A^2(\infty) = \infty\} \geq p_0 > 0$. This contradicts (3.29) proving the result. ■

Acknowledgment

The authors first met at the conference on Difference Equations and Applications, Munich, July 2005. They gratefully acknowledge the support of ISDE, the International Society of Difference Equations.

References

[1] J. Appleby, X. Mao, and A. Rodkina, On stochastic stabilization of difference equations. *Dynamics of Continuous and Discrete Systems*, 15(3):843–857, 2006.

[2] E.R. Caianiello and A. De Luca, Decision equation for binary systems: Application to neuronal behavior, *Kybernetik*, 3:33–44, 1966.

[3] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer Academic Publishers. 1992.

[4] K. Gopalsamy and I.K.C. Leung, Convergence under dynamical thresholds with delays, *IEEE Transactions on Neural Networks*, 8(2):341–348, 1997.

[5] Y. Hamaya and Shinsuke Sato, Global attractivity of some neural net models, *Inter. J. Pure and Appl. Math.*, 25(4):487–495, 2005.

[6] D.J. Higham, Mean-square and asymptotic stability of the stochastic theta method, *SIAM J. Numer. Anal.*, 38(3):753–769, 2003.

[7] D.J. Higham, X. Mao, and A.M. Stuart, Strong convergence of numerical methods for nonlinear stochastic differential equations, *SIAM J. Num. Anal.*, 40(3):1041–1063, 2002.

[8] D.J. Higham, X. Mao, and A.M. Stuart, Exponential mean-square stability of numerical solutions to stochastic differential equations, *LMS J. Comput. Math.*, 6:297–313, 2003.

[9] I. Karatzas and S.E. Shreve, Brownian Motion and Stochastic Calculus. Springer, New York, NY, 2nd ed. 1991.

[10] R.Z. Khasminskii, Stochastic Stability of Differential Equations, Sijthoff & Noordhoff, Alphen aan den Rijn, NL. 1980.

[11] P.E. Kloeden and E. Platen, Numerical Solution of Stochastic Differential Equations, Springer-Verlag, Berlin. 1992.

[12] V. Kolmanovskii and L. Shaikhet, General method of Lyapunov functionals construction for stability investigation of stochastic difference equations, *Dynamical Systems and Applications, World Sci. Ser. Appl. Anal.*, 4:397–439, 1995.

[13] V. Kolmanovskii and L. Shaikhet, Some peculiarities of the general method of Lyapunov functionals construction, *Appl. Math. Lett.*, 15(3):355–360, 2002.

[14] R.Sh. Liptser and A.N. Shiryayev, Theory of Martingales, Kluwer Academic Publisher, Dordrecht, NL. 1989.

[15] X. Mao, Stochastic Differential Equations and Applications, Horwood Series in Mathematics and Applications, Horwood, Chichester, 1997.

[16] G.N. Milstein, Numerical integration of stochastic differential equations, Kluwer Academic Publishers, 1995.

[17] H.A. El-Morshedy and K. Gopalsamy, On the oscillation and asymptotic behaviour of solutions of a neuronic equation, *Funkcialaj Ekvacioj*, 44:83–98, 2001.

[18] A. Rodkina, On asymptotic behaviour of solutions of stochastic difference equations, *Nonlinear Analysis*, 47:4719–4730, 2001.

[19] A. Rodkina and X. Mao, On Boundedness and Stability of Solutions of Nonlinear Difference Equation with Nonmartingale Type Noise, *J. Differ. Equations Appl.*, 7(4):529–550, 2001.

[20] A. Rodkina, X. Mao, and V. Kolmanovskii, On asymptotic behaviour of solutions of stochastic difference equations with Volterra type main term, *Stochastic Anal. Appl.*, 18(5):837–857, 2000.

[21] A. Rodkina and H. Schurz, On global asymptotic stability of solutions of some in-arithmetic-mean-sense monotone stochastic difference equations in R^1, *Int. J. Numer. Anal. Model.*, 2(3):355–366, 2005.

[22] A. Rodkina and H. Schurz, Global asymptotic stability of solutions to cubic stochastic difference equations, *Advances in Difference Equations*, 3:249–260, 2004.

[23] A. Rodkina and H. Schurz, A theorem on asymptotic stability of solutions of nonlinear stochastic difference equations with Volterra type noise, *SACTA*, 6(1):23–34, 2004.

[24] Y. Saito and T. Mitsui, Stability analysis of numerical schemes for stochastic differential equations, *SIAM J. Numer. Anal.*, 33:2254–2267, 1996.

[25] A. Shiryaev, Probability (2nd edition), Springer, Berlin, 1996.