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Oscillation of Two-term Sturm–Liouville Difference Equations*

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Abstract

We deal with oscillatory properties of the higher order Sturm–Liouville difference equation. We find the exact value of the constant in some oscillation criteria for this equation. We also determine the (non)oscillation of the Euler-type difference equation.

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1. Introduction

In this paper we study oscillatory behavior of the higher order Sturm–Liouville difference equation

$$(-1)^n \Delta^n \left(k^{(\alpha)} \Delta^n y_k\right) = q_k y_{k+n}, \tag{1.1}$$

where $\alpha \in \mathbb{R} \setminus \{1, 3, \dots, 2n-1\}$ and $k^{(\alpha)} := \dfrac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}$, $\Gamma(t) := \displaystyle\int_0^\infty e^{-s} s^{t-1} ds$ being the Gamma function.

We are motivated by several recent papers dealing with the differential equation

$$(-1)^n \left(t^\alpha y^{(n)}\right)^{(n)} = q(t) y, \quad \alpha \in \mathbb{R} \setminus \{1, 3, \dots, 2n-1\}, \tag{1.2}$$

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see [7, 11, 13, 14, 16–19], where (1.2) was viewed as a perturbation of the Euler-type differential equation

$$(-1)^n \left(t^\alpha y^{(n)}\right)^{(n)} - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y = 0.$$

Note that the "critical" constant $\gamma_{n,\alpha}$ is such that the Euler-type equation with γ instead of $\gamma_{n,\alpha}$ is nonoscillatory if and only if $\gamma \le \gamma_{n,\alpha}$.

In our paper we use the discrete analogy of the above mentioned approach.

We show that the discrete Euler-type equation

$$(-1)^n \Delta^n \left(k^{(\alpha)} \Delta^n y_k\right) - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)} y_{k+n} = 0, \tag{1.3}$$

where

$$\gamma_{n,\alpha} := \frac{1}{4^n} \prod_{j=1}^{n} (2j-1-\alpha)^2 \tag{1.4}$$

(the same constant as in the continuous case) and where $l \in \{0, \dots, n\}$ is such that

$$\begin{aligned} &\alpha \in (2l-1, 2l+1) \quad \text{for} \quad l = 1, \dots, n-1, \\ &l := 0 \quad \text{for} \quad \alpha < 1, \\ &l := n \quad \text{for} \quad \alpha > 2n-1 \end{aligned} \tag{1.5}$$

is nonoscillatory. Then we study (1.1) as a perturbation of (1.3), i.e., oscillation of (1.1) is measured by the sequence $q_k - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)}$ when this equation is written in the form

$$\begin{aligned} &(-1)^n \Delta^n \left(k^{(\alpha)} \Delta^n y_k\right) - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)} y_{k+n} \\ &\quad = \left(q_k - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)}\right) y_{k+n}. \end{aligned}$$

In particular, we obtain an improved version of the oscillatory counterpart of the following two results which are proved in [15].

Proposition 1.1. [15] Let $\alpha \in \mathbb{R} \setminus \{1, 3, \dots, 2n-1\}$. Equation (1.1) is nonoscillatory provided one of the following two conditions holds:

(i) $\alpha < 2n-1$ and

$$\lim_{k\to\infty} k^{2n-1-\alpha} \sum_{k}^{\infty} q_j^+ < \frac{\gamma_{n,\alpha}}{2n-1-\alpha},$$

(ii) $\alpha > 2n-1$ and

$$\lim_{k\to\infty} k^{2n-1-\alpha} \sum^{k} q_j^+ < \frac{\gamma_{n,\alpha}}{\alpha-2n+1}.$$

Proposition 1.2. [15] Let $\alpha \in \mathbb{R} \setminus \{1, 2, \dots, 2n-1\}$. Equation (1.1) is oscillatory provided one of the following two conditions holds:

(i) $\alpha < 2n-1$ and

$$\lim_{k\to\infty} k^{2n-1-\alpha}\sum_{k}^{\infty} q_j > (2n-1-\alpha)(n-1)! := \delta_{n,\alpha},$$

(ii) $\alpha > 2n-1$ and

$$\lim_{k\to\infty} k^{2n-1-\alpha}\sum^{k} q_j > \frac{\left[(\alpha-n^{(n)})\right]^2}{\alpha-2n+1} := \varrho_{n,\alpha}.$$

We show that under the additional condition $q_k \geq \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)}$ for large k, we are able to remove the gap between the "oscillation" and "nonoscillation" constants. We prove that the "oscillation" constants $\delta_{n,\alpha}$ and $\varrho_{n,\alpha}$ can be replaced by the "nonoscillation" constant $\dfrac{\gamma_{n,\alpha}}{|\alpha-2n+1|}$. Note that this result is an analogy to that in [10], where equation (1.2) was investigated. Note also that a similar result (with some modifications in the proof) dealing with equation (1.1) with $\alpha = 0$ was published in [12].

As a consequence of the presented criteria we derive that the constant $\gamma_{n,\alpha}$ is the "critical" constant in oscillation of the discrete Euler-type equation, similarly as in the continuous case. In the special case $n = 1$ and $\alpha = 0$, i.e., if (1.1) reduces to the second order difference equation $\Delta^2 y_k + q_k y_{k+1} = 0$, our oscillation constant $\gamma_{1,0} = \dfrac{1}{4}$ complies with the oscillation constant established in the previous papers [5, 22].

We use the methods based on the relationship between Sturm–Liouville difference equations and linear Hamiltonian difference systems and on the variation techniques.

The paper is organized as follows. In the next section we recall necessary definitions and some preliminary results. The nonoscillation of the Euler-type difference equation is studied in Section 3 and our main results, the oscillation criteria for (1.1), are contained in Section 4. In the last section we formulate some technical results needed in the proofs.

2. Preliminaries

Consider the general Sturm–Liouville difference equation

$$L_n(y) := \sum_{\nu=0}^{n}(-1)^\nu\Delta^\nu\left(r_k^{[\nu]}\Delta^\nu y_{k+n-\nu}\right) = 0, \tag{2.1}$$

where $r_k^{[\nu]}$, $\nu = 0,\dots,n$, are real-valued sequences, $r_k^{[n]} > 0$. Most of the results of this section remain to hold under the weaker assumption $r_k^{[n]} \neq 0$, but since $r_k^{[n]} = k^{(\alpha)} > 0$ in all our applications, we suppose that $r_k^{[n]} > 0$.

Oscillatory properties of (2.1) can be investigated within the scope of the oscillation theory of linear Hamiltonian difference system (further LHΔS)

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k, \tag{2.2}$$

where A, B, C are sequences of $n \times n$ matrices with B, C symmetric and the matrix $I - A$ is nonsingular.

Indeed, equation (2.1) can be written as (2.2) using the following substitution. If y_k is a solution of (2.1) and if we set

$$x_k = \left(x_k^{[1]}, \dots, x_k^{[n]}\right)^T, \; u_k = \left(u_k^{[1]}, \dots, u_k^{[n]}\right)^T,$$

where $x_k^{[\nu]} = \Delta^{\nu-1} y_{k+n-\nu}, \nu = 1, \dots, n$, $u_k^{[n]} = r_k^{[n]} \Delta^n y_k$, $u_k^{[n-\nu]} = r_k^{[n-\nu]} \Delta^{n-\nu} y_{k+\nu} - \Delta u_k^{[n-\nu+1]}, \nu = 1, \dots, n-1$, i.e.,

$$x_k = \begin{pmatrix} y_{k+n-1} \\ \Delta y_{k+n-2} \\ \vdots \\ \Delta^{n-1} y_k \end{pmatrix}, \quad u_k = \begin{pmatrix} (-1)^{n-1} \Delta^{n-1} \left(r_k^{[n]} \Delta^n y_k\right) + \dots + r_k^{[1]} \Delta y_{k+n-1} \\ \vdots \\ -\Delta \left(r_k^{[n]} \Delta^n y_k\right) + r_k^{[n-1]} \Delta^{n-1} y_{k+1} \\ r_k^{[n]} \Delta^n y_k \end{pmatrix},$$

then the pair of n-vectors (x_k, u_k) solves (2.2) with A, B, C given by

$$B_k = \operatorname{diag}\left\{0, \dots, 0, \frac{1}{r_k^{[n]}}\right\}, \quad C_k = \operatorname{diag}\left\{r_k^{[0]}, \dots, r_k^{[n-1]}\right\},$$
$$A = A_{i,j} = \begin{cases} 1, & \text{if } j = i+1, \; i = 1, \dots, n-1, \\ 0, & \text{elsewhere.} \end{cases}$$

Oscillatory properties of LHΔS are defined using the concept of the *focal points* of a conjoined basis of this system.

Let (X, U) be the solution of the matrix linear Hamiltonian difference system (referred to again as (2.2))

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \quad \Delta U_k = C_k X_{k+1} - A_k^T U_k,$$

where X, U are $n \times n$ matrices.

A solution (X, U) of (2.2) is said to be a *conjoined basis* of (2.2) if $X^T U = U^T X$ and $\operatorname{rank}\begin{pmatrix} X \\ U \end{pmatrix} = n$. For a conjoined basis (X, U) of (2.2), we say that an interval $(m, m+1]$ contains a *focal point* of (X, U) if

$$\operatorname{Ker} X_{m+1} \subseteq \operatorname{Ker} X_m \quad \text{and} \quad X_m X_{m+1}^\dagger (I - A_m)^{-1} B_m \geq 0$$

fails to hold. Here $\operatorname{Ker}, \dagger$ and $\geq$ stand for the kernel, the Moore–Penrose generalized inverse and nonnegative definiteness of the matrix indicated.

System (2.2) is said to be *nonoscillatory* if there exists $N \in \mathbb{N}$ such that the solution (X, U) given by the initial condition $X_N = 0, U_N = I$ has no focal point in (N, ∞). System (2.2) is said to be *oscillatory* in the opposite case.

Consequently, by the above mentioned relationship between (2.1) and (2.2), we say that (2.1) is *nonoscillatory (oscillatory)* if the associated LHΔS has the corresponding property.

Recall also that a conjoined basis (X, U) of (2.2) is said to be *recessive* if X_k is nonsingular, $X_{k+1}^{-1}(I - A_k)B_k X_k^{T-1} \geq 0$, both for large k, and

$$\lim_{k\to\infty}\left(\sum^{k} X_{j+1}^{-1}(I - A_j)B_j\left(X_j^T\right)^{-1}\right)^{-1} = 0. \tag{2.3}$$

The following statement, the so called variational principle, which relates nonoscillation of (2.1) to positivity of a certain discrete quadratic functional, plays an important role in the proofs of our criteria. Denote

$$\begin{aligned}\mathcal{D}_n(N) \ := \ &\{y = \{y_k\}_{k=N}^{\infty} : y_k = 0, k = N, \dots, N+n-1,\\ &\exists m \in \mathbb{N}, m \geq N : y_k = 0, k \geq m\}.\end{aligned}$$

Lemma 2.1. [2] Equation (2.1) is nonoscillatory if and only if there exists $N \in \mathbb{N}$ such that

$$\mathcal{F}(y; N, \infty) := \sum_{k=N}^{\infty}\left[\sum_{\nu=0}^{n} r_k^{[\nu]}\left(\Delta^\nu y_{k+n-\nu}\right)^2\right] > 0$$

for every nontrivial $y \in \mathcal{D}_n(N)$.

We will also need a following statement.

Lemma 2.2. [20] Let L_1 be the second order linear difference operator given in (2.1), i.e., $n = 1$ in (2.1). If there exists a sequence $z = z_k$ such that $z_k z_{k+1} > 0$ and $L_1(z) \geq 0$, both for large k, then the equation $L_1(y) = 0$ is nonoscillatory.

3. Nonoscillation of the Euler Type Equation

In this section we deal with the Euler-type equation (1.3). As a first step we investigate the second order equation, i.e., the case when $n = 1$.

Lemma 3.1.
(i) Equation

$$\Delta\left(k^{(\alpha)}\Delta y_k\right) + \frac{(1-\alpha)^2}{4}(k-1)^{(\alpha-2)}y_{k+1} = 0 \tag{3.1}$$

is nonoscillatory for $\alpha > 1$.
(ii) Equation

$$\Delta\left(k^{(\alpha)}\Delta y_k\right) + \frac{(1-\alpha)^2}{4}(k+1)^{(\alpha-2)}y_{k+1} = 0 \tag{3.2}$$

is nonoscillatory for $\alpha < 1$.

Proof. To prove both the statements (i) and (ii), we will use Lemma 2.2 with the sequence

$$z_k := (k-l)^{\left(\frac{1-\alpha}{2}\right)},\ l \in \mathbb{R}. \tag{3.3}$$

Since

$$(k+1)^{(\alpha)} = \frac{\Gamma(k+2)}{\Gamma(k+2-\alpha)} = \frac{(k+1)\Gamma(k+1)}{\Gamma(k+2-\alpha)} = (k+1)k^{(\alpha-1)}$$

and

$$\begin{aligned}(k-l)^{\left(\frac{-1-\alpha}{2}\right)} &= \frac{\Gamma(k-l+1)}{\Gamma(k-l+\frac{3+\alpha}{2})} = \frac{(k-l+\frac{3+\alpha}{2})\Gamma(k-l+1)}{\Gamma(k-l+\frac{5+\alpha}{2})} \\ &= (k-l+\frac{3+\alpha}{2})(k-l)^{\left(\frac{-3-\alpha}{2}\right)},\end{aligned}$$

we have

$$\begin{aligned}\Delta\left(k^{(\alpha)}\Delta z_k\right) &= \frac{1-\alpha}{2}\Delta\left[k^{(\alpha)}(k-l)^{\left(\frac{-1-\alpha}{2}\right)}\right] \\ &= \frac{1-\alpha}{2}\left[\alpha k^{(\alpha-1)}(k-l)^{\left(\frac{-1-\alpha}{2}\right)} - \frac{1+\alpha}{2}(k+1)^{(\alpha)}(k-l)^{\left(\frac{-3-\alpha}{2}\right)}\right] \\ &= \frac{1-\alpha}{2}k^{(\alpha-1)}(k-l)^{\left(\frac{-3-\alpha}{2}\right)}\left[\alpha\left(k-l+\frac{3+\alpha}{2}\right) - \frac{1+\alpha}{2}(k+1)\right]\end{aligned}$$

and

$$\begin{aligned}z_{k+1} &= \frac{\Gamma(k-l+2)}{\Gamma\left(k-l+\frac{3+\alpha}{2}\right)} \\ &= \frac{\left(k-l+\frac{3+\alpha}{2}\right)(k-l+1)\Gamma(k-l+1)}{\Gamma\left(k-l+\frac{5+\alpha}{2}\right)} \\ &= (k-l+1)\left(k-l+\frac{3+\alpha}{2}\right)(k-l)^{\left(\frac{-3-\alpha}{2}\right)}.\end{aligned}$$

Hence

$$\frac{\Delta\left(k^{(\alpha)}\Delta z_k\right)}{z_{k+1}} = -\frac{(1-\alpha)^2}{4}k^{(\alpha-1)}\frac{k+\frac{\alpha^2+2(1-l)\alpha-1}{\alpha-1}}{(k-l+1)\left(k-l+\frac{3+\alpha}{2}\right)}. \tag{3.4}$$

(i) Let $\alpha > 1$. To finish the proof of the first statement, it suffices to take $l = 0$ in (3.3) and consequently also in (3.4) and to verify that

$$\frac{(1-\alpha)^2}{4}k^{(\alpha-1)}\frac{k+\frac{\alpha^2+2\alpha-1}{\alpha-1}}{(k+1)\left(k+\frac{3+\alpha}{2}\right)} - \frac{(1-\alpha)^2}{4}(k-1)^{(\alpha-2)} \geq 0$$

holds for large k. Then, nonoscillation of (3.1) will follow from Lemma 2.2. Since

$$k^{(\alpha-1)} = \frac{\Gamma(k+1)}{\Gamma(k+2-\alpha)} = \frac{k\Gamma(k)}{\Gamma(k+2-\alpha)} = k(k-1)^{(\alpha-2)},$$

we have to show that

$$k\frac{k+\frac{\alpha^2+2\alpha-1}{\alpha-1}}{(k+1)\left(k+\frac{3+\alpha}{2}\right)}-1\geq 0.$$

This inequality is satisfied, since for $\alpha>1$

$$k\left(k+\frac{\alpha^2+2\alpha-1}{\alpha-1}\right)-(k+1)\left(k+\frac{3+\alpha}{2}\right)=\frac{\alpha^2+3}{2(\alpha-1)}k-\frac{3+\alpha}{2}\geq 0.$$

(ii) Let $\alpha<1$, and take $l\geq\frac{7}{4}$ in (3.3). Similarly as in (i) we show that

$$\frac{(1-\alpha)^2}{4}k^{(\alpha-1)}\frac{k+\frac{\alpha^2+2(1-l)\alpha-1}{\alpha-1}}{(k-l+1)\left(k-l+\frac{3+\alpha}{2}\right)}-\frac{(1-\alpha)^2}{4}(k+1)^{(\alpha-2)}\geq 0$$

for large k. We have

$$\begin{aligned}k^{(\alpha-1)} &= \frac{\Gamma(k+1)}{\Gamma(k+2-\alpha)}=\frac{(k+3-\alpha)(k+2-\alpha)\Gamma(k+2)}{(k+1)\Gamma(k+4-\alpha)}\\ &= \frac{(k+3-\alpha)(k+2-\alpha)}{k+1}(k+1)^{(\alpha-2)},\end{aligned}$$

and hence we need to verify that

$$\frac{(k+3-\alpha)(k+2-\alpha)\left(k+\frac{\alpha^2+2(1-l)\alpha-1}{\alpha-1}\right)}{(k+1)(k-l+1)\left(k-l+\frac{3+\alpha}{2}\right)}-1\geq 0. \tag{3.5}$$

By a direct computation we obtain

$$\begin{aligned}&(k+3-\alpha)(k+2-\alpha)\left(k+\frac{\alpha^2+2(1-l)\alpha-1}{\alpha-1}\right)\\ &-(k+1)(k-l+1)\left(k-l+\frac{3+\alpha}{2}\right)=\frac{3\alpha^2-12\alpha+5+4l}{2(1-\alpha)}k^2+O(k),\end{aligned}$$

as $k\to\infty$, which is positive for $l\geq\frac{7}{4}$ and $\alpha<1$, and hence (3.5) is satisfied. ■

Now, using Lemma 2.1, the previous statement can be extended to equation (1.3) with general n.

Theorem 3.2. Let $\alpha\in\mathbb{R}\setminus\{1,3,\dots,2n-1\}$ and $l\in\{0,\dots,n\}$ be such that conditions (1.5) are satisfied, $\gamma_{n,\alpha}$ be given by (1.4). Then equation (1.3) is nonoscillatory.

Proof. Let $y=\{y_k\}_N^\infty$, $N\in\mathbb{N}$, be arbitrary nontrivial sequence such that $y\in\mathcal{D}_n(N)$. Note that then $\Delta^{n-j}y_{k+j-1}\in\mathcal{D}_1(N)$, $j=1,\dots,n$.

Let $l \in \{0,\dots,n\}$ be fixed. It follows from conditions (1.5) that $\alpha-2j+2>1$ for $j=1,\dots,l$ and $\alpha-2j+2<1$ for $j=l+1,\dots,n$. Hence, equation (3.1) with $\alpha-2j+2$ instead of α is nonoscillatory for $j=1,\dots,l$ and equation (3.2) with $\alpha-2j+2$ instead of α is nonoscillatory for $j=l+1,\dots,n$. Using Lemma 2.1 we obtain

$$\begin{aligned}&\sum_{k=N}^{\infty}(k-j+1)^{(\alpha-2j+2)}\left(\Delta^{n-j+1}y_{k+j-1}\right)^2\\&\quad>\frac{(2j-1-\alpha)^2}{4}\sum_{k=N}^{\infty}(k-j)^{(\alpha-2j)}\left(\Delta^{n-j}y_{k+j}\right)^2, \quad j=1,\dots,l,\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=N}^{\infty}(k+j-1-2l)^{(\alpha-2j+2)}\left(\Delta^{n-j+1}y_{k+j-1}\right)^2\\&\quad>\frac{(2j-1-\alpha)^2}{4}\sum_{k=N}^{\infty}(k+j-2l)^{(\alpha-2j)}\left(\Delta^{n-j}y_{k+j}\right)^2, \quad j=l+1,\dots,n.\end{aligned}$$

Summarizing all these inequalities for $j=1,\dots,n$, we obtain

$$\begin{aligned}\sum_{k=N}^{\infty}k^{(\alpha)}\left(\Delta^{n}y_{k}\right)^2&>\frac{(1-\alpha)^2}{4}\sum_{k=N}^{\infty}(k-1)^{(\alpha-2)}\left(\Delta^{n-1}y_{k+1}\right)^2\\&>\frac{(1-\alpha)^2(3-\alpha)^2}{4^2}\sum_{k=N}^{\infty}(k-2)^{(\alpha-4)}\left(\Delta^{n-2}y_{k+2}\right)^2\\&\;\vdots\\&>\frac{1}{4^l}\prod_{j=1}^{l}(2j-1-\alpha)^2\sum_{k=N}^{\infty}(k-l)^{(\alpha-2l)}\left(\Delta^{n-l}y_{k+l}\right)^2\\&>\frac{1}{4^{l+1}}\prod_{j=1}^{l+1}(2j-1-\alpha)^2\sum_{k=N}^{\infty}(k-l+1)^{(\alpha-2l-2)}\left(\Delta^{n-l-1}y_{k+l+1}\right)^2\\&\;\vdots\\&>\frac{1}{4^n}\prod_{j=1}^{n}(2j-1-\alpha)^2\sum_{k=N}^{\infty}(k+n-2l)^{(\alpha-2n)}y_{k+n}^2\\&=\gamma_{n,\alpha}\sum_{k=N}^{\infty}(k+n-2l)^{(\alpha-2n)}y_{k+n}^2,\end{aligned}$$

which means, according to Lemma 2.1 again, that equation (1.3) is nonoscillatory. ■

4. Oscillation Criteria for (1.1)

Theorem 4.1. Let $\alpha \in \mathbb{R} \setminus \{1, 3, \ldots, 2n-1\}$ and $l \in \{0, \ldots, n\}$ satisfy conditions (1.5), $\gamma_{n,\alpha}$ be given by (1.4) and $q_k - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)} \geq 0$ for large k. If

$$\sum^{\infty} \left(q_k - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)}\right) k^{2n-1-\alpha} = \infty, \tag{4.1}$$

then equation (1.1) is oscillatory.

Proof. Let $N_0, K, L, M, N \in \mathbb{N}$, $N_0 < K < L < M < N$ (N_0 is arbitrary, the other values will be specified later). We show that for M, N sufficiently large, there exists a sequence $y_k \in \mathcal{D}_n(N_0)$ such that

$$\mathcal{F}(y; N_0, \infty) := \sum_{k=N_0}^{\infty} \left[k^{(\alpha)} \left(\Delta^n y_k\right)^2 - q_k y_{k+n}^2\right] \leq 0$$

and then, oscillation of (1.1) will be a consequence of Lemma 2.1. We construct the sequence y_k as follows:

$$y_k = \begin{cases} 0, & k < K, \\ f_k, & k = K, \ldots, L-1, \\ h_k, & k = L, \ldots, M-1, \\ g_k, & k = M, \ldots, N-1, \\ 0, & k \geq N, \end{cases}$$

where

$$h_k = k^{\left(\frac{2n-1-\alpha}{2}\right)},$$

f_k is any sequence such that

$$\Delta^j f_{K+n-1-j} = 0, \ \Delta^j f_{L+n-1-j} = \Delta^j h_{L+n-1-j}, \ j = 0, \ldots, n-1$$

and g is a solution of (1.3) satisfying the boundary conditions

$$\Delta^j g_{M+n-1-j} = \Delta^j h_{M+n-1-j}, \ \Delta^j g_{N+n-1-j} = 0, \ j = 0, \ldots, n-1. \tag{4.2}$$

(The existence of this solution is shown e.g., in [8].)
Denote

$$c_1 := \sum_{k=K}^{L-1} \left[k^{(\alpha)} \left(\Delta^n f_k\right)^2 - q_k f_{k+n}^2\right].$$

It follows from later given Lemma 5.1 that

$$\frac{h_{k+n}^2}{k^{2n-1-\alpha}} \to 1 > 0 \quad \text{as} \quad k \to \infty$$

and thus condition (4.1) implies that

$$\sum^{\infty} \left(q_k - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)}\right) h_{k+n}^2 = \infty.$$

Hence, by Lemma 5.2 and condition (4.1), it is possible to choose M so large that

$$\sum_{k=L}^{\infty} \left[k^{(\alpha)}\left(\Delta^n h_k\right)^2 - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)} h_{k+n}^2\right] =: c_2 \in \mathbb{R}$$

and

$$\sum_{k=L}^{M-1} \left[\gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)} - q_k\right] h_{k+n}^2 \le -(c_1 + c_2 + 1).$$

Hence

$$\begin{aligned}
\sum_{k=L}^{M-1} \left[k^{(\alpha)}\left(\Delta^n h_k\right)^2 - q_k h_{k+n}^2\right] &= \sum_{k=L}^{M-1} \left[k^{(\alpha)}\left(\Delta^n h_k\right)^2 - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)} h_{k+n}^2\right] \\
&\quad + \sum_{k=L}^{M-1} \left[\gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)} h_{k+n}^2 - q_k h_{k+n}^2\right] \\
&\le -(c_1 + 1).
\end{aligned}$$

Since $q_k - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)} \ge 0$ for large k, we have

$$\sum_{k=M}^{N-1} \left[k^{(\alpha)}\left(\Delta^n g_k\right)^2 - q_k g_{k+n}^2\right] \le \sum_{k=M}^{N-1} \left[k^{(\alpha)}\left(\Delta^n g_k\right)^2 - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)} g_{k+n}^2\right].$$

Next we use the relationship between (1.3) and corresponding LHΔS (2.2). Denote $(x_k^{[g]}, u_k^{[g]})$ the solution of (2.2) associated with the solution g_k of (1.3). We have

$$x_k^{[g]} = \begin{pmatrix} g_{k+n-1} \\ \Delta g_{k+n-2} \\ \vdots \\ \Delta^n g_k \end{pmatrix}, \quad u_k^{[g]} = \begin{pmatrix} (-1)^{n-1}\Delta^{n-1}(k^{(\alpha)}\Delta^n g_k) \\ \vdots \\ -\Delta(k^{(\alpha)}\Delta^n g_k) \\ k^{(\alpha)}\Delta^n g_k \end{pmatrix}$$

and

$$B_k = \operatorname{diag}\left\{0, \dots, 0, \frac{1}{k^{(\alpha)}}\right\}, \quad C_k = \operatorname{diag}\{-\gamma_{n,\alpha}(k+n)^{(\alpha-2n)}, 0, \dots, 0\}.$$

Then, using summation by parts and conditions (4.2),

$$\begin{aligned}
\sum_{k=M}^{N-1} \left(u_k^{[g]}\right)^T \Delta x_k^{[g]} &= \left(u_k^{[g]}\right)^T x_k^{[g]}\Big|_{k=M}^{N} - \sum_{k=M}^{N-1} \Delta \left(u_k^{[g]}\right)^T x_{k+1}^{[g]} \\
&= -\left(u_M^{[g]}\right)^T x_M^{[g]} - \sum_{k=M}^{N-1} \left(x_{k+1}^{[g]}\right)^T \Delta u_k^{[g]}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\sum_{k=M}^{N-1} \left[k^{(\alpha)} \left(\Delta^n g_k\right)^2 - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)} g_{k+n}^2\right] \\
&= \sum_{k=M}^{N-1} \left[\left(u_k^{[g]}\right)^T B_k u_k^{[g]} + \left(x_{k+1}^{[g]}\right)^T C_k x_{k+1}^{[g]}\right] \\
&= \sum_{k=M}^{N-1} \left(u_k^{[g]}\right)^T \Delta x_k^{[g]} + \sum_{k=M}^{N-1} \left[\left(x_{k+1}^{[g]}\right)^T C_k x_{k+1}^{[g]} - \left(u_k^{[g]}\right)^T A x_{k+1}^{[g]}\right] \\
&= -\left(u_M^{[g]}\right)^T x_M^{[g]} + \sum_{k=M}^{N-1} \left(x_{k+1}^{[g]}\right)^T \left[C_k x_{k+1}^{[g]} - A^T u_k^{[g]} - \Delta u_k^{[g]}\right] = -\left(u_M^{[g]}\right)^T x_M^{[g]}.
\end{aligned}
$$

Further, let (X_k, U_k) be the recessive solution of the LHΔS associated with (1.3). The existence of this solution is established in [1], see also [9]. If we denote

$$
\tilde{h}_k := (h_{k+n-1}, \Delta h_{k+n-2}, \ldots, \Delta^{n-1} h_k)^T, \quad \tilde{B}_j := X_{j+1}^{-1}(I-A)^{-1} B_j (X_j^T)^{-1},
$$

then the solution $\left(x_k^{[g]}, u_k^{[g]}\right)$ can be written in the form

$$
\begin{aligned}
x_k^{[g]} &= X_k \left(\sum_{j=k}^{N-1} \tilde{B}_j\right) \left(\sum_{j=M}^{N-1} \tilde{B}_j\right)^{-1} X_j^{-1} \tilde{h}_M, \\
u_k^{[g]} &= \left(U_k \sum_{j=k}^{N-1} \tilde{B}_j - (X_k^T)^{-1}\right) \left(\sum_{j=M}^{N-1} \tilde{B}_j\right)^{-1} X_M^{-1} \tilde{h}_M,
\end{aligned}
$$

see [6], and hence (according to conditions (4.2))

$$
-(u_M^{[g]})^T x_M^{[g]} = -\tilde{h}_M^T U_M X_M^{-1} \tilde{h}_M + \tilde{h}_M^T (X_M^T)^{-1} \left(\sum_{j=M}^{N-1} \tilde{B}_j\right)^{-1} X_M^{-1} \tilde{h}_M.
$$

Using the fact, that (X_k, U_k) is the recessive solution, it follows from (2.3) that we can choose $N > M$ such that

$$
\tilde{h}_M (X_M^T)^{-1} \left(\sum_{j=M}^{N-1} \tilde{B}_j\right)^{-1} X_M^{-1} \tilde{h}_M \leq 1.
$$

Finally, $U_M X_M^{-1} \geq 0$, since $C_k \leq 0$ and it means that

$$
\tilde{h}_M^T U_M X_M^{-1} \tilde{h}_M \geq 0.
$$

The proof of this fact can be found in [4]. If we summarize all the above computations, we obtain

$$\mathcal{F}(y; N_0, \infty) \leq c_1 - (c_1 + 1) + 1 = 0.$$

■

Corollary 4.2. Let $\alpha \in \mathbb{R} \setminus \{1, 3, \dots, 2n-1\}$ and $l \in \{0, \dots, n\}$ satisfy conditions (1.5), $\gamma_{n,\alpha}$ be given by (1.4). Equation

$$(-1)^n \Delta^n \left(k^{(\alpha)} \Delta^n y_k\right) - \gamma (k + n - 2l)^{(\alpha - 2n)} y_{k+n} = 0 \tag{4.3}$$

is nonoscillatory for $\gamma \leq \gamma_{n,\alpha}$ and it is oscillatory for $\gamma > \gamma_{n,\alpha}$.

Proof. Nonoscillation of (4.3) for $\gamma \leq \gamma_{n,\alpha}$ is a consequence of Lemma 2.1, since nonoscillation of (1.3) and Lemma 2.1 imply that there exists $N \in \mathbb{N}$ such that for any $y \in \mathcal{D}_n(N)$

$$\begin{aligned} 0 &< \sum_{k=N}^{\infty} \left[k^{(\alpha)} \left(\Delta^n y_k\right)^2 - \gamma_{n,\alpha} (k + n - 2l)^{(\alpha-2n)} y_{k+n}^2\right] \\ &< \sum_{k=N}^{\infty} \left[k^{(\alpha)} \left(\Delta^n y_k\right)^2 - \gamma (k + n - 2l)^{(\alpha-2n)} y_{k+n}^2\right]. \end{aligned}$$

If $\gamma > \gamma_{n,\alpha}$, then condition (4.1) for (4.3) is of the form

$$(\gamma - \gamma_{n,\alpha}) \sum^{\infty} (k + n - 2l)^{(\alpha-2n)} k^{2n-1-\alpha} = \infty,$$

and it holds, since using Lemma 5.1

$$\lim_{k \to \infty} \frac{(k + n - 2l)^{(\alpha-2n)} k^{2n-1-\alpha}}{k^{-1}} = 1 > 0 \quad \text{and} \quad \sum^{\infty} k^{-1} = \infty.$$

Oscillation then follows from Theorem 4.1. ■

Theorem 4.3. Let $\alpha \in \mathbb{R} \setminus \{1, 2, \dots, 2n-1\}$ and $l \in \{0, \dots, n\}$ satisfy conditions (1.5), $\gamma_{n,\alpha}$ be given by (1.4) and $q_k - \gamma_{n,\alpha}(k + n - 2l)^{(\alpha-2n)} \geq 0$ for large k. Equation (1.1) is oscillatory provided one of the following two conditions holds:

(i) $\alpha < 2n - 1$ and

$$\lim_{k \to \infty} k^{2n-1-\alpha} \sum_{j=k}^{\infty} q_j > \frac{\gamma_{n,\alpha}}{2n - 1 - \alpha},$$

(ii) $\alpha > 2n - 1$ and

$$\lim_{k \to \infty} k^{2n-1-\alpha} \sum^{k} q_j > \frac{\gamma_{n,\alpha}}{\alpha + 1 - 2n}.$$

Proof. According to Theorem 4.1, it is sufficient to show that condition (4.1) is satisfied, and this holds (using Lemma 5.1) if and only if

$$\sum^{\infty} \left(q_k - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)}\right)(k+n-2l)^{(2n-1-\alpha)} = \infty. \tag{4.4}$$

Consider first case (i). Denote $L := \lim_{k\to\infty} k^{2n-1-\alpha} \sum_{j=k}^{\infty} q_j$ and suppose that $L < \infty$. Using Lemma 5.1, we have

$$\begin{aligned} L &= \lim_{k\to\infty}(k+1)^{2n-1-\alpha} \sum_{j=k+1}^{\infty} q_j = \lim_{k\to\infty}(k+n-2l)^{(2n-1-\alpha)} \sum_{j=k+1}^{\infty} q_j \\ &= \lim_{k\to\infty}(k+n-2l)(k+n-2l)^{(2n-2-\alpha)} \sum_{j=k+1}^{\infty} q_j. \end{aligned}$$

Since $L > \dfrac{\gamma_{n,\alpha}}{2n-1-\alpha}$, there exists $\varepsilon_1 > 0$ and $N_1 \in \mathbb{N}$ such that for $k \geq N_1$

$$(k+n-2l)^{(2n-2-\alpha)} \sum_{j=k+1}^{\infty} q_j > \frac{\frac{\gamma_{n,\alpha}}{2n-1-\alpha} + \varepsilon_1}{k+n-2l}. \tag{4.5}$$

Next denote

$$a_k := (k+n-2l)^{(\alpha-2n)}(k+n-2l)^{(2n-1-\alpha)} \tag{4.6}$$

and let ε_2 be arbitrary real constant satisfying $0 < \varepsilon_2 < \dfrac{2n-1-\alpha}{\gamma_{n,\alpha}}\varepsilon_1$. We have

$$(k+n-2l)a_k \to 1 \quad \text{as} \quad k \to \infty$$

and hence there exists $N_2 \in \mathbb{N}$ such that $(k+n-2l)a_k < 1 + \varepsilon_2$ for $k \geq N_2$, i.e.,

$$-a_k > -\frac{1+\varepsilon_2}{k+n-2l}. \tag{4.7}$$

Let $N = \max\{N_1, N_2\}$ and $M > N$. Summation by parts and using inequalities (4.5) and (4.7) we obtain

$$
\begin{aligned}
&\sum_{k=N}^{M}\left[q_k-\gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)}\right](k+n-2l)^{(2n-1-\alpha)}\\
&= \sum_{k=N}^{M} q_k(k+n-2l)^{(2n-1-\alpha)}-\gamma_{n,\alpha}\sum_{k=N}^{M} a_k\\
&= -(k+n-2l)^{(2n-1-\alpha)}\sum_{j=k}^{\infty} q_j\Big|_{k=N}^{M+1}\\
&\quad +(2n-1-\alpha)\sum_{k=N}^{M}(k+n-2l)^{(2n-2-\alpha)}\sum_{j=k+1}^{\infty} q_j-\gamma_{n,\alpha}\sum_{k=N}^{M} a_k\\
&> -(k+n-2l)^{(2n-1-\alpha)}\sum_{j=k}^{\infty} q_j\Big|_{k=N}^{M+1}\\
&\quad +[\gamma_{n,\alpha}+(2n-1-\alpha)\varepsilon_1]\sum_{k=N}^{M}\frac{1}{k+n-2l}-\gamma_{n,\alpha}(1+\varepsilon_2)\sum_{k=N}^{M}\frac{1}{k+n-2l}\\
&= (N+n-2l)^{(2n-1-\alpha)}\sum_{j=N}^{\infty} q_j-(M+1+n-2l)^{(2n-1-\alpha)}\sum_{j=M+1}^{\infty} q_j\\
&\quad +[(2n-1-\alpha)\varepsilon_1-\gamma_{n,\alpha}\varepsilon_2]\sum_{k=N}^{M}\frac{1}{k+n-2l}.
\end{aligned}
$$

Since $L<\infty$ and $(2n-1-\alpha)\varepsilon_1-\gamma_{n,\alpha}\varepsilon_2>0$, we obtain

$$
\sum_{k=N}^{M}\left[q_k-\gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)}\right](k+n-2l)^{(2n-1-\alpha)}\to\infty \quad \text{as} \quad M\to\infty.
$$

Thus (4.4), and consequently also (4.1), is satisfied. If $L=\infty$, then oscillation of (1.1) follows from Proposition 1.2, part (i).

Concerning case (ii),

$$
\begin{aligned}
L &:= \lim_{k\to\infty} k^{2n-1-\alpha}\sum^{k} q_j=\lim_{k\to\infty}(k-n)^{(2n-1-\alpha)}\sum^{k} q_j\\
&= \lim_{k\to\infty}(k-n)(k-n)^{(2n-2-\alpha)}\sum^{k} q_j
\end{aligned}
$$

and similarly as in the previous case let $L<\infty$. In the opposite case, the oscillation of (1.1) is a consequence of Proposition 1.2, part (ii). Since $L>\dfrac{\gamma_{n,\alpha}}{\alpha+1-2n}$, there exists

$\varepsilon_1 > 0$ and $N_1 \in \mathbb{N}$ such that for $k \geq N_1$

$$(k-n)^{(2n-2-\alpha)} \sum^{k} q_j > \frac{\frac{\gamma_{n,\alpha}}{\alpha+1-2n} + \varepsilon_1}{k-n}. \tag{4.8}$$

Let a_k be given by (4.6) (note that $l = n$ according to conditions (1.5)) and let ε_2 be such that $0 < \varepsilon_2 < \dfrac{\alpha+1-2n}{\gamma_{n,\alpha}}\varepsilon_1$. Then there exists $N_2 \in \mathbb{N}$ such that

$$-a_k > -\frac{1+\varepsilon_2}{k-n}. \tag{4.9}$$

Again, using summation by parts and conditions (4.8) and (4.9), we obtain for $N = \max\{N_1, N_2\}$ and $M > N$

$$\begin{aligned}
&\sum_{k=N}^{M} \left[q_k - \gamma_{n,\alpha}(k-n)^{(\alpha-2n)}\right](k-n)^{(2n-1-\alpha)} \\
&= (k-n)^{(2n-1-\alpha)} \sum^{k-1} q_j \Bigg|_{k=N}^{M+1} + (\alpha+1-2n)\sum_{k=N}^{M}(k-n)^{(2n-2-\alpha)}\sum^{k} q_j \\
&\quad -\gamma_{n,\alpha}\sum_{k=N}^{M} a_k \\
&> (M+1-n)^{(2n-1-\alpha)}\sum^{M} q_j - (N-n)^{(2n-1-\alpha)}\sum^{N-1} q_j \\
&\quad + \left[(\alpha+1-2n)\varepsilon_1 - \gamma_{n,\alpha}\varepsilon_2\right]\sum_{k=N}^{M}\frac{1}{k-n} \to \infty,
\end{aligned}$$

as $M \to \infty$. This means that (4.1) holds and hence equation (1.1) is oscillatory. ■

Remark 4.4. Note that if the limits in Theorem 4.3 are finite, we do not need to exclude the case when $\alpha \in \{2, 4, \ldots, 2n-2\}$ (since we do not need to use Proposition 1.2) and we can suppose that $\alpha \in \mathbb{R} \setminus \{1, 3, \ldots, 2n-1\}$.

5. Technical Results

The proof of the following technical result was suggested by M. Bohner [3].

Lemma 5.1. We have

$$\lim_{k\to\infty} \frac{k^{\alpha}}{k^{(\alpha)}} = 1, \quad \text{where} \quad k^{(\alpha)} := \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}.$$

Proof. Stirling's formula [21, Chapter 8] says that

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}} = 1.$$

Thus

$$a_k := \frac{\Gamma(k+1-\alpha)\left(\frac{k}{e}\right)^k\sqrt{2\pi k}}{\Gamma(k+1)\left(\frac{k-\alpha}{e}\right)^{k-\alpha}\sqrt{2\pi(k-\alpha)}} \to 1 \quad \text{as} \quad k\to\infty.$$

Thus we can conclude that

$$\begin{aligned}
\frac{k^\alpha}{k^{(\alpha)}} &= k^\alpha a_k \frac{\left(\frac{k-\alpha}{e}\right)^{k-\alpha}\sqrt{2\pi(k-\alpha)}}{\left(\frac{k}{e}\right)^k\sqrt{2\pi k}} \\
&= a_k \frac{\left(\frac{k-\alpha}{e}\right)^{k-\alpha}\sqrt{k-\alpha}}{e^{-\alpha}\sqrt{k}} \\
&= a_k e^\alpha \sqrt{1-\frac{\alpha}{k}}\left(1-\frac{\alpha}{k}\right)^{k-\alpha} \\
&\to 1\cdot e^\alpha \cdot 1 \cdot e^{-\alpha} = 1 \quad \text{as} \quad k\to\infty.
\end{aligned}$$

■

Lemma 5.2. Let $\alpha \in \mathbb{R}\setminus\{1,3,\dots,2n-1\}$, $l\in\{0,1,\dots,n\}$ and $\gamma_{n,\alpha}$ be given by (1.4). Then

$$\sum^{\infty}\left[k^{(\alpha)}(\Delta^n h_k)^2 - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)}h_{k+n}^2\right] < \infty,$$

where $h_k = k^{\left(\frac{2n-1-\alpha}{2}\right)}$.

Proof. We have

$$\Delta^n h_k = \frac{1}{2^n}\prod_{j=1}^n (2j-1-\alpha)k^{\left(\frac{-1-\alpha}{2}\right)}$$

and hence

$$\begin{aligned}
&k^{(\alpha)}(\Delta^n h_k)^2 - \gamma_{n,\alpha}(k+n-2l)^{(\alpha-2n)}h_{k+n}^2 \\
&= \gamma_{n,\alpha}\left\{k^{(\alpha)}\left[k^{\left(\frac{-1-\alpha}{2}\right)}\right]^2 - (k+n-2l)^{(\alpha-2n)}\left[(k+n)^{\left(\frac{2n-1-\alpha}{2}\right)}\right]^2\right\} \\
&= \gamma_{n,\alpha}\left\{k^{(\alpha)}\left[k^{\left(\frac{-1-\alpha}{2}\right)}\right]^2 - \frac{\Gamma(k+n+1-2l)\Gamma^2(k+n+1)}{\Gamma(k+3n+1-\alpha-2l)\Gamma^2\left(k+\frac{3}{2}+\frac{\alpha}{2}\right)}\right\} \\
&= \gamma_{n,\alpha}k^{(\alpha)}\left[k^{\left(\frac{-1-\alpha}{2}\right)}\right]^2\left\{1-\frac{\Gamma(k+n+1-2l)\Gamma^2(k+n+1)\Gamma(k+1-\alpha)}{\Gamma(k+3n+1-\alpha-2l)\Gamma^3(k+1)}\right\} \\
&= \gamma_{n,\alpha}k^{(\alpha)}\left[k^{\left(\frac{-1-\alpha}{2}\right)}\right]^2 \times \\
&\qquad \times\left\{1-\frac{\Gamma(k+n+1-2l)\Gamma(k+1-\alpha)(k+n)^2\cdots(k+1)^2}{\Gamma(k+3n+1-\alpha-2l)\Gamma(k+1)}\right\}.
\end{aligned}$$

Denote

$$c_k := 1 - \frac{\Gamma(k+n+1-2l)\Gamma(k+1-\alpha)(k+n)^2\cdots(k+1)^2}{\Gamma(k+3n+1-\alpha-2l)\Gamma(k+1)}.$$

If $n > 2l$, then

$$\begin{aligned} c_k &= 1 - \frac{(k+n-2l)\cdots(k+1)(k+n)^2\cdots(k+1)^2}{(k+3n-2l-\alpha)\cdots(k+1-\alpha)} \\ &= 1 - \frac{k^{3n-2l} + O\left(k^{3n-2l-1}\right)}{k^{3n-2l} + O\left(k^{3n-2l-1}\right)} = \frac{O\left(k^{3n-2l-1}\right)}{k^{3n-2l} + O\left(k^{3n-2l-1}\right)} \quad \text{as} \quad k \to \infty; \end{aligned}$$

if $n = 2l$, then

$$\begin{aligned} c_k &= 1 - \frac{(k+n)^2\cdots(k+1)^2}{(k+2n-\alpha)\cdots(k+1-\alpha)} \\ &= 1 - \frac{k^{2n} + O\left(k^{2n-1}\right)}{k^{2n} + O\left(k^{2n-1}\right)} = \frac{O\left(k^{2n-1}\right)}{k^{2n} + O\left(k^{2n-1}\right)} \quad \text{as} \quad k \to \infty; \end{aligned}$$

and if $n < 2l$, then

$$\begin{aligned} c_k &= 1 - \frac{(k+n)^2\cdots(k+1)^2(k-\alpha)\cdots(k-\alpha+1+n-2l)}{k(k-1)\cdots(k+1+n-2l)(k+3n-\alpha-2l)\cdots(k+1-\alpha+n-2l)} \\ &= 1 - \frac{k^{n+2l-1} + O\left(k^{n+2l-2}\right)}{k^{n+2l-1} + O\left(k^{n+2l-2}\right)} = \frac{O\left(k^{n+2l-2}\right)}{k^{n+2l-1} + O\left(k^{n+2l-2}\right)} \quad \text{as} \quad k \to \infty. \end{aligned}$$

Using Lemma 5.1,

$$\lim_{k\to\infty} \frac{\gamma_{n,\alpha} k^{(\alpha)} \left[k^{\left(\frac{-1-\alpha}{2}\right)}\right]^2 |c_k|}{k^{-\frac{3}{2}}} = 0 < 1$$

holds in all three cases, and since $\sum^{\infty} k^{-\frac{3}{2}} < \infty$, we have the result of this lemma. ■

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