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Existence of Solutions to First-Order Dynamic Boundary Value Problems*

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Abstract

This article investigates the existence of solutions to boundary value problems (BVPs) involving systems of first-order dynamic equations on time scales subject to two-point boundary conditions. The methods involve novel dynamic inequalities and fixed-point theory to yield new theorems guaranteeing the existence of at least one solution.

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1. Introduction

This paper considers the existence of solutions to the first-order dynamic equation of the type

$$x^{\Delta} + b(t)x = h(t, x), \quad t \in [a, c]_{\mathbb{T}} := [a, c] \cap \mathbb{T}, \tag{1.1}$$

subject to the boundary conditions

$$G(x(a), x(\sigma(c))) = 0, \quad a, c \in \mathbb{T}, \tag{1.2}$$

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where $h : [a, c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous, nonlinear function; t is from a so-called “time scale” $\mathbb{T}$ (which is a nonempty closed subset of $\mathbb{R}$); x^Δ is the generalized derivative of x; the function $b : [a, c]_{\mathbb{T}} \to \mathbb{R}$; $a < c$ are given constants in $\mathbb{T}$; and G is some known function describing a linear set of boundary conditions. Equation (1.1) subject to (1.2) is known as a dynamic boundary value problem (BVP) on time scales.

If $\mathbb{T} = \mathbb{R}$, then $x^\Delta = x'$ and (1.1), (1.2) become the following BVP for ordinary differential equations

$$x' + b(t)x = h(t, x), \qquad t \in [a, c], \tag{1.3}$$

$$G(x(a), x(c)) = 0, \tag{1.4}$$

If $\mathbb{T} = \mathbb{Z}$, then $x^\Delta = \Delta x$ and (1.1), (1.2) become the following BVP for difference equations

$$\Delta x + b(t)x = h(t, x), \qquad t \in \{a, a+1, \dots, c\}, \tag{1.5}$$

$$G(x(a), x(c+1)) = 0, \qquad a, c \in \mathbb{Z}. \tag{1.6}$$

There are many more time scales than just $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ and hence many more dynamic equations.

The field of dynamic equations on time scales provides a natural framework for:

1. establishing new insight into the theories of non-classical difference equations;

2. forming novel knowledge about “differential-difference” equations;

3. advancing, in their own right, each of the theories of differential equations and (classical) difference equations.

In cases 1 and 2, interested researchers tend to analyze known results for differential equations and/or (classical) difference equations and then extend these ideas to the more general time scale setting. Above, “non-classical” difference equations include, for example, the rapidly developing q-difference equations [11], used in physics. “Differential-difference” equations feature both differential equations and difference equations. These type of equations appear in models where time flows continuously and discretely at different periods, see Example 5.1.

In situation 3, researchers desire to formulate new results in the general time scale setting, with particular significance being found when special cases of the new results are novel, even for the differential or difference equation case.

Motivated by the above, and also by [17, 18], this article investigates the existence of solutions to systems of dynamic equations in the general time scale setting. Some sufficient conditions, in terms of dynamic inequalities on h, are presented that ensure the existence of at least one solution to the dynamic BVP under consideration. The main tools involve fixed-point methods and the Nonlinear Alternative.

This article advances all three situations raised above. In Sections 2 and 3, the interest is in the first two situations. In Section 4, the interest lies in the third situation.

To understand the notation used above, some preliminary definitions are needed.

Definition 1.1. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$.

Since a time scale may or may not be connected, the concept of the jump operator is useful, and we will use it to define the generalized derivative x^Δ of the function x.

Definition 1.2. The forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ (respectively $\rho(t)$ at t for $t > \inf \mathbb{T}$) is given by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\}) \text{ for all } t \in \mathbb{T}.$$

Define the graininess function $\mu : \mathbb{T} \to [0, \infty)$ as $\mu(t) = \sigma(t) - t$.

Throughout this work the assumption is made that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$.

Definition 1.3. The jump operators σ and ρ allow the classification of points in a time scale in the following way: If $\sigma(t) > t$, then the point t is called right-scattered; while if $\rho(t) < t$, then t is termed left-scattered. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then the point t is called right-dense; while if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then we say t is left-dense.

If $\mathbb{T}$ has a left-scattered maximum value m, then we define $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 1.4. Fix $t \in \mathbb{T}^\kappa$ and let $x : \mathbb{T} \to \mathbb{R}^n$. Define $x^\Delta(t)$ to be the vector (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood U of t with

$$|[x_i(\sigma(t)) - x_i(s)] - x_i^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s| \quad \text{for all } s \in U \text{ and each } i = 1, \ldots, n.$$

Call $x^\Delta(t)$ the delta derivative of $x(t)$ and say that x is delta differentiable.

Definition 1.5. If $K^\Delta(t) = k(t)$, then define the delta integral by

$$\int_a^t k(s)\Delta s = K(t) - K(a).$$

If $\mathbb{T} = \mathbb{R}$, then $\displaystyle\int_a^t k(s)\Delta s = \int_a^t k(s)ds$, while if $\mathbb{T} = \mathbb{Z}$, then $\displaystyle\int_a^t k(s)\Delta s = \sum_a^{t-1} k(s)$. Once again, there are many more time scales than just $\mathbb{R}$ and $\mathbb{Z}$ and hence there are many more delta integrals. For a more general definition of the delta integral see [2].

Theorem 1.6. [9] Assume that $k : \mathbb{T} \to \mathbb{R}^n$ and let $t \in \mathbb{T}^\kappa$.

(i) If k is delta differentiable at t, then k is continuous at t.

(ii) If k is continuous at t and t is right-scattered, then k is delta differentiable at t with

$$k^{\Delta}(t) = \frac{k(\sigma(t)) - k(t)}{\sigma(t) - t}.$$

(iii) If k is delta differentiable and t is right-dense, then

$$k^{\Delta}(t) = \lim_{s \to t} \frac{k(t) - k(s)}{t - s}.$$

(iv) If k is delta differentiable at t, then $k(\sigma(t)) = k(t) + \mu(t)k^{\Delta}(t)$.

The relatively young theory of time scales dates back to Hilger [9]. The monographs [2] and [12] also provide an excellent introduction. For more recent developments in dynamic equations on time scales, the reader is referred to [1, 3, 5–8, 10, 14, 15, 19, 20].

A solution to (1.1), (1.2) is a continuous function $x : [a, \sigma(c)]_{\mathbb{T}} \to \mathbb{R}^n$ (denoted by $x \in C([a, \sigma(c)]_{\mathbb{T}}; \mathbb{R}^n)$) that satisfies (1.1) and (1.2).

In what follows, if $y, z \in \mathbb{R}^n$, then $\langle y, z \rangle$ denotes the usual inner product and $\|z\|$ denotes the Euclidean norm of z on $\mathbb{R}^n$.

2. Existence for the Non-Periodic Case

This section considers the existence of solutions to the first-order dynamic equation

$$x^{\Delta} = f(t, x), \quad t \in [a, c]_{\mathbb{T}}, \tag{2.1}$$

subject to the boundary conditions

$$Mx(a) + Rx(\sigma(c)) = 0, \quad a, c \in \mathbb{T}, \tag{2.2}$$

where $f : [a, c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous, nonlinear function; $a < c$ are given constants in $\mathbb{T}$; and M, R are given constants in $\mathbb{R}$.

Throughout this section, assume

$$M + R \neq 0. \tag{2.3}$$

Lemma 2.1. Suppose (2.3) holds. The BVP (2.1), (2.2) is equivalent to the integral equation

$$x(t) = \int_a^t f(s, x(s))\, \Delta s - (M + R)^{-1} R \int_a^{\sigma(c)} f(s, x(s))\, \Delta s, \quad t \in [a, c]_{\mathbb{T}}. \tag{2.4}$$

Proof. Let $x : [a, \sigma(c)]_{\mathbb{T}} \to \mathbb{R}^n$ satisfy (2.1) and (2.2). It is easy to see that

$$x(t) = x(a) + \int_a^t f(s, x(s))\, \Delta s, \quad t \in [a, \sigma(c)]_{\mathbb{T}}, \tag{2.5}$$

and

$$x(\sigma(c)) = x(a) + \int_a^{\sigma(c)} f(s, x(s)) \, \Delta s.$$

So (2.2) gives

$$0 = Mx(a) + R\left(x(a) + \int_a^{\sigma(c)} f(s, x(s)) \, \Delta s\right) \tag{2.6}$$

and rearranging (2.6) yields

$$x(a) = -(M+R)^{-1} R \int_a^{\sigma(c)} f(s, x(s)) \, \Delta s. \tag{2.7}$$

So substituting (2.7) into (2.5) gives, for $t \in [a, \sigma(c)]_{\mathbb{T}}$,

$$x(t) = -(M+R)^{-1} R \int_a^{\sigma(c)} f(s, x(s)) \, \Delta s + \int_a^t f(s, x(s)) \, \Delta s. \tag{2.8}$$

If x is a solution to (2.4), then is it easy to show that (2.1) and (2.2) hold by direct calculation. ■

The following is the main result of this section.

Theorem 2.2. Suppose (2.3) holds and $f : [a, c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. If there exist non-negative constants α and K such that

$$\|f(t, q)\| \le 2\alpha \langle q, f(t, q)\rangle + K, \qquad \forall (t, q) \in [a, c]_{\mathbb{T}} \times \mathbb{R}^n, \tag{2.9}$$

$$\text{and} \quad |M/R| \le 1, \tag{2.10}$$

then the BVP (2.1), (2.2) has at least one solution.

Proof. By Lemma 2.1, we want to show that there exists at least one solution to (2.4), which is equivalent to showing that (2.1), (2.2) has at least one solution. To do this, we use the Nonlinear Alternative.

Consider the map $T : C([a, \sigma(c)]_{\mathbb{T}}; \mathbb{R}^n) \to C([a, \sigma(c)]_{\mathbb{T}}; \mathbb{R}^n)$ defined by

$$Tx(t) = -(M+R)^{-1} R \int_a^{\sigma(c)} f(s, x(s)) \, \Delta s + \int_a^t f(s, x(s)) \, \Delta s, \quad \forall t \in [a, \sigma(c)]_{\mathbb{T}}.$$

Thus, our problem is reduced to proving the existence of at least one x such that

$$x = Tx. \tag{2.11}$$

With this in mind, consider the family of equations associated with (2.11) given by

$$x = \lambda Tx, \quad \lambda \in [0, 1]. \tag{2.12}$$

We show that

$$x \neq \lambda T x, \quad x \in \partial B_P, \quad \lambda \in [0,1], \tag{2.13}$$

for some suitable ball $B_P \subset C([a,\sigma(c)]_{\mathbb{T}};\mathbb{R}^n)$ with radius $P > 0$. Let

$$B_P = \left\{ x \in C([a,\sigma(c)]_{\mathbb{T}};\mathbb{R}^n) \mid \max_{t\in[a,\sigma(c)]_{\mathbb{T}}} \|x(t)\| < P \right\},$$
$$P = \left[1 + |(M+R)^{-1}R|\right] K(\sigma(c)-a) + 1.$$

Let x be a solution to (2.12) and see that, by Lemma 2.1, x must also be a solution to

$$x^{\Delta} = \lambda f(t,x), \quad t \in [a,c]_{\mathbb{T}}, \qquad \lambda \in [0,1], \tag{2.14}$$
$$Mx(a) + Rx(\sigma(c)) \;=\; 0. \tag{2.15}$$

Consider $r(t) := \|x(t)\|^2$ for all $t \in [a,\sigma(c)]_{\mathbb{T}}$. By the product rule [2, Theorem 1.20 (iii)] and Theorem 1.6 (iv) we have

$$\begin{aligned} r^{\Delta}(t) &= 2\langle x(t), x^{\Delta}(t)\rangle + \mu(t)\|x^{\Delta}(t)\|^2, \quad t \in [a,c]_{\mathbb{T}}, \\ &= 2\langle x(t), \lambda f(t,x(t))\rangle + \mu(t)\|\lambda f(t,x(t))\|^2 \\ &\geq 2\langle x(t), \lambda f(t,x(t))\rangle. \end{aligned} \tag{2.16}$$

From (2.16) and (2.9) obtain

$$\begin{aligned} \|\lambda f(t,q)\| &\leq 2\alpha\langle q, \lambda f(t,q)\rangle + \lambda K \\ &\leq \alpha r^{\Delta}(t) + K, \end{aligned} \tag{2.17}$$

Also, (2.10) implies

$$r(\sigma(c)) \leq r(a) \tag{2.18}$$

since (2.2) gives

$$\|x(\sigma(c))\| \leq |M/R| \, \|x(a)\| \leq \|x(a)\|.$$

Let $H := 1 + |(M+R)^{-1}R|$. We show that $\|\lambda T x\| < P$ for all $\|x\| \leq P$ and thus (2.13) will hold. With this in mind, consider

$$\begin{aligned} \|\lambda T x(t)\| &= \left\| -(M+R)^{-1}R \int_a^{\sigma(c)} \lambda f(s,x(s))\,\Delta s + \int_a^t \lambda f(s,x(s))\,\Delta s \right\| \\ &\leq H \int_a^{\sigma(c)} \|\lambda f(s,x(s))\|\,\Delta s \\ &\leq H \int_a^{\sigma(c)} \left[\alpha r^{\Delta}(s) + K\right]\,\Delta s, \quad \text{from (2.17)} \\ &= H\left[\alpha(r(\sigma(c)) - r(a)) + K(\sigma(c)-a)\right] \\ &\leq H\left[K(\sigma(c)-a)\right], \quad \text{from (2.18)} \\ &< P \end{aligned}$$

and thus $T : \overline{B_P} \to C([a, \sigma(c); \mathbb{R}^n)$ satisfies (2.13).

The operator $T : \overline{B_P} \to C([a, \sigma(c); \mathbb{R}^n)$ is compact by the Arzela–Ascoli theorem (because it is a completely continuous map restricted to a closed ball).

The Nonlinear Alternative ensures the existence of at least one solution in B_P to (2.4) and hence to (2.1), (2.2). ■

If $M = 1 = N$, then the boundary conditions (2.2) become the so-called anti-periodic boundary conditions

$$x(a) = -x(\sigma(c)) \tag{2.19}$$

and the following corollary to Theorem 2.2 easily follows.

Corollary 2.3. Let $f : [a, c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. If there exist non-negative constants α and K such that (2.9) holds, then the anti-periodic BVP (2.1), (2.19) has at least one solution.

Proof. It is easy to see that for $M = 1 = R$, all of the conditions of Theorem 2.2 hold. Thus the result follows from Theorem 2.2. ■

Now consider (2.1), (2.2) with $n = 1$. For this case the following corollary to Theorem 2.2 is obtained.

Corollary 2.4. Suppose (2.3) holds and let $f : [a, c]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ be continuous. If there exist non-negative constants α and K such that

$$|f(t, q)| \leq 2\alpha q f(t, q) + K, \qquad \forall (t, q) \in [a, c]_{\mathbb{T}} \times \mathbb{R}, \tag{2.20}$$

$$\text{and} \quad |M/R| \leq 1, \tag{2.21}$$

then, for $n = 1$, the BVP (2.1), (2.2) has at least one solution.

Proof. It is easy to see that for $n = 1$: (2.9) becomes (2.20); and the result follows from Theorem 2.2. ■

3. Existence for the Periodic Case

This section considers the existence of solutions to the first-order system of periodic BVPs

$$x^{\Delta} + b(t)x = g(t, x), \quad t \in [a, c]_{\mathbb{T}}, \tag{3.1}$$

$$x(a) = x(\sigma(c)), \tag{3.2}$$

where $g : [a, c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ and $b : [a, c]_{\mathbb{T}} \to \mathbb{R}$ are both continuous functions, with b having no zeros on $[a, c]_{\mathbb{T}}$.

Also considered herein, is the existence of solutions to the first-order system

$$x^{\Delta} = f(t, x), \quad t \in [a, c]_{\mathbb{T}}, \tag{3.3}$$

subject to (3.2), where $f : [a, c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function.

We need a few more definitions to assist with our investigation. The following gives a generalized idea of continuity on time scales.

Definition 3.1. Assume $k : \mathbb{T} \to \mathbb{R}$. Define and denote $k \in C_{rd}(\mathbb{T};\mathbb{R})$ as right-dense continuous (rd-continuous) if: k is continuous at every right-dense point $t \in \mathbb{T}$; and $\lim_{s\to t^-} k(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

Now define the so-called set of regressive functions, $\mathcal{R}$, by

$$\mathcal{R} = \{p \in C_{rd}(\mathbb{T};\mathbb{R}) \text{ and } 1 + p(t)\mu(t) \neq 0 \text{ on } \mathbb{T}\}.$$

The methods in this section rely on a generalized exponential function $e_p(\cdot, t_0)$ on a time scale $\mathbb{T}$. For $p \in \mathcal{R}$, we define (see [2, Theorem 2.35]) the exponential function $e_p(\cdot, t_0)$ on the time scale $\mathbb{T}$ as the unique solution to the IVP

$$x^{\Delta} = p(t)x, \quad x(t_0) = x_0.$$

If $p \in \mathcal{R}$, then $e_p(\cdot, t_0)$ has no zeros.

More explicitly, the exponential function $e_p(\cdot, t_0)$ is given by

$$e_p(t, t_0) = \begin{cases} \exp\left(\int_{t_0}^{t} p(s)\Delta s\right), & \text{for } t \in \mathbb{T},\ \mu = 0, \\ \exp\left(\int_{t_0}^{t} \dfrac{\text{Log}(1 + \mu(s)p(s))}{\mu(t)}\Delta s\right), & \text{for } t \in \mathbb{T},\ \mu > 0, \end{cases}$$

where Log is the principal logarithm function.

Throughout this section assume

$$-b \in \mathcal{R}, \quad \text{and} \quad e_{-b}(\sigma(c), a) \neq 1. \tag{3.4}$$

Lemma 3.2. Let (3.4) hold. The BVP (3.1), (3.2) is equivalent to the integral equation

$$x(t) = e_{-b}(t, a)\left[\frac{1}{1 - e_{-b}(\sigma(c), a)}\int_a^{\sigma(c)} \frac{g(s, x(s))}{e_{-b}(\sigma(s), a)}\Delta s + \int_a^t \frac{g(s, x(s))}{e_{-b}(\sigma(s), a)}\Delta s\right] \tag{3.5}$$

for $t \in [a, \sigma(c)]_{\mathbb{T}}$.

Proof. Let x be a solution to (3.1), (3.2). By the quotient rule [2, Theorem 1.20 (v)], consider

$$\left[\frac{x(t)}{e_{-b}(t, a)}\right]^{\Delta} = \frac{x^{\Delta}(t) + b(t)x(t)}{e_{-b}(\sigma(t), a)} = \frac{g(t, x(t))}{e_{-b}(\sigma(t), a)}, \quad t \in [a, c]_{\mathbb{T}},$$

and hence integrating the above on $[a,t]_{\mathbb{T}}$ obtain

$$x(t) = e_{-b}(t,a)\left[x(a) + \int_a^t \frac{g(s,x(s))}{e_{-b}(\sigma(s),a)}\Delta s\right], \quad t \in [a,\sigma(c)]_{\mathbb{T}}. \tag{3.6}$$

Using (3.2) and (3.6), obtain

$$x(a) = x(\sigma(c)) = e_{-b}(\sigma(c),a)\left[x(a) + \int_a^{\sigma(c)} \frac{g(s,x(s))}{e_{-b}(\sigma(s),a)}\Delta s\right]$$

and a rearrangement leads to

$$x(a) = \frac{1}{1 - e_{-b}(\sigma(c),a)} \int_a^{\sigma(c)} \frac{g(s,x(s))}{e_{-b}(\sigma(s),a)}\Delta s. \tag{3.7}$$

So by substituting (3.7) into (3.6), the result follows.

If x is a solution to (3.5), then x satisfies (3.1), (3.2), which may be verified by direct computation. ■

The following is the main result of this section.

Theorem 3.3. Let (3.4) hold and $g : [a,c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. If there exist non-negative constants α and K such that

$$\frac{\|\lambda g(t,q)\|}{|e_{-b}(\sigma(t),a)|} \leq \alpha\left[2\langle q, \lambda g(t,q) - b(t)q\rangle\right] + K,$$
$$\forall (t,q,\lambda) \in [a,c]_{\mathbb{T}} \times \mathbb{R}^n \times [0,1], \tag{3.8}$$

then the BVP (3.1), (3.2) has at least one solution.

Proof. From Lemma 3.2 we see that the BVP (3.1), (3.2) is equivalent to the integral equation (3.5).

Define the map $T_1 : C([a,\sigma(c)]_{\mathbb{T}};\mathbb{R}^n) \to C([a,\sigma(c)]_{\mathbb{T}};\mathbb{R}^n)$ by

$$T_1 x(t) = e_{-b}(t,a)\left[\frac{1}{1 - e_{-b}(\sigma(c),a)} \int_a^{\sigma(c)} \frac{g(s,x(s))}{e_{-b}(\sigma(s),a)}\Delta s + \int_a^t \frac{g(s,x(s))}{e_{-b}(\sigma(s),a)}\Delta s\right]$$

for $t \in [a,\sigma(c)]_{\mathbb{T}}$.

Thus, our problem is reduced to proving the existence of at least one x such that

$$x = T_1 x. \tag{3.9}$$

With this in mind, it is sufficient to show that

$$\lambda T_1 x \neq x, \quad x \in \partial B_{P_1}, \quad \lambda \in [0,1], \tag{3.10}$$

for some suitable ball $B_{P_1} \subset C([a,\sigma(c)]_{\mathbb{T}};\mathbb{R}^n)$ with radius $P_1 > 0$. Let

$$B_{P_1} = \left\{ x \in C([a,\sigma(c)]_{\mathbb{T}};\mathbb{R}^n) \mid \max_{t\in[a,\sigma(c)]_{\mathbb{T}}} \|x(t)\| < P_1 \right\},$$
$$P_1 = \sup_{t\in[a,\sigma(c)]_{\mathbb{T}}} \left[|e_{-b}(t,a)| \left(\frac{1}{|1-e_{-b}(\sigma(c),a)|} + 1 \right) \right] K + 1.$$

The rest of the proof follows similar lines to the proof of Theorem 2.2 and so is only briefly sketched. Let x be a solution to $\lambda T_1 x = x$. Consider $r(t) := \|x(t)\|^2$ for all $t \in [a,\sigma(c)]_{\mathbb{T}}$. By the product rule [2, Theorem 1.20 (iii)] we have

$$\begin{aligned} r^{\Delta}(t) &= 2\langle x(t), x^{\Delta}(t)\rangle + \mu(t)\|x^{\Delta}(t)\|^2, \quad t \in [a,c]_{\mathbb{T}}, \\ &= 2\langle x(t), \lambda g(t,x(t)) - b(t)x(t)\rangle + \mu(t)\|\lambda g(t,x(t)) - b(t)x(t)\|^2. \\ &\geq 2\langle x(t), \lambda g(t,x(t)) - b(t)x(t)\rangle. \end{aligned} \tag{3.11}$$

Let

$$H_1(t) := \left[|e_{-b}(t,a)| \left(\frac{1}{|1-e_{-b}(\sigma(c),a)|} + 1 \right) \right], \quad t \in [a,\sigma(c)]_{\mathbb{T}}.$$

Now, for each $t \in [a,\sigma(c)]_{\mathbb{T}}$, consider

$$\begin{aligned} &\|\lambda T x(t)\| \\ &\leq |e_{-b}(t,a)| \left[\frac{1}{|1-e_{-b}(\sigma(c),a)|} \int_a^{\sigma(c)} \frac{\|\lambda g(s,x(s))\|}{|e_{-b}(\sigma(s),a)|} \Delta s + \int_a^t \frac{\|\lambda g(s,x(s))\|}{|e_{-b}(\sigma(s),a)|} \Delta s \right] \\ &\leq H_1(t) \int_a^{\sigma(c)} \frac{\|\lambda g(s,x(s))\|}{|e_{-b}(\sigma(s),a)|} \Delta s \\ &\leq H_1(t) \int_a^{\sigma(c)} 2\alpha[\langle x(s), \lambda g(s,x(s)) - b(t)x(s)\rangle + K]\, \Delta s \\ &\leq H_1(t) \int_a^{\sigma(c)} \left[\alpha r^{\Delta}(s) + K\right]\, \Delta s \\ &= H_1(t) \left[\alpha(\|x(\sigma(c))\|^2 - \|x(a)\|^2) + K\right] \\ &= H_1(t) K \\ &< P_1 \end{aligned}$$

and so (3.10) holds. The result now follows from the Nonlinear Alternative. ■

The following two corollaries give less technical conditions implying (3.8) which are easy to verify in practice.

Corollary 3.4. Let (3.4) hold with $b < 0$ and let $g : [a,c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. If there exist non-negative constants α and K such that

$$\frac{\|g(t,q)\|}{|e_{-b}(\sigma(t),a)|} \leq 2\alpha\langle q, g(t,q)\rangle + K, \quad \forall (t,q) \in [a,c]_{\mathbb{T}} \times \mathbb{R}^n, \tag{3.12}$$

then the BVP (3.1), (3.2) has at least one solution.

Proof. It is not difficult to show that if (3.12) holds, then, since $b < 0$, (3.8) must also hold and the result follows from Theorem 3.3. ■

Corollary 3.5. Let (3.4) hold and let $g : [a, c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. If g is bounded on $[a, c]_{\mathbb{T}} \times \mathbb{R}^n$, then the BVP (3.1), (3.2) has at least one solution.

Proof. Since g is bounded on $[a, c]_{\mathbb{T}} \times \mathbb{R}^n$, there exists a non-negative constant L such that

$$\sup_{(t,q)\in[a,c]_{\mathbb{T}}\times\mathbb{R}^n} \|g(t, q)\| \leq L.$$

Thus, (3.8) will hold for the choices $\alpha = 0$ and

$$K = \sup_{t\in[a,c]_{\mathbb{T}}} |e_{-b}(\sigma(t), a)|L.$$

The result then follows from Theorem 3.3. ■

Now consider (3.1), (3.2) with $n = 1$. For this case, the following corollary to Theorem 3.3 is obtained.

Corollary 3.6. Let g be a continuous, scalar-valued function and let (3.4) hold. If there exist non-negative constants α and K such that

$$\frac{|\lambda g(t, q)|}{|e_{-b}(\sigma(t), a)|} \leq 2\alpha[q\lambda g(t, q) - b(t)q^2] + K, \quad \forall(t, q, \lambda) \in [a, c]_{\mathbb{T}} \times \mathbb{R} \times [0, 1], \tag{3.13}$$

then, for $n = 1$, the BVP (3.1), (3.2) has at least one solution.

Proof. It is easy to see that for $n = 1$: (3.8) becomes (3.13); and the result follows from Theorem 3.3. ■

Attention is now turned to (3.3), (3.2). As it stands, the BVP (3.3), (3.2) is not guaranteed to be invertible. That is, we may be unable to reformulate it as an equivalent delta integral equation. However, the BVP

$$x^{\Delta} - x = f(t, x) - x, \quad t \in [a, c]_{\mathbb{T}}, \tag{3.14}$$

$$x(a) = x(\sigma(c)), \tag{3.15}$$

is invertible since Lemma 3.2 holds for the special cases $b = -1$ and $g(t, p) = f(t, p) - p$. We will use this to now formulate some existence theorems for (3.3), (3.2).

Theorem 3.7. Let $f : [a, c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and let (3.4) hold for $b = -1$. If there exist non-negative constants α and K such that

$$\frac{\|f(t, q) - q\|}{|e_{-1}(\sigma(t), a)|} \leq 2\alpha\langle q, f(t, q)\rangle + K, \quad \forall(t, q) \in [a, c]_{\mathbb{T}} \times \mathbb{R}^n, \tag{3.16}$$

then the BVP (3.3), (3.2) has at least one solution.

Proof. Consider the BVP (3.3), (3.2) rewritten as

$$x^{\Delta} - x = f(t, x) - x, \quad t \in [a, c]_{\mathbb{T}}, \tag{3.17}$$

$$x(a) = x(\sigma(c)). \tag{3.18}$$

The BVP (3.17), (3.18) is in the form (3.1), (3.2) with $b = -1$ and $g(t, p) = f(t, p) - p$. It is not difficult to show that (3.8) reduces to (3.16) for these special cases. Hence the result follows from Theorem 3.3. ■

Corollary 3.8. Let $f : [a, c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and let (3.4) hold for $b = -1$. If there exist non-negative constants α and K such that

$$\frac{\|f(t, q) - q\|}{|e_{-1}(\sigma(t), a)|} \leq 2\alpha\langle q, f(t, q)\rangle + K, \quad \forall (t, q) \in [a, c]_{\mathbb{T}} \times \mathbb{R}^n, \tag{3.19}$$

then the BVP (3.3), (3.2) has at least one solution.

Proof. The result follows from Corollary 3.8 with $b = -1$, $g(t, q) = f(t, q) - q$. ■

Corollary 3.9. Let $f : [a, c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. If $f(t, q) - q$ is bounded for $(t, q) \in [a, c]_{\mathbb{T}} \times \mathbb{R}^n$, then the BVP (3.3), (3.2) has at least one solution.

Proof. This is a special case of Corollary 3.9 with $b = -1$, $g(t, q) = f(t, q) - q$. ■

4. More on Existence for the Periodic Case

This section considers the existence of solutions to the first-order system of periodic BVPs

$$x^{\Delta} + b(t)x^{\sigma} = g(t, x^{\sigma}), \quad t \in [a, c]_{\mathbb{T}}, \tag{4.1}$$

$$x(a) = x(\sigma(c)), \tag{4.2}$$

where $g : [a, c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ and $b : [a, c]_{\mathbb{T}} \to \mathbb{R}$ are both continuous functions, with b having no zeros on $[a, c]_{\mathbb{T}}$ and $x^{\sigma}(t) := x(\sigma(t))$.

The reader may wonder why x^{σ} appears in (4.1). There are two reasons: firstly, the theory developed in this section is, in general, inapplicable to (3.1); and secondly, by studying (4.1), advancement in the theory of

$$x' + b(t)x = g(t, x), \quad t \in [a, c], \tag{4.3}$$

$$x(a) = x(c). \tag{4.4}$$

is still attainable (by taking $\mathbb{T} = \mathbb{R}$).

Throughout this section assume

$$b \in \mathcal{R}, \quad \text{and} \quad e_b(\sigma(c), a) \neq 1. \tag{4.5}$$

Lemma 4.1. Let (4.5) hold. The BVP (4.1), (4.2) is equivalent to the integral equation

$$x(t) = \frac{1}{e_b(t,a)} \left[\frac{1}{1 - e_b(\sigma(c), a)} \int_a^{\sigma(c)} g(s, x^\sigma(s)) e_b(s,a) \Delta s + \int_a^t g(s, x^\sigma(s)) e_b(s,a) \Delta s \right] \tag{4.6}$$

for $t \in [a, \sigma(c)]_{\mathbb{T}}$.

Proof. Let x be a solution to (3.1), (3.2). By the product rule, consider

$$[x(t)e_b(t,a)]^\Delta = [x^\Delta(t) + b(t)x^\sigma(t)]e_b(t,a) = g(t, x(t))e_b(t,a), \quad t \in [a,c]_{\mathbb{T}}.$$

Integrating the above on $[a,t]_{\mathbb{T}}$ and then using the boundary conditions, the result follows in a similar way to the proof of Lemma 3.2.

If x is a solution to (4.6), then x satisfies (4.1), (4.2), which may be verified by direct computation. ■

The following is the main result of this section.

Theorem 4.2. Let (4.5) hold and $g : [a,c]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. If there exist non-negative constants α and K such that

$$\|\lambda g(t,q) e_b(t,a)\| \le -2\alpha \langle q, \lambda g(t,q) - b(t)q \rangle + K, \quad \forall (t,q,\lambda) \in [a,c]_{\mathbb{T}} \times \mathbb{R}^n \times [0,1], \tag{4.7}$$

then the BVP (4.1), (4.2) has at least one solution.

Proof. From Lemma 4.1 we see that the BVP (4.1), (4.2) is equivalent to the integral equation (4.6).

Define the map $T_2 : C([a,\sigma(c)]_{\mathbb{T}}; \mathbb{R}^n) \to C([a,\sigma(c)]_{\mathbb{T}}; \mathbb{R}^n)$ by

$$T_2 x(t) = \frac{1}{e_b(t,a)} \left[\frac{1}{1 - e_b(\sigma(c), a)} \int_a^{\sigma(c)} g(s, x^\sigma(s)) e_b(s,a) \Delta s + \int_a^t g(s, x^\sigma(s)) e_b(s,a) \Delta s \right]$$

for $t \in [a, \sigma(c)]_{\mathbb{T}}$.

Thus, our problem is reduced to proving the existence of at least one x such that

$$x = T_2 x. \tag{4.8}$$

With this in mind, it is sufficient to show that

$$\lambda T_2 x \ne x, \quad x \in \partial B_{P_2}, \quad \lambda \in [0,1], \tag{4.9}$$

for some suitable ball $B_{P_2} \subset C([a,\sigma(c)]_{\mathbb{T}};\mathbb{R}^n)$ with radius $P_2 > 0$. Let

$$B_{P_2} = \left\{ x \in C([a,\sigma(c)]_{\mathbb{T}};\mathbb{R}^n) \mid \max_{t\in[a,\sigma(c)]_{\mathbb{T}}} \|x(t)\| < P_2 \right\},$$
$$P_2 = \sup_{t\in[a,\sigma(c)]_{\mathbb{T}}} \left[|e_b(t,a)| \left(\frac{1}{|1-e_b(\sigma(c),a)|} + 1 \right) \right] K + 1.$$

The rest of the proof follows similar lines to the proof of Theorem 3.3 and so is only briefly sketched. Let x be a solution to $\lambda T_2 x = x$. Consider $r(t) := \|x(t)\|^2$ for all $t \in [a,\sigma(c)]_{\mathbb{T}}$. By the product rule [2, Theorem 1.20 (iii)] and Theorem 1.6 (vi) we have, for $\alpha \geq 0$,

$$\begin{aligned} -\alpha r^{\Delta}(t) &= -\alpha \left[2\langle x^{\sigma}(t), x^{\Delta}(t)\rangle - \mu(t)\|x^{\Delta}(t)\|^2 \right], \quad t \in [a,c]_{\mathbb{T}}, \\ &\geq -2\alpha\langle x(t), \lambda g(t,x^{\sigma}(t)) - b(t)x^{\sigma}(t)\rangle. \end{aligned} \tag{4.10}$$

Let

$$H_2(t) := \left[|e_b(t,a)| \left(\frac{1}{|1-e_b(\sigma(c),a)|} + 1 \right) \right], \quad t \in [a,\sigma(c)]_{\mathbb{T}}.$$

Now, for each $t \in [a,\sigma(c)]_{\mathbb{T}}$, consider

$$\begin{aligned} &\|\lambda T x(t)\| \\ &\leq |e_b(t,a)| [\frac{1}{|1-e_b(\sigma(c),a)|} \int_a^{\sigma(c)} \|\lambda g(s,x(s))\| |e_b(s,a)| \Delta s \\ &\quad + \int_a^t \|\lambda g(s,x(s))\| |e_b(s,a)| \Delta s] \\ &\leq H_2(t) \int_a^{\sigma(c)} [-2\alpha\langle x(s), \lambda g(s,x(s)) - b(t)x(s)\rangle + K] \,\Delta s \\ &\leq H_2(t) \int_a^{\sigma(c)} \left[-\alpha r^{\Delta}(s) + K \right] \,\Delta s \\ &= H_2(t) \left[-\alpha(\|x(\sigma(c))\|^2 - \|x(a)\|^2) + K \right] \\ &= H_2(t) K \\ &< P_2 \end{aligned}$$

and so (4.9) holds. The result now follows from the Nonlinear Alternative. ■

As a special case of Theorem 4.2, the following result appears to be new.

Corollary 4.3. Let $e^{\int_a^c b(t)dt} \neq 1$ and $g : [a,c] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. If there exist non-negative constants α and K such that

$$\begin{aligned} \|\lambda g(t,q) e^{\int_a^t b(s)ds}\| &\leq -2\alpha\langle q, \lambda g(t,q) - b(t)q\rangle + K, \\ &\forall (t,q,\lambda) \in [a,c] \times \mathbb{R}^n \times [0,1], \end{aligned} \tag{4.11}$$

then the BVP (4.3), (4.4) has at least one solution.

Proof. Take $\mathbb{T} = \mathbb{R}$ in Theorem 4.2. ∎

5. An Example

An example is now provided to highlight some of the theory from previous sections.

Example 5.1. Consider the scalar dynamic BVP ($n = 1$) given by

$$x^{\Delta} = tx^3 + 1, \quad t \in [1, c]_{\mathbb{T}}, \tag{5.1}$$

$$x(1) = -x(\sigma(c)), \quad 1, c \in \mathbb{T}. \tag{5.2}$$

The claim is that (5.1), (5.2) has at least one solution for arbitrary $\mathbb{T}$.

Proof. Let $f(t, q) = tq^3 + 1$. It needs to be shown that (2.20) holds for non-negative constants α and K.

First note that $|f(t, q)| \leq c|q^3| + 1$ for $(t, q) \in [1, c]_{\mathbb{T}} \times \mathbb{R}$. Also note that for $(t, q) \in [1, c]_{\mathbb{T}} \times \mathbb{R}$

$$c(q^4 + q + 10) \geq c(|q^3| + 1) \geq c|q^3| + 1 \geq |f(t, q)|.$$

Hence for α and K to be chosen below

$$\begin{aligned} 2\alpha q f(t, q) + K &= 2\alpha q(tq^3 + 1) + K \\ &\geq 2\alpha(q^4 + q) + K \\ &= c(q^4 + q) + 10c, \quad \text{for the choice } \alpha = c/2, K = 10c \\ &\geq |g(t, q)|, \quad \text{for all } (t, q) \in [1, c]_{\mathbb{T}} \times \mathbb{R} \end{aligned}$$

and thus (2.20) holds for the choices $\alpha = c/2$ and $K = 10c$. Thus, all of the conditions of Corollary 2.4 hold and the BVP (5.1), (5.2) has at least one solution. ∎

To give the reader a flavour of the above result, consider (5.1), (5.2) on the time scale

$$\mathbb{T} := \{2 - 1/i\}_{i=1}^{\infty} \cup [2, \infty) = \left\{1, \frac{3}{2}, \frac{5}{3}, \ldots\right\} \cup [2, \infty).$$

It is not difficult, using Definition 1.2, to show that for the above time scale,

$$\sigma(t) = \begin{cases} \dfrac{4 - t}{3 - t}, & \text{for } t \in [1, 2)_{\mathbb{T}}, \\ t, & \text{for } t \in [2, \infty)_{\mathbb{T}}. \end{cases}$$

For this time scale, let $a = 1$ and $c = 3$. So our intervals of interest for (5.1), (5.2) become $[a, c]_{\mathbb{T}} = [1, 3]_{\mathbb{T}} = [a, \sigma(c)]_{\mathbb{T}}$. Note that by using Theorem 1.6 we obtain

$$x^{\Delta}(t) = \begin{cases} \dfrac{x(\sigma(t)) - x(t)}{\sigma(t) - t}, & \text{for } t \in [1, 2)_{\mathbb{T}}, \\ \\ x'(t), & \text{for } t \in [2, \infty)_{\mathbb{T}}. \end{cases}$$

For this special time scale, (5.1) and (5.2) have at least one solution.

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