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# Hyers–Ulam Stability for a Continuous Time Scale with Discrete Uniform Jumps

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Dedicated to Johnny Henderson on the occasion of his 70th birthday.

**Abstract**

We investigate the Hyers–Ulam stability (HUS) of a certain first-order linear complex constant coefficient dynamic equation on the time scale $\mathbb{P}_{\alpha,h}$, which has continuous intervals of length $\alpha > 0$ followed by discrete jumps of length $h > 0$. In particular, we establish results in the case of this specific time scale, for coefficient values in the complex plane, including where the exponential function alternates in sign. In our analysis, we employ the Lambert $W$ function. For increasing jump size $h$ relative to $\alpha$, we prove that the complex constant coefficient undergoes a bifurcation in its parameter space. We establish interesting results for both the delta dynamic equation and the nabla dynamic equation.

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# 1 Content Prelude

Imagine a data burst or transmission signal broadcast over a short time period, and then repeated, or a device or system that runs continuously for a fixed time, shuts off, and then runs again. In biology, think of an organism that lives a fixed unit of time, followed by hibernation or dormancy, and then is active again, and so on. These scenarios may be modeled by a specific time scale $\mathbb{T}$ with fixed jump size that displays both continuous

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and discrete properties, see Bohner and Peterson [8, Examples 1.38–1.40], where a time scale is any closed subset of the real line $\mathbb{R}$. In particular, let $\mathbb{T} = \mathbb{P}_{\alpha,h}$ for continuous interval length $\alpha > 0$ and discrete jump size $h > 0$, namely

$$\mathbb{P}_{\alpha,h} = \bigcup_{k=0}^{\infty} [k(\alpha+h), k(\alpha+h)+\alpha],$$

and consider the differential operator defined by

$$x^{\Delta}(t) = \begin{cases} \dfrac{d}{dt}x(t) & \text{for } t \in [k(\alpha+h), k(\alpha+h)+\alpha) \\ \dfrac{x(t+h)-x(t)}{h} & \text{for } t = k(\alpha+h)+\alpha. \end{cases}$$

We will be investigating some stability questions for this time scale and this derivative operator, in the Hyers-Ulam sense, defined below. Note that this problem is explored briefly but incompletely in [2, Example 4.1]. Our aim is to give a more robust analysis of the situation in this work.

Time scales were introduced by Hilger [12] to unify continuous and discrete analysis. Hyers–Ulam stability was initiated by Ulam [27], followed by Hyers [13] and Rassias [25]. Some early work on differential equations and this type of stability include Miura et al. [18, 19] and Jung et al. [14–17]. Some recent papers on HUS and difference equations or more generally time scales include Anderson [1], Anderson and Onitsuka [2–6], Andras et al. [7], Brzdek et al. [9], Buse et al. [10], Nam [20–22], Onitsuka [23, 24], and Shen [26].

**Definition 1.1** (Hyers-Ulam Stability)**.** We say that

$$x^{\Delta}(t) = \lambda x(t), \quad \lambda \in \mathbb{C} \backslash \left\{ -\frac{1}{h} \right\}, \quad t \in \mathbb{T} \tag{1.1}$$

has Hyers–Ulam stability on $\mathbb{T}$ if and only if there exists a constant $K > 0$ with the following property. For arbitrary $\varepsilon > 0$, if a function $\phi : \mathbb{T} \to \mathbb{C}$ satisfies

$$|\phi^{\Delta}(t) - \lambda\phi(t)| \le \varepsilon, \quad t \in \mathbb{T}, \tag{1.2}$$

then there exists a solution $x : \mathbb{T} \to \mathbb{C}$ of (1.1) such that $|\phi(t) - x(t)| \le K\varepsilon$ for all $t \in \mathbb{T}$. Such a constant $K$ is called an HUS constant for (1.1) on $\mathbb{T}$.

In this work, we consider the time scale $\mathbb{T} = \mathbb{P}_{\alpha,h}$, and the time scale eigenvalue problem given in (1.1). For $t \in \mathbb{T}$, we have the forward jump operator $\sigma$ defined by

$$\sigma(t) := \begin{cases} t & \text{for } t \in [k(\alpha+h),\ k(\alpha+h)+\alpha), \\ t+h & \text{for } t = k(\alpha+h)+\alpha. \end{cases}$$

For $\lambda \in \mathbb{C} \backslash \left\{-\frac{1}{h}\right\}$, the exponential function $e_\lambda(t,0)$ is given by

$$e_\lambda(t,0) = (1+h\lambda)^k e^{\lambda(t-hk)}, \quad t \in \big[k(\alpha+h), k(\alpha+h)+\alpha\big], \quad k \in \mathbb{N}_0, \tag{1.3}$$

which can also be written as

$$e_\lambda(t,0) = \left[(1+h\lambda)e^{\alpha\lambda}\right]^k e^{\lambda j}, \quad t = k(\alpha+h)+j, \quad j \in [0,\alpha].$$

Clearly, the exponential function in (1.3) is well defined for $\lambda \in \mathbb{C} \backslash \left\{-\frac{1}{h}\right\}$. Notice that

$$x(t) = x_0 e_\lambda(t,0), \quad t \in \mathbb{T}, \tag{1.4}$$

is the general solution of (1.1), for the exponential function $e_\lambda$ given in (1.3). Moreover, for given $\varepsilon > 0$, the function

$$\phi(t) = \phi_0 e_\lambda(t,0) + e_\lambda(t,0)\int_0^t \frac{q(s)}{e_\lambda(\sigma(s),0)}\Delta s, \quad |q(s)| \le \varepsilon \ \forall s \in \mathbb{T}, \tag{1.5}$$

where

$$\int_{k(\alpha+h)}^t \frac{q(s)}{e_\lambda(\sigma(s),0)}\Delta s = \int_{k(\alpha+h)}^t \frac{q(s)}{e_\lambda(s,0)}ds, \qquad t \in [k(\alpha+h), k(\alpha+h)+\alpha]$$

and

$$\int_{k(\alpha+h)-h}^{k(\alpha+h)} \frac{q(s)}{e_\lambda(\sigma(s),0)}\Delta s = \frac{hq(k(\alpha+h)-h)}{e_\lambda(k(\alpha+h),0)},$$

is the solution of (1.2) by the variation of parameters formula, see Bohner and Peterson [8, Theorem 2.77].

Throughout the paper, we will need to employ the Lambert $W$ function, see Corless et al. [11], which we denote by $W_z$, where $W_z$ satisfies $W_z(y)e^{W_z(y)} = y$, for every $z \in \mathbb{Z}$. For example, using (1.3) and $t = k(\alpha+h)$, we have

$$e_\lambda(k(\alpha+h),0) = [(1+h\lambda)e^{\alpha\lambda}]^k.$$

To prevent this from vanishing, we always assume $\lambda \neq -\frac{1}{h}$. Moreover, we will see that other key values for $\lambda \in \mathbb{R}$ include when the base $(1+h\lambda)e^{\alpha\lambda} = \pm 1$. If $\lambda = 0$, then $(1+h\lambda)e^{\alpha\lambda} = 1$, but, for $\lambda \in \mathbb{R}$, we note here that for the branches $z = -1, 0$ of the Lambert $W = W_z$ function,

$$(1+h\lambda)e^{\alpha\lambda} = -1 \iff \lambda = \frac{-1}{h} + \frac{1}{\alpha}W_0\left(-\frac{\alpha}{h}e^{\frac{\alpha}{h}}\right) \text{ and } h \ge \frac{\alpha}{W_0(e^{-1})} \approx 3.59112\alpha,$$

where $h > 0$ is the jump size, and $W_0$ is the principal branch of the Lambert $W$ function. In particular, if $h = \frac{\alpha}{W_0(e^{-1})}$ and $\lambda = -\frac{1}{\alpha} - \frac{1}{h} \approx \frac{-1.27846}{\alpha}$, then $e_\lambda(k(\alpha+h),0) = (-1)^k$.

## 2 Hyers–Ulam Stability on $\mathbb{P}_{\alpha,h}$

We now give our first new results, when the eigenvalue $\lambda$ is a real number; in a later section, we will consider the more general case of $\lambda \in \mathbb{C}$. Moreover, we will fix $\alpha > 0$ and let the jump size $h > 0$ range over all positive real numbers in relation to $\alpha$. Of course, one could also fix $h > 0$ and let $\alpha > 0$ vary, as well. For the sake of completeness, we will include the details of proofs for this specific time scale $\mathbb{T} = \mathbb{P}_{\alpha,h}$. We will refer to the following constant,

$$K_{\mathbb{R}} = \frac{1}{-\lambda}\left(\frac{e^{\alpha\lambda}(1+h\lambda)-(1+2h\lambda)}{1+e^{\alpha\lambda}(1+h\lambda)}\right), \tag{2.1}$$

throughout the remainder of this section.

**Theorem 2.1** (Delta equation)**.** *Fix $\alpha > 0$, and let $\lambda \in \mathbb{R}\backslash\left\{-\frac{1}{h}\right\}$. Also, let $K_{\mathbb{R}}$ be given as in* (2.1)*. We have the following cases.*

*(i) Suppose $0 < h < \frac{\alpha}{W_0(e^{-1})}$.*

- *(a) If $\lambda \in \left(-\frac{1}{h}, 0\right) \cup (0,\infty)$, then* (1.1) *is Hyers–Ulam stable, with best HUS constant $K = \frac{1}{|\lambda|}$.*
- *(b) If $\lambda = 0$, then* (1.1) *is not Hyers–Ulam stable.*
- *(c) If $\lambda \in \left(-\infty,\ -\frac{1}{h}\right)$, then* (1.1) *is HUS, with best HUS constant $K = K_{\mathbb{R}}$.*

*(ii) Suppose $h = \frac{\alpha}{W_0(e^{-1})}$. Then, $(1+h\lambda)e^{\alpha\lambda} = -1$ at $\lambda = -\frac{1}{\alpha}\left(1+W_0(e^{-1})\right)$, and we have the following subcases.*

- *(a) If $\lambda \in \left(\frac{W_0(e^{-1})}{-\alpha}, 0\right) \cup (0,\infty)$, then*(1.1) *is HUS, with best HUS constant $K = \frac{1}{|\lambda|}$.*
- *(b) If $\lambda = 0$ or $\lambda = -\frac{1}{\alpha}\left(1+W_0(e^{-1})\right)$, then* (1.1) *is not HUS.*
- *(c) If $\lambda \in \left(-\infty,\ -\frac{1}{\alpha}\left(1+W_0(e^{-1})\right)\right) \cup \left(-\frac{1}{\alpha}\left(1+W_0(e^{-1})\right),\ \frac{W_0(e^{-1})}{-\alpha}\right)$, then* (1.1) *is HUS, with best HUS constant $K = K_{\mathbb{R}}$ as in* (2.1).

*(iii) Suppose* $h > \dfrac{\alpha}{W_0(e^{-1})}$. *Then,* $(1+h\lambda)e^{\alpha\lambda} = -1$ *at*

$$\lambda_{-1} := -\frac{1}{h} + \frac{1}{\alpha}W_{-1}\left(-\frac{\alpha}{h}e^{\frac{\alpha}{h}}\right) \quad \textit{and} \quad \lambda_0 := -\frac{1}{h} + \frac{1}{\alpha}W_0\left(-\frac{\alpha}{h}e^{\frac{\alpha}{h}}\right),$$

*and we have the following subcases.*

*(a) If* $\lambda \in \left(-\dfrac{1}{h}, 0\right) \cup (0,\infty)$, *then* (1.1) *is HUS, with best HUS constant* $K = \dfrac{1}{|\lambda|}$.

*(b) If* $\lambda = \lambda_{-1}$, $\lambda = \lambda_0$, *or* $\lambda = 0$, *then* (1.1) *is not HUS.*

*(c) If* $\lambda \in (-\infty,\ \lambda_{-1}) \cup (\lambda_{-1}, \lambda_0) \cup \left(\lambda_0,\ -\dfrac{1}{h}\right)$, *then* (1.1) *is HUS, with best HUS constant* $K = |K_{\mathbb{R}}|$ *as in* (2.1).

*Proof.* Cases (i)(a), (ii)(a), and (iii)(a) all follow from [2, Corollary 3.8], while cases (i)(b), (ii)(b), and (iii)(b) all follow from [2, Theorem 3.10(ii)].

Case (i)(c). Suppose $\lambda < -\dfrac{1}{h}$. Since $0 < h < \dfrac{\alpha}{W_0(e^{-1})}$, the base of the exponential function (1.3) satisfies $(1+h\lambda)e^{\alpha\lambda} \in (-1, 0)$. Consequently, as $e_\lambda(t,0) = (1+h\lambda)^k e^{\lambda(t-hk)}$, we have an exponential function that changes sign; in particular, $e_\lambda(t,0) < 0$ for all $t \in [k(\alpha+h), k(\alpha+h)+\alpha]$ when $k$ is odd. Let $\phi$ satisfy the perturbed equation (1.2), and note that

$$x(t) = \phi_0 e_\lambda(t,0)$$

is a well-defined solution of (1.1). Then, for $t \in [k(\alpha+h), k(\alpha+h)+\alpha]$, we have $t = k(\alpha+h)+j$ for some $j \in [0,\alpha]$, and

$$\begin{aligned}
|\phi(t) - x(t)| &= \left|\phi_0 e_\lambda(t,0) + e_\lambda(t,0)\int_0^t \frac{q(s)}{e_\lambda(\sigma(s),0)}\Delta s - \phi_0 e_\lambda(t,0)\right| \\
&\le \varepsilon |e_\lambda(t,0)| \int_0^t \frac{1}{|e_\lambda(\sigma(s),0)|}\Delta s \\
&= \varepsilon |e_\lambda(t,0)| \left(\int_0^\alpha + \int_\alpha^{\alpha+h} + \int_{\alpha+h}^{2\alpha+h} + \cdots \right.\\
&\qquad \left. + \int_{k(\alpha+h)-h}^{k(\alpha+h)} + \int_{k(\alpha+h)}^{t}\right) \frac{1}{|e_\lambda(\sigma(s),0)|}\Delta s \\
&= \varepsilon |e_\lambda(t,0)| \left[\left(\int_0^\alpha + \int_{\alpha+h}^{2\alpha+h} + \cdots + \int_{(k-1)(\alpha+h)}^{k(\alpha+h)-h}\right)\frac{1}{|e_\lambda(s,0)|}ds \right.\\
&\qquad + \left(\int_\alpha^{\alpha+h} + \int_{2\alpha+h}^{2(\alpha+h)} + \cdots + \int_{k(\alpha+h)-h}^{k(\alpha+h)}\right)\frac{\Delta s}{|e_\lambda(\sigma(s),0)|}
\end{aligned}$$

$$
\begin{aligned}
& \qquad + \left.\int_{k(\alpha+h)}^{k(\alpha+h)+j} \frac{ds}{|e_\lambda(s,0)|}\right] \\
= & \quad \varepsilon|1+h\lambda|^k e^{\lambda(k\alpha+j)} \left( \sum_{m=0}^{k-1} \frac{e^{\alpha\lambda}-1}{\lambda|1+h\lambda|^m e^{\alpha\lambda(m+1)}} \right. \\
& \quad \left. + \sum_{m=1}^{k} \frac{h}{|1+h\lambda|^m e^{\alpha\lambda m}} - \frac{1}{\lambda|1+h\lambda|^k} \left( \frac{1}{e^{\lambda(k\alpha+j)}} - \frac{1}{e^{\lambda k\alpha}} \right) \right) \\
\leq & \quad \varepsilon \left( \frac{1+e^{\alpha\lambda}(1+h\lambda)-2e^{j\lambda}(1+h\lambda)}{-\lambda\left(1+e^{\alpha\lambda}(1+h\lambda)\right)} \right) \\
\leq & \quad \frac{\varepsilon}{-\lambda} \left( \frac{e^{\alpha\lambda}(1+h\lambda)-(1+2h\lambda)}{1+e^{\alpha\lambda}(1+h\lambda)} \right),
\end{aligned}
$$

where the penultimate line is the result of taking $k \to \infty$, and the last line follows by letting $j = 0$. This shows that (1.1) has HUS with HUS constant at most

$$
K_{\mathbb{R}} = \frac{1}{-\lambda} \left( \frac{e^{\alpha\lambda}(1+h\lambda)-(1+2h\lambda)}{1+e^{\alpha\lambda}(1+h\lambda)} \right),
$$

whenever $\lambda < -\dfrac{1}{h}$ and we assume $0 < h < \dfrac{\alpha}{W_0(e^{-1})}$. On the other hand, given any $\varepsilon > 0$, let

$$
q(s) := \frac{\varepsilon\; e_\lambda(\sigma(s),0)}{|e_\lambda(\sigma(s),0)|}, \quad s \in \mathbb{T}.
$$

Clearly $|q(s)| = \varepsilon$ for all $s \in \mathbb{T}$. Using this $q$ in a function $\phi$ of the form (1.5), we have that

$$
\phi(t) = \phi_0 e_\lambda(t,0) + \varepsilon\; e_\lambda(t,0) \int_0^t \frac{\Delta s}{|e_\lambda(\sigma(s),0)|},
$$

and $\phi$ satisfies (1.2). Let $x$ the solution of (1.1) with $x_0 = \phi_0$. Let $t = k(\alpha + h)$ $(j = 0)$ for arbitrarily large $k \in \mathbb{N}_0$. Then, similar to the calculations done above,

$$
\begin{aligned}
|\phi(t)-x(t)| \quad = & \quad \varepsilon|e_\lambda(t,0)| \int_0^t \frac{1}{|e_\lambda(\sigma(s),0)|}\Delta s \\
= & \quad \varepsilon|1+h\lambda|^k e^{\lambda(k\alpha)} \left( \sum_{m=0}^{k-1} \frac{e^{\alpha\lambda}-1}{\lambda|1+h\lambda|^m e^{\alpha\lambda(m+1)}} \right. \\
& \quad \left. + \sum_{m=1}^{k} \frac{h}{|1+h\lambda|^m e^{\alpha\lambda m}} \right) \\
= & \quad \varepsilon \frac{(h\lambda+(-1+e^{\alpha\lambda})|1+h\lambda|)(-1+e^{k\alpha\lambda}|1+h\lambda|^k)}{\lambda(-1+e^{\alpha\lambda}|1+h\lambda|)},
\end{aligned}
$$

so that, again with $t = k(\alpha + h)$, $k \in \mathbb{N}_0$ large, and $\lambda < -\dfrac{1}{h}$ with $0 < h < \dfrac{\alpha}{W_0(e^{-1})}$, we have

$$\begin{aligned}\lim_{t\to\infty} |\phi(t) - x(t)| &= \lim_{k\to\infty} \varepsilon \frac{(h\lambda + (-1 + e^{\alpha\lambda})|1 + h\lambda|)(-1 + e^{k\alpha\lambda}|1 + h\lambda|^k)}{\lambda(-1 + e^{\alpha\lambda}|1 + h\lambda|)} \\ &= \frac{\varepsilon}{-\lambda}\left(\frac{e^{\alpha\lambda}(1 + h\lambda) - (1 + 2h\lambda)}{1 + e^{\alpha\lambda}(1 + h\lambda)}\right),\end{aligned}$$

the same constant as above in (2.1). This proves $K_{\mathbb{R}}$ in (2.1) is the best possible constant.

Case (ii)(c). Let $\lambda \in \left(-\dfrac{1}{\alpha}\left(1 + W_0(e^{-1})\right), \dfrac{W_0(e^{-1})}{-\alpha}\right)$. As $h = \dfrac{\alpha}{W_0(e^{-1})}$, we have $(1 + h\lambda)e^{\alpha\lambda} \in (-1, 0)$; to see this, set $f(\lambda) := (1 + h\lambda)e^{\alpha\lambda} = \left(1 + \dfrac{\alpha\lambda}{W_0(e^{-1})}\right)e^{\alpha\lambda}$. Then,

$$\begin{aligned}f\left(-\frac{1}{\alpha}\left(1 + W_0(e^{-1})\right)\right) &= \left(1 + \frac{-1 - W_0(e^{-1})}{W_0(e^{-1})}\right)e^{-1-W_0(e^{-1})} \\ &= \frac{-1}{eW_0(e^{-1})e^{W_0(e^{-1})}} = \frac{-1}{e(e^{-1})} = -1\end{aligned}$$

by the property of the Lambert $W$ function; also, $f\left(\dfrac{W_0(e^{-1})}{-\alpha}\right) = 0$. This implies, if $t = k(\alpha + h) + j$ for $j \in [0, \alpha]$, then

$$|e_\lambda(t, 0)| = \left|\left[(1 + h\lambda)e^{\alpha\lambda}\right]^k e^{j\lambda}\right| \le \left|(1 + h\lambda)e^{\alpha\lambda}\right|^k \le 1$$

for any $k \in \mathbb{N}_0$, and thus for all $t \in \mathbb{T}$, and

$$\lim_{t\to\infty} |e_\lambda(t, 0)| = 0.$$

The proof of the rest of this case is similar to the proof above of case (i)(c), leading to $K_{\mathbb{R}}$ as in (2.1) using the fact that $\alpha e^{1+W_0(e^{-1})} = \dfrac{\alpha}{W_0(e^{-1})} = h$ by the Lambert $W$ function properties. Clearly this is the same $K$ value as found earlier, in Theorem 2.1 (iii). For $\lambda \in \left(-\infty, -\dfrac{1}{\alpha}\left(1 + W_0(e^{-1})\right)\right)$, we also have $(1+h\lambda)e^{\alpha\lambda} \in (-1, 0)$. Thus, case (ii)(c) holds.

Case (iii)(c). Let the exponential function be given by (1.3). For $\lambda \in (\lambda_{-1}, \lambda_0)$, the base of the exponential function satisfies $(1 + h\lambda)e^{\alpha\lambda} < -1$. If $\phi$ satisfies (1.2), then $\phi$ has the form given in (1.5), and

$$\int_0^\infty \frac{q(s)}{e_\lambda(\sigma(s), 0)}\Delta s$$

exists and is finite. Note that

$$x(t) = x_0 e_\lambda(t,0) \quad \text{for} \quad x_0 = \phi_0 + \int_0^\infty \frac{q(s)}{e_\lambda(\sigma(s),0)} \Delta s$$

is a well-defined solution of (1.1), and

$$\begin{aligned}
|\phi(t) - x(t)| &= |e_\lambda(t,0)| \left| - \int_t^\infty \frac{q(s)}{e_\lambda(\sigma(s),0)} \Delta s \right| \\
&\leq \varepsilon |e_\lambda(t,0)| \int_t^\infty \frac{1}{|e_\lambda(\sigma(s),0)|} \Delta s \\
&= \varepsilon |1+h\lambda|^k e^{\lambda(k\alpha+j)} \left( \sum_{m=k+1}^\infty \frac{e^{\alpha\lambda}-1}{\lambda|1+h\lambda|^m e^{\alpha\lambda(m+1)}} \right. \\
&\qquad \left. + \sum_{m=k+1}^\infty \frac{h}{|1+h\lambda|^m e^{\alpha\lambda m}} - \frac{1}{\lambda|1+h\lambda|^k} \left( \frac{1}{e^{\lambda(k\alpha+\alpha)}} - \frac{1}{e^{\lambda(k\alpha+j)}} \right) \right) \\
&= \frac{\varepsilon \left( -1 + e^{j\lambda}(1+h\lambda) + (e^{\alpha\lambda} - e^{j\lambda})|1+h\lambda| \right)}{\lambda \left( -1 + e^{\alpha\lambda}|1+h\lambda| \right)} \\
&\leq \frac{\varepsilon \left( (1+2h\lambda) - e^{\alpha\lambda}(1+h\lambda) \right)}{-\lambda \left( 1 + e^{\alpha\lambda}(1+h\lambda) \right)} \\
&= |K_{\mathbb{R}}| \, \varepsilon
\end{aligned}$$

for $K = K_{\mathbb{R}}$ in (2.1), having taken $j = 0$ to get the penultimate line. □

**Example 2.2.** Let $\lambda = -\dfrac{2}{h}$ for $0 < h < \dfrac{\alpha}{W_0(e^{-1})}$. By Theorem 2.1(i)(c), (1.1) is Hyers–Ulam stable, with minimal HUS constant

$$K = \frac{h(3e^{\frac{2\alpha}{h}} - 1)}{2(e^{\frac{2\alpha}{h}} - 1)} = \frac{h}{2}\left(2 + \coth\left(\frac{\alpha}{h}\right)\right),$$

where we have used $K = K_{\mathbb{R}}$ in (2.1).

*Remark* 2.3. Fix the jump size $h > \dfrac{\alpha}{W_0(e^{-1})}$, as above in Theorem 2.1(iii), and let $\alpha$ tend to 0. Then,

$$\lim_{\alpha \to 0} \mathbb{P}_{\alpha,h} = h\mathbb{Z}.$$

Note that, for $\alpha = 0$, $(1+h\lambda) = -1$ at $\lim\limits_{\alpha\to 0} \lambda_0 := \lim\limits_{\alpha \to 0} -\dfrac{1}{h} + \dfrac{1}{\alpha} W_0\left(-\dfrac{\alpha}{h} e^{\frac{\alpha}{h}}\right) = -\dfrac{2}{h}$, $\lim\limits_{\alpha\to 0} \lambda_{-1} = -\infty$, and we have the following subcases from Theorem 2.1(iii), with $\alpha = 0$.

(a) If $\lambda \in \left(-\dfrac{1}{h}, 0\right) \cup (0,\infty)$, then (1.1) is HUS, with best HUS constant $K = \dfrac{1}{|\lambda|}$.

(b) If $\lambda = -\dfrac{2}{h}$, and $\lambda = 0$, then (1.1) is not HUS.

(c) If $\lambda \in \left(-\infty, -\dfrac{2}{h}\right) \cup \left(-\dfrac{2}{h},\ -\dfrac{1}{h}\right)$, then (1.1) is HUS, with best HUS constant $K = \dfrac{1}{\left|\lambda + \frac{2}{h}\right|}$.

Case (iii)(c) recovers the $K$ value found in the $h\mathbb{Z}$ case in Onitsuka [23, Remark 4.6]; see also [4, Theorem 2.6].

*Remark* 2.4. If one does the analogous analysis on $\mathbb{P}_{\alpha,h}$ using the nabla backward difference operator instead of the Delta forward difference operator, then similarly interesting results are obtained. In the nabla case, the nabla differential operator is defined by

$$x^{\nabla}(t) = \begin{cases} \dfrac{d}{dt}x(t) & : t \in (k(\alpha+h), k(\alpha+h)+\alpha] \\ \dfrac{x(t)-x(t-h)}{h} & : t = k(\alpha+h), \end{cases}$$

and the nabla exponential function is given via

$$\widehat{e}_{\lambda}(t,0) = \frac{e^{\lambda t}}{[(1-h\lambda)e^{h\lambda}]^{k}}, \quad \lambda \in \mathbb{C} \backslash \left\{\frac{1}{h}\right\}. \tag{2.2}$$

If we take

$$\widehat{K}_{\mathbb{R}} = \frac{h\lambda - 1 + e^{\alpha\lambda}(1-2h\lambda)}{\lambda\left(h\lambda - 1 - e^{\alpha\lambda}\right)}. \tag{2.3}$$

for the nabla dynamic equation

$$x^{\nabla}(t) = \lambda x(t), \quad \lambda \in \mathbb{C} \backslash \left\{\frac{1}{h}\right\}, \quad t \in \mathbb{T}, \tag{2.4}$$

compare the following theorem with Theorem 2.1.

**Theorem 2.5** (Nabla equation)**.** *Fix $\alpha > 0$, and let $\lambda \in \mathbb{R} \backslash \{1/h\}$. Also, let $\widehat{K}_{\mathbb{R}}$ be given as in* (2.3)*. We have the following cases.*

*(i) Suppose $0 < h < \dfrac{\alpha}{W_0(e^{-1})}$.*

*(a) If $\lambda \in (-\infty, 0) \cup \left(0, \dfrac{1}{h}\right)$, then* (2.4) *is Hyers–Ulam stable, with best HUS constant $K = \dfrac{1}{|\lambda|}$.*

*(b) If $\lambda = 0$, then* (2.4) *is not Hyers–Ulam stable.*

(c) *If* $\lambda \in \left(\frac{1}{h}, \infty\right)$*, then* (2.4) *is HUS, with best HUS constant* $K = \widehat{K}_{\mathbb{R}}$.

(ii) *Suppose* $h = \dfrac{\alpha}{W_0(e^{-1})}$*. Then,* $(1-h\lambda)^{-1}e^{\alpha\lambda} = -1$ *at* $\lambda = \dfrac{1}{\alpha}\left(1 + W_0(e^{-1})\right)$*, and we have the following subcases.*

(a) *If* $\lambda \in (-\infty, 0) \cup \left(0, \dfrac{W_0(e^{-1})}{\alpha}\right)$*, then* (2.4) *is HUS, with best HUS constant* $K = \dfrac{1}{|\lambda|}$.

(b) *If* $\lambda = 0$ *or* $\lambda = \dfrac{1}{\alpha}\left(1 + W_0(e^{-1})\right)$*, then* (2.4) *is not HUS.*

(c) *If* $\lambda \in \left(\dfrac{W_0(e^{-1})}{\alpha},\ \dfrac{1}{\alpha}\left(1 + W_0(e^{-1})\right)\right) \cup \left(\dfrac{1}{\alpha}\left(1 + W_0(e^{-1})\right),\ \infty\right)$*, then* (2.4) *is HUS, with best HUS constant* $K = \widehat{K}_{\mathbb{R}}$ *as in* (2.3).

(iii) *Suppose* $h > \dfrac{\alpha}{W_0(e^{-1})}$*. Then,* $(1-h\lambda)^{-1}e^{\alpha\lambda} = -1$ *at*

$$\widehat{\lambda}_{-1} := \frac{1}{h} - \frac{1}{\alpha}W_{-1}\left(-\frac{\alpha}{h}e^{\frac{\alpha}{h}}\right) \quad \textit{and} \quad \widehat{\lambda}_0 := \frac{1}{h} - \frac{1}{\alpha}W_0\left(-\frac{\alpha}{h}e^{\frac{\alpha}{h}}\right),$$

*and we have the following subcases.*

(a) *If* $\lambda \in (-\infty, 0) \cup \left(0, \dfrac{1}{h}\right)$*, then* (2.4) *is HUS, with best HUS constant* $K = \dfrac{1}{|\lambda|}$.

(b) *If* $\lambda = 0$*,* $\lambda = \widehat{\lambda}_0$*, or* $\lambda = \widehat{\lambda}_{-1}$*, then* (2.4) *is not HUS.*

(c) *If* $\lambda \in \left(\dfrac{1}{h},\ \widehat{\lambda}_0\right) \cup \left(\widehat{\lambda}_0, \widehat{\lambda}_{-1}\right) \cup \left(\widehat{\lambda}_{-1},\ \infty\right)$*, then* (2.4) *is HUS, with best HUS constant* $K = \left|\widehat{K}_{\mathbb{R}}\right|$ *as in* (2.3).

*Remark* 2.6. If we compare $K_{\mathbb{R}}$ in (2.1) with $\widehat{K}_{\mathbb{R}}$ in (2.3), we see that

$$|K_{\mathbb{R}}(\lambda)| = \left|\widehat{K}_{\mathbb{R}}(-\lambda)\right|,$$

where we have made them into functions of the parameter $\lambda$.

# 3 Complex Eigenvalues

In this section, we extend the considered values of the eigenvalue to $\lambda \in \mathbb{C}\backslash\left\{-\frac{1}{h}\right\}$ on the time scale $\mathbb{P}_{\alpha,h}$ for continuous interval size $\alpha > 0$ and discrete jump size $h > 0$. To further motivate our use of the Lambert $W$ function, consider the exponential function given in (1.3). Set the base of the exponential function as follows, $(1+h\lambda)e^{\alpha\lambda} = Re^{i\theta}$, for $R > 0$, $i = \sqrt{-1}$, and $\theta \in (-\pi, \pi]$. Let $w = \frac{\alpha}{h} + \alpha\lambda$. Then, the following are equivalent:

$$\begin{aligned}
(1+h\lambda)e^{\alpha\lambda} &= Re^{i\theta} \\
\left(\frac{\alpha}{h} + \alpha\lambda\right)e^{\alpha\lambda} &= \frac{R\alpha}{h}e^{i\theta} \\
we^{w} &= \frac{R\alpha}{h}e^{\frac{\alpha}{h}+i\theta} \\
w &= W_z\left(\frac{R\alpha}{h}e^{\frac{\alpha}{h}+i\theta}\right),
\end{aligned}$$

so that

$$\lambda = -\frac{1}{h} + \frac{1}{\alpha}W_z\left(\frac{R\alpha}{h}e^{\frac{\alpha}{h}+i\theta}\right), \quad \theta \in (-\pi, \pi], \quad R > 0, \quad h > 0, \tag{3.1}$$

for various branches of the Lambert $W$ function in the complex plane determined by $z \in \mathbb{Z}$, for $\theta \in (-\pi, \pi]$, with a branch cut along the negative real axis, and principal branch $W_0$.

**Theorem 3.1** (Delta equation)**.** *Let* $\lambda \in \mathbb{C}\backslash\left\{-\frac{1}{h}\right\}$ *have the form* (3.1)*, and let* $W_z$ *be the Lambert* $W$ *function for any* $z \in \mathbb{Z}$*.*

*(i) If* $R = 1$*, then* (1.1) *is not Hyers–Ulam stable.*

*(ii) If* $R > 1$*, then* (1.1) *is Hyers–Ulam stable, with HUS constant at most*

$$K_{\mathbb{C}} := \max_{j\in[0,\alpha]} \frac{R - 1 - Re^{(j-\alpha)\operatorname{Re}(\lambda)} + e^{j\operatorname{Re}(\lambda)}\left(1 + h\operatorname{Re}(\lambda)\right)}{(R-1)\operatorname{Re}(\lambda)}, \tag{3.2}$$

*or* $K_{\mathbb{C}} = \frac{h + R\alpha}{R-1}$ *if* $\operatorname{Re}(\lambda) = 0$*.*

*(iii) If* $0 < R < 1$*, then* (1.1) *is Hyers–Ulam stable, with HUS constant at most*

$$|K_{\mathbb{C}}| = \max_{j\in[0,\alpha]} \frac{R - 1 - Re^{(j-\alpha)\operatorname{Re}(\lambda)} + e^{j\operatorname{Re}(\lambda)}(1 + h\operatorname{Re}(\lambda))}{(1-R)\operatorname{Re}(\lambda)}. \tag{3.3}$$

*Proof.* Let $\lambda \in \mathbb{C} \backslash \left\{-\dfrac{1}{h}\right\}$ have the form (3.1), and let $W_z$ be the Lambert $W$ function for any $z \in \mathbb{Z}$.

Case (i). If $R = 1$, then $\lambda = -\dfrac{1}{h} + \dfrac{1}{\alpha} W_z \left(\dfrac{\alpha}{h} e^{\frac{\alpha}{h}+i\theta}\right)$ for $\theta \in (-\pi, \pi]$ and for fixed $z \in \mathbb{Z}$. Let the exponential function be given by (1.3). Then, for $t = k(\alpha + h) + j \in [k(\alpha+h), k(\alpha+h)+\alpha]$ and $j \in [0, \alpha]$,

$$e_\lambda(t,0) \quad = \quad \left[(1+h\lambda)e^{\alpha\lambda}\right]^k e^{j\lambda} = e^{j\lambda + ik\theta}.$$

Note that, for all $j \in [0, \alpha]$ and $\theta \in (-\pi, \pi]$, and for any fixed $z \in \mathbb{Z}$, the real part of $\lambda$ satisfies $\operatorname{Re}(\lambda) \leq 0$, and

$$|e_\lambda(t,0)| = e^{j \operatorname{Re}(\lambda)} \in \left[e^{\alpha \operatorname{Re}(\lambda)},\ 1\right].$$

So, with $e_\lambda(t,0) = e^{j\lambda+ik\theta}$ for $t = k(\alpha+h)+j$, $j \in [0,\alpha]$, and $\theta \in (-\pi,\pi]$, set $\phi(t) = \varepsilon t e_\lambda(t,0)$. Then, we have

$$|\phi^\Delta(t) - \lambda\phi(t)| = |\varepsilon\lambda t e_\lambda(t,0) + \varepsilon e_\lambda^\sigma(t,0) - \varepsilon\lambda t e_\lambda(t,0)| = \varepsilon|e_\lambda^\sigma(t,0)| \leq \varepsilon$$

implies that $\phi$ satisfies (1.2), so that

$$|\phi(t) - x(t)| = |e_\lambda(t,0)||\varepsilon t - x_0| \geq e^{\alpha \operatorname{Re}(\lambda)}|\varepsilon t - x_0| \to \infty$$

for any possible initial condition $x_0$, meaning (1.1) is not HUS for $R = 1$, that is when $\lambda = -\dfrac{1}{h} + \dfrac{1}{\alpha} W_z \left(\dfrac{\alpha}{h} e^{\frac{\alpha}{h}+i\theta}\right)$ for any $\theta \in (-\pi, \pi]$, $h > 0$, and for any fixed $z \in \mathbb{Z}$.

Case (ii). Let $R > 1$, that is, let $\lambda = -\dfrac{1}{h} + \dfrac{1}{\alpha} W_z \left(\dfrac{R\alpha}{h} e^{\frac{\alpha}{h}+i\theta}\right)$, initially with $\operatorname{Re}(\lambda) \neq 0$, for $\theta \in (-\pi, \pi]$ and $z \in \mathbb{Z}$. Let the exponential function be given by (1.3), and let $\phi$ satisfy (1.2). Then, $\phi$ has the form given in (1.5), and again,

$$\int_0^\infty \frac{q(s)}{e_\lambda(\sigma(s),0)} \Delta s$$

exists and is finite, as $|e_\lambda(t,0)| = R^k e^{j \operatorname{Re}(\lambda)}$ for $R > 1$ and $t = k(\alpha+h)+j$, $j \in [0,\alpha]$. Note that

$$x(t) = x_0 e_\lambda(t,0), \qquad x_0 = \phi_0 + \int_0^\infty \frac{q(s)}{e_\lambda(\sigma(s),0)} \Delta s$$

is a well-defined solution of (1.1). Now, to integrate from $s = 0$ to $s = t = k(\alpha+h)+j$ for some $k \in \{0, 1, 2, \ldots\}$ and $j \in [0, \alpha]$, we see that there are $k$ continuous intervals and $k$ gaps to integrate over, plus the final partial interval (continuous), so that

$$\int_0^t \frac{\Delta s}{|e_\lambda(\sigma(s),0)|} \quad = \quad \sum_{m=0}^{k-1} \left( \int_{m(\alpha+h)}^{m(\alpha+h)+\alpha} \frac{ds}{|e_\lambda(s,0)|} \right)$$

$$
\begin{aligned}
&+\sum_{m=1}^{k}\left(\int_{m(\alpha+h)-h}^{m(\alpha+h)}\frac{\Delta s}{|e_\lambda(\sigma(s),0)|}\right)+\int_{k(\alpha+h)}^{t}\frac{ds}{|e_\lambda(s,0)|}\\
=\;&\sum_{m=0}^{k-1}\left(\int_0^\alpha\frac{e^{-j\operatorname{Re}(\lambda)}}{R^m}dj\right)+\sum_{m=1}^{k}\frac{h}{R^m}+\int_0^j\frac{e^{-\ell\operatorname{Re}(\lambda)}}{R^k}d\ell\\
=\;&\sum_{m=0}^{k-1}\frac{-1}{R^m\operatorname{Re}(\lambda)}e^{-j\operatorname{Re}(\lambda)}\Big|_{j=0}^{\alpha}+\sum_{m=1}^{k}\frac{h}{R^m}+\frac{-1}{R^k\operatorname{Re}(\lambda)}e^{-\ell\operatorname{Re}(\lambda)}\Big|_{\ell=0}^{j}\\
=\;&\frac{R(R^k-1)(1-e^{-\alpha\operatorname{Re}(\lambda)}))}{R^k(R-1)\operatorname{Re}(\lambda)}+\frac{h(R^k-1)}{R^k(R-1)}+\frac{1-e^{-j\operatorname{Re}(\lambda)}}{R^k\operatorname{Re}(\lambda)} \qquad (3.4)
\end{aligned}
$$

and

$$
\int_0^\infty\frac{\Delta s}{|e_\lambda(\sigma(s),0)|}=\lim_{t\to\infty}\int_0^t\frac{\Delta s}{|e_\lambda(\sigma(s),0)|}=\frac{R(1-e^{-\alpha\operatorname{Re}(\lambda)}))}{(R-1)\operatorname{Re}(\lambda)}+\frac{h}{R-1}.
$$

Using these two integral values, we have

$$
\begin{aligned}
|\phi(t)-x(t)| &= |e_\lambda(t,0)|\left|-\int_t^\infty\frac{q(s)}{e_\lambda(\sigma(s),0)}\Delta s\right|\\
&\le \varepsilon|e_\lambda(t,0)|\int_t^\infty\frac{1}{|e_\lambda(\sigma(s),0)|}\Delta s\\
&= \varepsilon|e_\lambda(t,0)|\left(\int_0^\infty-\int_0^t\right)\frac{1}{|e_\lambda(\sigma(s),0)|}\Delta s\\
&= \varepsilon\left(\frac{R-1+e^{j\operatorname{Re}(\lambda)}-Re^{(j-\alpha)\operatorname{Re}(\lambda)}+e^{j\operatorname{Re}(\lambda)}h\operatorname{Re}(\lambda)}{(R-1)\operatorname{Re}(\lambda)}\right)
\end{aligned}
$$

for $j\in[0,\alpha]$, and for fixed $z\in\mathbb{Z}$, $R>1$, $\theta\in(-\pi,\pi]$ that determine $\lambda\in\mathbb{C}$ with $\operatorname{Re}(\lambda)\neq 0$. Set $K$ as in (3.2), that is,

$$
K_{\mathbb{C}}:=\max_{j\in[0,\alpha]}\frac{R-1+e^{j\operatorname{Re}(\lambda)}-Re^{(j-\alpha)\operatorname{Re}(\lambda)}+e^{j\operatorname{Re}(\lambda)}h\operatorname{Re}(\lambda)}{(R-1)\operatorname{Re}(\lambda)}.
$$

Therefore, (1.1) has HUS for $\lambda=-\frac{1}{h}+\frac{1}{\alpha}W_z\left(\frac{R\alpha}{h}e^{\frac{\alpha}{h}+i\theta}\right)$ with $\operatorname{Re}(\lambda)\neq 0$ and $R>1$, with HUS constant at most $K_{\mathbb{C}}$. If $\operatorname{Re}(\lambda)=0$, then

$$
\begin{aligned}
\lim_{\operatorname{Re}(\lambda)\to 0}K_{\mathbb{C}} &= \lim_{\operatorname{Re}(\lambda)\to 0}\max_{j\in[0,\alpha]}\frac{R-1+e^{j\operatorname{Re}(\lambda)}-Re^{(j-\alpha)\operatorname{Re}(\lambda)}+e^{j\operatorname{Re}(\lambda)}h\operatorname{Re}(\lambda)}{(R-1)\operatorname{Re}(\lambda)}\\
&= \max_{j\in[0,\alpha]}\frac{h+R\alpha+j(1-R)}{R-1}\\
&= \frac{h+R\alpha}{R-1},
\end{aligned}
$$

since $R > 1$ in this case. As a result, (1.1) is HUS with HUS constant at most $K_{\mathbb{C}} = \dfrac{h + R\alpha}{R-1}$, for $R > 1$ and $\mathrm{Re}(\lambda) = 0$. In either instance, case (ii) holds.

Case (iii). Finally, let $\lambda = -\dfrac{1}{h} + \dfrac{1}{\alpha} W_z\left(\dfrac{R\alpha}{h} e^{\frac{\alpha}{h} + i\theta}\right)$ for $\theta \in (-\pi, \pi]$ and $R \in (0,1)$, let the exponential function be given by (1.3), and let $\phi$ satisfy (1.2). Using $R \in (0,1)$ and (3.4), as well as $t = k(\alpha + h) + j$ for $j \in [0, \alpha]$, we can modify (3.4) to get

$$\int_0^t \frac{\Delta s}{|e_\lambda(\sigma(s), 0)|} = \frac{R(1 - R^k)(1 - e^{-\alpha \mathrm{Re}(\lambda)})}{R^k(1-R)\mathrm{Re}(\lambda)} + \frac{h(1 - R^k)}{R^k(1-R)} + \frac{1 - e^{-j \mathrm{Re}(\lambda)}}{R^k \mathrm{Re}(\lambda)}. \tag{3.5}$$

If $\phi$ satisfies the perturbed equation (1.2), then $\phi$ is again given as in (1.5). Let $x$ be a solution of (1.1) with form (1.4), where

$$x_0 = \phi_0 - \varepsilon\left(\frac{h}{1-R} + \frac{R(1 - e^{-\alpha \mathrm{Re}(\lambda)})}{(1-R)\mathrm{Re}(\lambda)}\right);$$

note that both fractions in the parentheses here are positive, due to $R \in (0,1)$ and $\mathrm{Re}(\lambda) < 0$ in this case. Employing (3.5) with $t = k(\alpha + h) + j$, we see that

$$\begin{aligned}
|\phi(t) - x(t)| &= |e_\lambda(t,0)| \Bigg| \phi_0 + \int_0^t \frac{q(s)}{e_\lambda(\sigma(s),0)} \Delta s \\
&\quad - \left(\phi_0 - \varepsilon\left(\frac{h}{1-R} + \frac{R(1 - e^{-\alpha \mathrm{Re}(\lambda)})}{(1-R)\mathrm{Re}(\lambda)}\right)\right)\Bigg| \\
&= |e_\lambda(t,0)| \left| \int_0^t \frac{q(s)}{e_\lambda(\sigma(s),0)} \Delta s + \varepsilon\left(\frac{h}{1-R} + \frac{R(1 - e^{-\alpha \mathrm{Re}(\lambda)})}{(1-R)\mathrm{Re}(\lambda)}\right)\right| \\
&\le \varepsilon |e_\lambda(t,0)| \left(\int_0^t \frac{1}{|e_\lambda(\sigma(s),0)|} \Delta s + \frac{h}{1-R} + \frac{R(1 - e^{-\alpha \mathrm{Re}(\lambda)})}{(1-R)\mathrm{Re}(\lambda)}\right) \\
&= \varepsilon R^k e^{j \mathrm{Re}(\lambda)} \left(\frac{R(1 - e^{-\alpha \mathrm{Re}(\lambda)})}{R^k(1-R)\mathrm{Re}(\lambda)} + \frac{h}{R^k(1-R)} + \frac{1 - e^{-j \mathrm{Re}(\lambda)}}{R^k \mathrm{Re}(\lambda)}\right) \\
&= \varepsilon e^{j \mathrm{Re}(\lambda)} \left(\frac{R(1 - e^{-\alpha \mathrm{Re}(\lambda)})}{(1-R)\mathrm{Re}(\lambda)} + \frac{h}{1-R} + \frac{1 - e^{-j \mathrm{Re}(\lambda)}}{\mathrm{Re}(\lambda)}\right) \\
&\le \varepsilon \left(\frac{1 - R + Re^{(j-\alpha)\mathrm{Re}(\lambda)} - e^{j \mathrm{Re}(\lambda)}(1 + h \mathrm{Re}(\lambda))}{(-1+R)\mathrm{Re}(\lambda)}\right)
\end{aligned}$$

for $j \in [0, \alpha]$, as $R \in (0,1)$. Therefore (1.1) has HUS for $\lambda = -\dfrac{1}{h} + \dfrac{1}{\alpha} W_z\left(\dfrac{R\alpha}{h} e^{\frac{\alpha}{h} + i\theta}\right)$ for $R \in (0,1)$, with HUS constant given by at most $K = |K_{\mathbb{C}}|$ given in (3.3), for $K_{\mathbb{C}}$ as in (3.2). This ends the proof. □

*Remark* 3.2. In Figure 3.1, we illustrate the effects of an increasing jump size $h$, relative to $\alpha$, on the eigenvalues $\lambda$ as parameterized curves in the complex plane. Here, $\alpha = 1$,

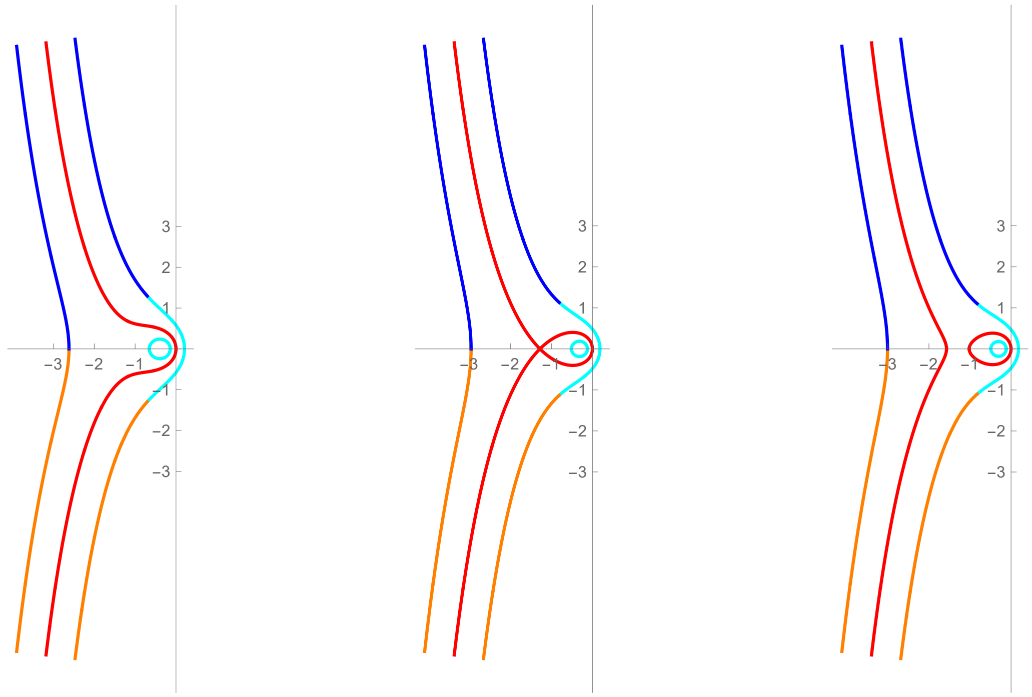

Figure 3.1: Delta case: Let $\alpha = 1$, and let $\lambda$ be as in (3.1), for all $\theta \in (-\pi, \pi]$ and $z = -1$ (orange), $z = 0$ (cyan), $z = 1$ (blue). Let $R = \frac{1}{2}$ (left-hand curve and oval), $R = 1$ (red middle curve, unstable manifold), and $R = 2$ (right-hand curve). **Left Graph:** $\boldsymbol{h = 3.0}$ The parameterized values of $\lambda \in \mathbb{C}$ before the bifurcation in the unstable manifold has occurred. **Middle Graph:** $\boldsymbol{h = \frac{1}{W_0(e^{-1})} \approx 3.59112}$ (the bifurcation value) The unstable manifold is the homoclinic orbit given by the parameterized graph of $\lambda = -W_0(e^{-1}) + W_z(e^{1+i\theta})$. **Right Graph:** $\boldsymbol{h = 3.7}$ The parameterized values of $\lambda \in \mathbb{C}$ after the bifurcation in the unstable manifold has occurred. End of caption.

and $h$ increases from $h = 3.0$, through the bifurcation value of $h = \dfrac{1}{W_0(e^{-1})}$, to $h = 3.7$, after the bifurcation in the parameter space has occurred. In Figure 3.2, the complex eigenvalues for the nabla equation are likewise illustrated.

*Remark* 3.3. For $\lambda$ as given in (3.1), note that $R = |1 + h\lambda|\, e^{\alpha \operatorname{Re}(\lambda)}$. As the jump size $h > 0$ approaches zero with $\alpha > 0$ fixed, $R = e^{\alpha \operatorname{Re}(\lambda)}$ implies $\operatorname{Re}(\lambda) = \dfrac{1}{\alpha} \ln R$. If we write $\lambda = \operatorname{Re}(\lambda) + \operatorname{Im}(\lambda) i$, where $\operatorname{Re}$ and $\operatorname{Im}$ are the real and imaginary parts of $\lambda \in \mathbb{C} \backslash \left\{-\dfrac{1}{h}\right\}$, respectively, then

$$R\cos(\theta) = e^{\alpha \operatorname{Re}(\lambda)} \left[ (1 + h \operatorname{Re}(\lambda)) \cos(\operatorname{Im}(\lambda)) - h \operatorname{Im}(\lambda) \sin(\operatorname{Im}(\lambda)) \right],$$

$$R\sin(\theta) = e^{\alpha\,\mathrm{Re}(\lambda)}\left[(1+h\,\mathrm{Re}(\lambda))\sin(\mathrm{Im}(\lambda)) + h\,\mathrm{Im}(\lambda)\cos(\mathrm{Im}(\lambda))\right];$$

taking $h$ to zero, we see that $\sin(\theta) = \sin(\mathrm{Im}(\lambda))$ and $\cos(\theta) = \cos(\mathrm{Im}(\lambda))$. Summarizing, we have that for fixed $\alpha > 0$,

$$\lim_{h\to 0^+} \mathbb{P}_{\alpha,h} = \mathbb{R}$$

and

$$\lim_{h\to 0^+} \lambda = \frac{1}{\alpha}\ln R + (\theta + 2\pi z)i, \quad R > 0, \quad \theta \in (-\pi, \pi], \quad z \in \mathbb{Z}.$$

In particular, note that for $R = 1$, the eigenvalues $\lambda$ in (3.1) converge to purely imaginary points in the complex plane as the jump size $h$ goes to zero, which corresponds to the known fact that the Hyers–Ulam instability region for (1.1) with $\mathbb{T} = \mathbb{R}$ is the imaginary axis. See Theorem 3.1(i).

*Remark* 3.4. Similar to Remark 2.3 earlier, fix the jump size $h > \dfrac{\alpha}{W_0(e^{-1})}$, and let $\alpha$ tend to 0. Then, $\lim\limits_{\alpha\to 0} \mathbb{P}_{\alpha,h} = h\mathbb{Z}$, and we have the following cases from Theorem 3.1, with $\alpha = 0$, along the principal branch of the Lambert $W$ function, $W_0$.

(i) If $R = 1$, then (1.1) is not Hyers–Ulam stable.

(ii) If $R > 1$, then (1.1) is Hyers–Ulam stable, with HUS constant at most

$$K_{\mathbb{C}} = \frac{h}{R-1} = \frac{h}{|1+h\lambda| - 1}.$$

(iii) If $0 < R < 1$, then (1.1) is Hyers–Ulam stable, with HUS constant at most

$$K = |K_{\mathbb{C}}| = \frac{h}{1-R} = \frac{h}{1 - |1+h\lambda|}.$$

Thus, we recover the (best) $K$ value found in the $h\mathbb{Z}$ case in [4, Theorem 2.6]. Since $|K_{\mathbb{C}}| = \dfrac{h}{||1+h\lambda| - 1|} = \dfrac{1}{|\,\mathrm{Re}_h(\lambda)|}$, which is the absolute value of the reciprocal of the Hilger-real part of $\lambda$, we can view $K_{\mathbb{C}}^{-1}$ as the $\mathbb{P}_{\alpha,h}$-real part of $\lambda$ in some sense.

*Remark* 3.5. Consider the nabla case with $\lambda \in \mathbb{C}\backslash\{1/h\}$ on the time scale $\mathbb{P}_{\alpha,h}$, for continuous interval size $\alpha > 0$ and discrete jump size $h > 0$. With the nabla exponential function given in (2.2), set the base of the exponential function as follows, $(1-h\lambda)^{-1}e^{\alpha\lambda} = Re^{i\theta}$, for $R > 0$, $i = \sqrt{-1}$, and $\theta \in (-\pi, \pi]$. Let $w = \dfrac{\alpha}{h} - \alpha\lambda$. Then, the following are equivalent:

$$\begin{aligned}(1-h\lambda)^{-1}e^{\alpha\lambda} &= Re^{i\theta} \\ \left(\frac{\alpha}{h} - \alpha\lambda\right)e^{-\alpha\lambda} &= \frac{\alpha}{hR}e^{-i\theta}\end{aligned}$$

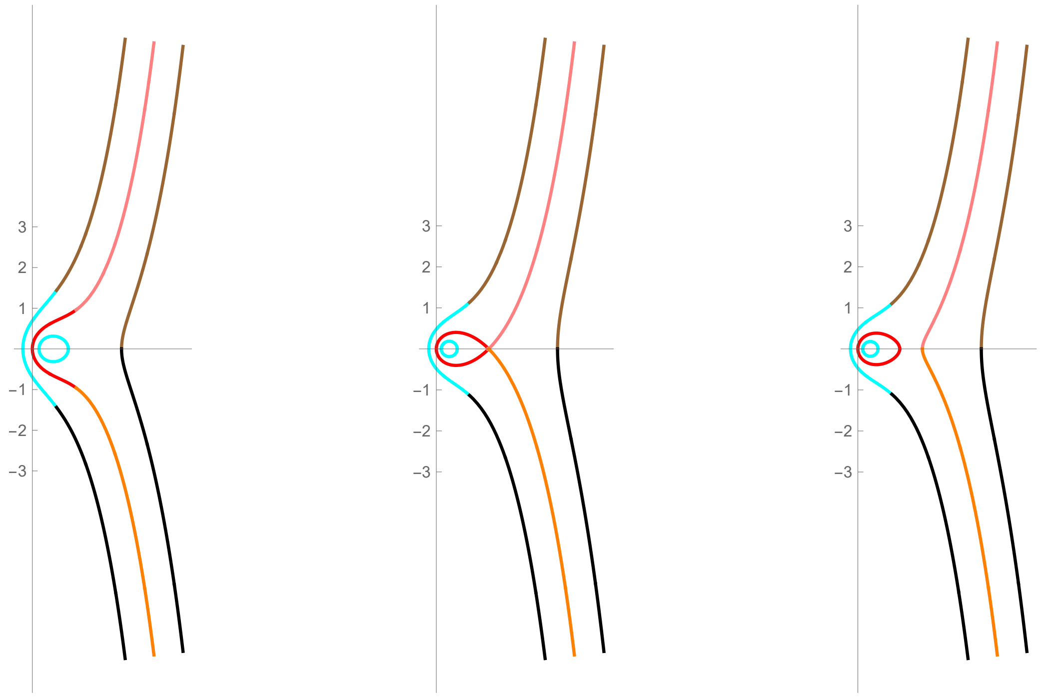


Figure 3.2: Nabla case: Let $\alpha = 1$, and let $\lambda$ be as in (3.6), for all $\theta \in (-\pi, \pi]$ and $z = -1$ (black), $z = 0$ (cyan), $z = 1$ (brown). Let $R = \dfrac{1}{2}$ (left-hand curve), $R = 1$ (orange/red/pink middle curves, unstable manifold), and $R = 2$ (oval and right-hand curve). **Left Graph:** $\boldsymbol{h = 3.0}$ The parameterized values of $\lambda \in \mathbb{C}$ before the bifurcation in the unstable manifold has occurred. **Middle Graph:** $\boldsymbol{h = \dfrac{1}{W_0(e^{-1})} \approx 3.59112}$ (the bifurcation value) The unstable manifold is the homoclinic orbit given by the parameterized graph of $\lambda = -W_0(e^{-1}) + W_z(e^{1+i\theta})$. **Right Graph:** $\boldsymbol{h = 3.7}$ The parameterized values of $\lambda \in \mathbb{C}$ after the bifurcation in the unstable manifold has occurred. End of caption.

$$\begin{aligned} we^w &= \frac{\alpha}{hR} e^{\frac{\alpha}{h} - i\theta} \\ w &= W_z\left(\frac{\alpha}{hR} e^{\frac{\alpha}{h} - i\theta}\right), \end{aligned}$$

so that

$$\lambda = \frac{1}{h} - \frac{1}{\alpha} W_z\left(\frac{\alpha}{hR} e^{\frac{\alpha}{h} - i\theta}\right), \quad \theta \in (-\pi, \pi], \quad R > 0, \quad h > 0, \tag{3.6}$$

for various branches of the Lambert $W$ function in the complex plane determined by $z \in \mathbb{Z}$, for $\theta \in (-\pi, \pi]$, with a branch cut along the negative real axis, and principal branch $W_0$. Compare the following theorem with Theorem 3.1.

**Theorem 3.6** (Nabla equation)**.** *Let $\lambda \in \mathbb{C}\backslash\{1/h\}$ have the form* (3.6)*, and let $W_z$ be the Lambert $W$ function for any $z \in \mathbb{Z}$.*

*(i) If $R = 1$, then* (2.4) *is not Hyers–Ulam stable.*

*(ii) If $R > 1$, then* (2.4) *is Hyers–Ulam stable, with HUS constant at most*

$$\widehat{K}_{\mathbb{C}} := \max_{j\in[0,\alpha]} \frac{R - 1 - Re^{(j-\alpha)\operatorname{Re}(\lambda)} + e^{j\operatorname{Re}(\lambda)}\left(1 + Rh\operatorname{Re}(\lambda)\right)}{(R-1)\operatorname{Re}(\lambda)}, \tag{3.7}$$

*or* $\widehat{K}_{\mathbb{C}} = \dfrac{R(h+\alpha)}{R-1}$ *if* $\operatorname{Re}(\lambda) = 0$.

*(iii) If $0 < R < 1$, then* (2.4) *is Hyers–Ulam stable, with HUS constant at most*

$$K = \left|\widehat{K}_{\mathbb{C}}\right| := \max_{j\in[0,\alpha]} \frac{R - 1 - Re^{(j-\alpha)\operatorname{Re}(\lambda)} + e^{j\operatorname{Re}(\lambda)}(1 + Rh\operatorname{Re}(\lambda))}{(1-R)\operatorname{Re}(\lambda)}. \tag{3.8}$$

# 4 Related Time Scales

Related to the time scale $\mathbb{T} = \mathbb{P}_{\alpha,h}$ are time scales with continuous intervals broken up by isolated points. For example, consider the time scale

$$\begin{aligned}\mathbb{T} = \mathbb{P}_{\alpha,\beta,\gamma,\delta} := \quad & \bigcup_{k=0}^{\infty}[k(\alpha,\beta,\gamma,\delta), k(\alpha,\beta,\gamma,\delta)+\alpha] \\ & \cup\{k(\alpha,\beta,\gamma,\delta)+(\alpha+\beta)\} \cup \{k(\alpha,\beta,\gamma,\delta)+(\alpha+\beta+\gamma)\},\end{aligned}$$

which one can think of as dash-dot-dot, dash-dot-dot, and so on, a continuous dash or interval of length $\alpha$, followed by jumps of length $\beta, \gamma$ to two isolated points, respectively, followed by a jump of length $\delta$ to the next continuous interval, repeated.

**Theorem 4.1.** *Let $\mathcal{I}_k = [k(\alpha,\beta,\gamma,\delta), k(\alpha,\beta,\gamma,\delta)+\alpha]$. The solution to*

$$x^{\Delta}(t) = \lambda x(t), \quad t \in \mathbb{P}_{\alpha,\beta,\gamma,\delta},$$

*is given by the exponential function*

$$e_\lambda(t,0) = \begin{cases} \left(\dfrac{(1+\beta\lambda)(1+\gamma\lambda)(1+\delta\lambda)}{e^{(\beta+\gamma+\delta)\lambda}}\right)^k e^{\lambda t} & \text{if } t \in \mathcal{I}_k \\ (1+\beta\lambda)^{k+1}\left((1+\gamma\lambda)(1+\delta\lambda)\right)^k e^{\alpha\lambda(k+1)} & \text{if } t = T_{k,\alpha,\beta} \\ \left((1+\beta\lambda)(1+\gamma\lambda)\right)^{k+1}(1+\delta\lambda)^k e^{\alpha\lambda(k+1)} & \text{if } t = T_{k,\alpha,\beta,\gamma} \end{cases}$$

*for each fixed $k \in \mathbb{N}_0$ and $t \in \mathbb{P}_{\alpha,\beta,\gamma,\delta}$, where $T_{k,\alpha,\beta} = k(\alpha,\beta,\gamma,\delta) + (\alpha+\beta)$ and $T_{k,\alpha,\beta,\gamma} = k(\alpha,\beta,\gamma,\delta) + (\alpha+\beta+\gamma)$.*

The HUS analysis for this time scale would clearly track with the analysis earlier in this work, and involve the Lambert $W$ function.

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