International Journal of Difference Equations
ISSN 0973-6069, Volume 15, Number 2, pp. 511–530 (2020)
http://campus.mst.edu/ijde

Existence Theory and Topological Aspects of the Solution Set of Integrodifferential Equations

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Abstract

The present research work is devoted to study the existence of solutions and related results for the integrodifferential equation, where all the functions take values in a Banach space while the used integral is Henstock–Kurzweil–Pettis. Moreover, in order to exhibit the structure of the solution set some topological properties will be determined for the underlying problem.

AMS Subject Classifications: 26A33, 34A08, 35S11.
Keywords: Integrodifferential equation, Henstock–Kurzweil–Pettis integral, measure of noncompactness, pseudo solution.

1 Introduction and Background Material

If an equation involves both differential operator and integral operator, then such an equation is called an integrodifferential equation. Applications of such kinds of equations appear in various branches of science like fluid dynamics, biological models, and chemical kinetics. The most common example of such an equation occurs in “Basic

Received June 21, 2020; Accepted July 24, 2020
Communicated by Paul Eloe

Electric Circuit Analysis." For the last few years this theory has received ample attention from researchers and scientists. Keeping in view the applications of Banach spaces, the theory for initial value problems and for integrodifferential equations has been extended to vector valued functions. Using these new tools, some significant work has been done, and we suggest the readers see the papers [5–7, 11–13, 19], and references therein. In reality, mathematical modeling of various epidemiological and other essential phenomena require integrodifferential equations. From an application point of view, different investigations of such kinds of equations lead to inspection of dynamical behavior of various models. Physical systems that are characterized via Levy jumps, can be modeled using these equations.

For a quantum system the fractional-Schrödinger equation can also be solved using integrodifferential equations [9]. R.C. Maccamy in [14] established an asymptotic stability theory for a certain type of integrodifferential equation. It has been shown that the underlying problem is associated with heat flow theory in materials along with memory. Another application of integrodifferential equations can be seen in [2]. Initially, the authors in [2] studied theoretical aspects of the proposed integrodifferential equation. Results related to local existence and uniqueness of solutions, as well as continuous dependence, of an abstract integrodifferential equation were obtained. Moreover, the authors provided applications of their obtained results, including results that are connected with strongly damped plate equations with memory. Various theories have applied to study integrodifferential equations. Among these, the theory of semigroups of bounded and linear operators is deeply associated with solutions of integrodifferential equations in Banach spaces. In the recent past, assuming the ground set to be a Banach space, this theory has applied to family of nonlinear differential equations. On the basis of this theory, Pazy in [15] studied the existence, and uniqueness of mild solutions. In addition, strong as well as classical solutions of semilinear evolution equations have been explored.

A literature review of integrodifferential equations reveals some valuable theoretical work about the study of integrodifferential equations. In this regard the authors in [13] studied the following semilinear integrodifferential equation along with a nonlocal subsidiary initial condition,

$$\begin{cases} \dfrac{dw(s)}{ds} = A\left[w(s) + \displaystyle\int_{s_0}^{s} f\,(s-\tau)\,w(\tau)d\tau\right] + F(s,\, w(s)), \quad s \in [s_0,\ s_0+T] \\ w(s_0) = w_0 - h(s_1, s_2, \ldots, s_p,\ w), \end{cases} \tag{1.1}$$

where A is a generator for a C_0 semigroup on some Banach space X. The operator $f(\cdot)$ is bounded and linear and is defined on a Banach space X satisfying some auxiliary conditions. The function h is continuous and takes on vector values. The authors in [13] established new results related to existence and uniqueness of solutions of their underlying nonlocal problem. Outcomes of their finding were the unification and extension of previous work carried out for differential or integrodifferential equations. Some illustrative examples related to heat conduction in materials with memory were given to show that their outcomes fulfilled these gaps.

Following the work of [13] several authors have studied different classes of integrodifferential equations. In [17] the author obtained existence results for the following integrodifferential equations via a new kind of vector valued integral,

$$\begin{cases} \dfrac{du(t)}{dt} = f\left(t, u(t), \displaystyle\int_0^t K(t,s,u(s))ds\right), \\ u(0) = u_0, \qquad\qquad t \in [0,a], a > 0, u_0 \in E. \end{cases} \tag{1.2}$$

Here, the underlying ground set is taken to be a Banach space E, and the integral for this model is the Henstock–Lebesgue (HL) integral. Using the measure of noncompactness and some fixed point theorem, the author obtained existence results for (1.2). The purpose of using the HL Integral was to construct equations that include highly oscillatory functions. Dealing with such kinds of integration the existence theory of Riemann and Lebesgue is insufficient. Therefore, Henstock and Kurzweil independently defined a new integral, now known as the Henstock–Kurzweil integral [16]. In addition, the author in [18] reconsidered the problem studied in [17] while using the theory of time scales calculus. In fact the purpose of this theory was to unify discrete and continuous calculus.

Motivated by the literature review, the present work will provide existence results as well as some topological aspects of the solution set for the integrodifferential equation in Banach spaces given as,

$$\begin{cases} \dfrac{dw(t)}{dt_p} = f\Big(t,\, w(t), (HKP)\displaystyle\int_0^t k_1\,(t,s)\,w(s)ds, \\ \qquad\qquad\qquad (HKP)\displaystyle\int_0^t k_2\,(t,s)\,w(s)ds\Big), \\ w(0) = w_0. \end{cases} \tag{1.3}$$

with $t \in [0,\, b]$, involving the Henstock–Kurzweil–Pettis integral.

2 Henstock–Kurzweil–Pettis Integral and Related Results in Banach Spaces

In this section we include some definitions and properties needed to establish our main results. These include the Henstock–Kurzweil–Pettis integral that generalizes both the Pettis integral and the Henstock–Kurzweil integral. The following fundamental results and definitions have taken from [8, 16].

Definition 2.1 (See [8])**.** Let $\delta : [a,b] \to \mathbb{R}$ be a positive function. A tagged interval $(\tau, [p,q])$ consists of an interval $[p,q]$ and a point $\tau \in [p,q]$ where, $[p,q]$ is subset of $[a,b]$. An interval with a tag "τ", expressed as $(\tau, [p,q])$ is subordinate to δ, if $[p,q] \subseteq$

$(\tau - \delta(\tau), \tau + \delta(\tau))$. If we denote by $P = \{(\tau_i, [p_i, q_i])|1 \leq i \leq n, n \in \mathbb{N}\}$ such a collection in $[a, b]$, then

(a) the set of points, $\{\tau_i|1 \leq i \leq n\}$ are named the tags of the partition P;

(b) the collection of subintervals $\{[p_i, q_i]|1 \leq i \leq n\}$ are named the $intervals$ of P;

(c) the partition P will be known as $sub - \delta$, if $\{(\tau_i, [p_i, q_i])|1 \leq i \leq n, n \in \mathbb{N}\}$ is subordinate to the gauge value δ for every i;

(d) the partition P is known as a tagged partition of $[a, b]$, whenever

$$[a, b] = \cup_{i=1}^{n} [p_i, q_i];$$

(e) if h is a function with domain $[a, b]$ and range set E, then

$$h(P) = \Sigma_{i=1}^{n} h(\tau_i)(q_i - p_i);$$

(f) if a function G is defined on the subintervals $[p_i, q_i]$ of $[a, b]$, then

$$G(P) = \Sigma_{i=1}^{n} G([p_i, q_i]) = \Sigma_{i=1}^{n} [G(q_i) - G(p_i)].$$

If $G : [a, b] \to E$, then G can be considered as a function of intervals, and defined as $G([c, d]) = h(q) - h(p)$. In such kind of conditions the function G has the form $G(P) = h(q) - h(p)$ where P is a tagged partition of $[a, b]$.

Definition 2.2 (See [8])**.** A mapping h defined on $[a, b]$ into $\mathbb{R}$ is Henstock–Kurzweil integrable if for a given real number I with the property: If for each $\epsilon > 0$ we can find a positive function $\delta : [a, b] \to \mathbb{R}^+$ such that $|h(P) - I| < \epsilon$ where P is a tagged partition of $[a, b]$ that is subordinate to δ. This integral is denoted by $(HK) \displaystyle\int_a^b h(t)dt$.

Lemma 2.3 (See [4])**.** *Let $U \subset C([a, b], E)$ be bounded and equicontinuous. Then*

1. *the function $t \to \beta(U(t))$ is continuous on $[a, b]$,*
2. *$\beta_c(U) = \sup_{t \in [a,b]} \beta(U(t))$, here β_c denote the measure of weak noncompactness on $C([a, b], E)$, while $U(t) = \{u(t); u \in U\}, t \in [a, b]$.*

Definition 2.4 (See [8])**.** A function $h : [a, b] \to E$ is Henstock–Kurzweil integrable on $[a, b]$ if there exists a vector $I \in E$ with the following property: if for each $\epsilon \in (0, \infty)$ there exists a positive function δ on $[a, b]$ such that $||h(P) - I|| < \epsilon$ whenever P is a tagged partition of $[a, b]$ $sub - \delta$. The function h is Henstock–Kurzweil integrable on a measurable set $A \subseteq [a, b]$, if $h\chi_A$ is Henstock-Kurzweil integrable on $[a, b]$. The vector I is the Henstock–Kurzweil integral of h.

Definition 2.5 (See [16]). A function h defined on $[a, b]$ into the Banach space E is Pettis integrable if:

(i) for each functional, $x^* \in E^*$, the function x^*h is Lebesgue integrable on $[a, b]$;

(ii) for all measurable subsets A of $[a, b]$, there exists an element g of E such that for all $x^* \in E^*$, $x^*g = (L)\displaystyle\int_A x^*h(s)ds$."

We now provide a definition, which extends both Pettis and Henstock–Kurzweil integrals to a new integral.

Definition 2.6 (See [8]). A function $h : [a, b] \to E$ is said to be (HKP) Henstock–Kurzweil–Pettis integrable if there exists a function $g : [a, b] \to E$ with the following properties:

(i) for all $x^* \in E^*$, x^*h is Henstock–Kurzweil integrable on $[a, b]$; and

(ii) for each element $t \in [a, b]$, there exists $x^* \in E^*$ such that

$$x^*g(t) = HK\int_0^t x^*h(s)ds.$$

Here, the function g is called a primitive of h and we denote by $g(t) = \displaystyle\int_0^t h(s)ds$, the Henstock–Kurzweil–Pettis integral of h on the interval $[a, b]$.

Theorem 2.7 (See [10]). *Assume E is a locally convex topological vector space that is metrizable. Let D be a closed convex subset of E, and G be a weakly-weakly sequentially continuous map of D into itself. If for some $y \in D$, the implication*

$$\overline{\Gamma} = \overline{\text{conv}}\left(y \cup G(\Gamma)\right),\ implies\ \Gamma\ is\ relatively\ weakly\ compact$$

holds for every subset Γ of D, then G has a fixed point.

For the sequel we provide a list of some properties of the HKP integral which are important for studying existence results of differential equations via the HKP integral.

Theorem 2.8 (See [8]). *Assume $h : [a, b] \to E$ is an HKP integrable function, and let*

$$F(x) = \int_a^x h(s)ds,\quad x \in [a, b].$$

Then

(i) *for each $x^* \in E^*$, the real valued function x^*h is a Henstock–Kurzweil integrable function on the same interval $[a, b]$ and*

$$(HK)\int_a^x x^*(h(s))ds = x^*(F(x));$$

(ii) *the function h is the pseudo-derivative of F on $[a,b]$, while F is a weakly continuous function on $[a,b]$.*

Theorem 2.9 (See [16])**.** *Assume $h : [a,b] \to E$. If $h = 0$ a.e. on $[a,b]$, then h is HKP integrable on $[a,b]$ and $\int_a^b h(t)dt = 0$.*

Remark 2.10. The mean value theorem plays an important role in the study of differential equations. In the present study we deal with the HKP integral, and therefore, we provide a mean value theorem for the HKP integral.

Theorem 2.11 (See [8])**.** *If a given function $h : [a,b] \to E$ is HKP integrable, then*

$$\int_I h(t)dt \in |I| \cdot \overline{\text{conv}}\{h(I)\},$$

where I is an arbitrary subinterval of $[a,b]$ and $|I|$ represents the length of I.

Theorem 2.12 (See [8,16])**.** *Assume $h : [a,b] \to E$ and suppose $h_n : [a,b] \to E, n \in \mathbb{N}$, is a sequence of HKP integrable functions. Let F_n be the relative primitives of h_n. If we suppose:*

(i) *for each $x^* \in E^*$, $x^*(h_n(t)) \to x^*(h(t))$ a.e. on $[a,b]$,*

(ii) *for each $x^* \in E^*$, the family $G = \{x^* F_n | n = 1,2,3,\ldots\}$ is uniformly ACG^* on $[a,b]$, and*

(iii) *for each $x^* \in E^*$ the set G is equicontinuous on $[a,b]$,*

then h is HKP integrable on $[a,b]$ and $\int_0^t h_n(s)ds$ approaches weakly in E to $\int_0^t h(s)ds$ for each $t \in [a,b]$.

Definition 2.13 (See [16])**.** Let $G : [a,b] \to \mathbb{R}$ be a function and let F be a subset of the interval $[a,b] \subset \mathbb{R}$. Then G is absolutely continuous in the generalized sense (ACG^*) on F, if G is continuous on F and if F can be expressed as a countable union of sets on each of which G is absolutely continuous.

In the next section, we will study solutions of an integrodifferential equation. Then in a later section, we will study topological properties of the set of solutions.

3 Weak Topology and Existence Results for Integrodifferential Equations

In this section, we will establish existence results for solutions of the integrodifferential equation,

$$\begin{cases} \dfrac{dw(t)}{dt_p} = f\Big(t, w(t), (HKP)\displaystyle\int_0^t k_1\,(t,s)\,w(s)ds, \\ \qquad\qquad\qquad\qquad (HKP)\displaystyle\int_0^t k_2\,(t,s)\,w(s)ds\Big), \\ w(0) = w_0, \end{cases} \tag{3.1}$$

where $t \in [0,\, b]$. The functions $f(\cdot,\cdot,\cdot,\cdot)$, $k_1(\cdot,\cdot)$ and $k_2(\cdot,\cdot)$ take values in a Banach space. The integral is the Henstock–Kurzweil–Pettis integral. In addition to the above, assume

$$a_1 = \sup_{t\in[0,b]} \left\| \int_0^t k_1(t,s)ds \right\|,$$
$$a_2 = \sup_{t\in[0,b]} \left\| \int_0^t k_2(t,s)ds \right\|.$$

The following problem is considered while fixing the functional $w^* \in E^*$,

$$\begin{cases} \dfrac{d}{dt_p}\langle w^*, w(t)\rangle = \Big\langle w^*, f\Big(t,\, w(t),\, (HKP)\displaystyle\int_0^t k_1\,(t,s)\,w(s)ds, \\ \qquad\qquad\qquad\qquad (HKP)\displaystyle\int_0^t k_2\,(t,s)\,w(s)ds\Big)\Big\rangle, \\ \qquad w(0) = w_0, \end{cases}$$

where the operator $\dfrac{d}{dt_p}$ denotes the pseudo-derivative of the function $w(\cdot)$.

In order to establish a solution of the equation (3.1), we define some sets that will be used in the sequel,

$$\begin{aligned} H &:= \{w \in E : \ \|w\|_E \le \|w_0\|_E + h,\ \ h \in (0,\infty)\}, \\ \widetilde{H} &:= \{w \in (C[I_b,\, E], \omega) : \ \ w(0) = w_0,\ \ \|w\|_E \le \|w_0\|_E + h,\ \ h \in (0,\infty)\}. \end{aligned}$$

In addition to, we define an operator,

$$\begin{aligned} Gw(t) &= w_0 + (HKP)\int_0^t f\Big(z,\, w(z),\, (HKP)\int_0^z k_1\,(z,s)\,w(s)ds, \\ &\qquad\qquad (HKP)\int_0^z k_1\,(z,s)\,w(s)ds\Big)\,dz. \end{aligned}$$

To avoid tedious calculations, we define some additional sets given below,

$$X \ := \ \Big\{G(w) : \ \ w \in \widetilde{H}\Big\},$$

$$X_1 := \left\{(HKP)\int_0^z k_1(z,s)\,w(s)ds:\ z\in[0,t],\ t\in[0,\,b],\ w\in\widetilde{H}\right\},$$
$$X_2 := \left\{(HKP)\int_0^z k_2(z,s)\,w(s)ds:\ z\in[0,t],\ t\in[0,\,b],\ w\in\widetilde{H}\right\}.$$

To discuss the existence of solutions of the proposed integrodifferential equation (3.1), we consider the problem,

$$w(t) = w_0 + (HKP)\int_0^t f\Bigg(z,\, w(z),\, (HKP)\int_0^z k_1(z,s)\,w(s)ds, \\ (HKP)\int_0^z k_1(z,s)\,w(s)ds\Bigg)\,dz.$$

Prior to showing the existence of a solution of the integrodifferential equation (3.1), as well as to inspect the solution set structure, we provide the definition of a solution of (3.1).

Definition 3.1. A weakly continuous function $w(\cdot)$ be a solution of (3.1) if it satisfies the following conditions.

1. $w(\cdot)$ should be an absolutely continuous function in the generalized sense;
2. $w(0) = w_0$;
3. There exists a set $A(w^*)$ of Lebesgue measure zero, where $w^* \in E^*$, and for every $t \notin A(w^*)$ the relation holds,

$$\frac{d}{dt_p}\langle w^*, w(t)\rangle = \Bigg\langle w^*, f\Bigg(t,\, w(t),\, (HKP)\int_0^t k_1(t,s)\,w(s)ds, \\ (HKP)\int_0^t k_2(t,s)\,w(s)ds\Bigg)\Bigg\rangle,$$

where $\frac{d}{dt_p}$ denotes pseudo derivatives of the solution.

Theorem 3.2. *Suppose the following hold:*

1. *For each* $t \in I_b = [0, b]$, *the functions* $k_1(t,\cdot)\,w(\cdot)$, $k_2(t,\cdot)\,w(\cdot)$ *and*

$$f\left(\cdot,\, w(\cdot), (HKP)\int_0^{(\cdot)} k_1(\cdot,s)\,w(s)ds,\ (HKP)\int_0^{(\cdot)} k_1(\cdot,s)\,w(s)\right)$$

are HKP integrable functions for every function $w(\cdot): I_b \to E$ *which is uniformly* ACG^**.*

2. *For each $t \in I_b$, the function $f\,(t,\,\cdot,\,\cdot,\,\cdot)$ is a weakly-weakly sequentially continuous function.*

3. *For each $t \in I_b$, the functions $k_i(t,\cdot) \in BV(I_b,\mathbb{R})$, for each $i = 1, 2$, and the applications $t \mapsto k_i(t,\cdot)$ are $\|\cdot\|_{BV}$-continuous.*

4. *With β denoting the measure of noncompactness, there exist positive constants q_1, q_2, q_3, q_4 and q_5 such that*

$$\beta\left(f\left(J,\,D_1,\,D_2,\,D_3\right)\right) \leq q_1\beta\left(D_1\right) + q_2\beta\left(D_2\right) + q_3\beta\left(D_3\right)$$

where $D_1,\,D_2$ are proper subsets of H and $\mathrm{J} \subset \mathrm{I_b}$,

$$\beta\left(k_1\left(J,\,J\right)w(D_4)\right) \leq q_4\beta\left(D_4\right) \quad \text{where } \mathrm{D_4} \subseteq \mathrm{H} \text{ and } \mathrm{J} \subset \mathrm{I_b},$$

$$\beta\left(k_2\left(J,\,J\right)w(D_5)\right) \leq q_5\beta\left(D_5\right) \quad \text{where } \mathrm{D_5} \subseteq \mathrm{H} \text{ and } \mathrm{J} \subset \mathrm{I_b},$$

and where the entities, $f\,(J,D_1,D_2,D_3)$, $k_1\,(J,J)\,w(D_4)$ and $k_2\,(J,J)\,w(D_5)$ are defined as,

$$\begin{aligned} f\,(J,D_1,D_2,D_3) &:= \left\{f\,(t,y_1,y_2,y_3) : |\,(t,y_1,y_2,y_3) \in J \times D_1 \times D_2 \times D_3\right\} \\ k_i(J,\,J) &:= \left\{k_i(t,\,s) : |\,(t,\,s) \in J \times J\right\},\ \ i = 1,\,2. \end{aligned}$$

5. *The sets X, X_1 and X_2 are uniformly ACG^* and equicontinuous on I_b.*

Then the problem (3.1) *has a pseudo solution on I_λ for some $0 < \lambda \leq b$ and $q_1\lambda + q_2q_4\lambda^2 + q_3q_5\lambda^2 \in (0,\,1)$.*

Proof. Assume the spaces,

$$\begin{aligned} H &:= \left\{w \in E : \ \|w\|_E \leq \|w_0\|_E + h,\ \ h \in (0,\infty)\right\}, \\ \widetilde{H} &:= \left\{w \in (C[I_d,\,E],\omega) : \ \ w(0) = w_0,\ \ \|w\|_E \leq \|w_0\|_E + h,\ \ h \in (0,\infty)\right\}, \end{aligned}$$

for some fixed $h \geq 0$, and for some suitable d. By the definition of equicontinuity, for each positive value of ϵ, there exist $\delta > 0$ such that

$$\left\|\int_\tau^t f\left(y,\,w(y),\,(HKP)\int_0^y k_1\left(y,s\right)w(s)ds,\,(HKP)\int_0^t k_1\left(y,s\right)w(s)\right)dy\right\| < \epsilon,$$

for $w \in \widetilde{H}$, where $|t-\tau| < \delta$ and $\tau, t \in [0,b]$. Following the preceding fact, we can find a number $\lambda \in (0,\,b]$ such that

$$\left\|\int_0^t f\left(y,w(y),(HKP)\int_0^y k_1\left(y,s\right)w(s)ds,(HKP)\int_0^y k_2\left(y,s\right)w(s)ds\right)dy\right\| \leq h,$$

for I_λ and $w \in \widetilde{H}$.

Now we show that the operator $G(\cdot)$ is well defined and maps the space $\widetilde{H}$ into $\widetilde{H}$. For this, let us assume there exists a functional $w^* \in E^*$ such that $||w^*|| \leq 1$. For any $t \in I_\lambda$ and $w \in \widetilde{H}$, we have

$$\begin{aligned}
& \Big| \langle w^*, G(w)(t) \rangle \Big| \\
\leq \; & \Big| \langle w^*, w_0 \rangle \Big| + \left| \left\langle w^*, \int_0^t f\left(y, w(y), (HKP) \int_0^y k_1(y,s)\, w(s) ds, \right.\right.\right. \\
& \left.\left.\left. \qquad (HKP) \int_0^y k_2(y,s)\, w(s) ds \right) dy \right\rangle \right| \\
\leq \; & \|w^*\| \, \|w_0\| + \|w^*\| \left\| \int_0^t f\left(y, w(y), (HKP) \int_0^y k_1(y,s)\, w(s) ds, \right.\right. \\
& \left.\left. \qquad (HKP) \int_0^y k_2(y,s)\, w(s) ds \right) dy \right\| \\
\leq \; & \|w_0\| + h,
\end{aligned}$$

which implies that

$$\sup \{ \langle w^*, \, G(w)(t) \rangle : \; w^* \in E^*, \, ||w^*|| \leq 1 \} \leq \|w_0\| + h.$$

Finally, it follows that

$$\|G(w)\| \leq \|w_0\| + h.$$

Thus, we obtain that $G(w) \in \widetilde{H}$.

Now we need to show that the operator $G(\cdot)$ is weakly-weakly sequentially continuous. To do this, we assume there exists a sequence $(w_n(t))_{n \in \mathbb{N}}$ in $C(I_\lambda, \, E)$ which converges weakly to $w(t)$ for $t \in I_\lambda$. Let $s \in [0, b]$. Then $k_i(t, \, s) w_n(s) \to k_i(t, \, s) w(s)$, as $n \to \infty$ for $i = 1\,, 2$. Using the assumptions on $\widetilde{X}_i$ we can obtain

$$\lim_{n \to \infty} \left\langle w^*, \int_0^t k_i(t, \, s) w_n(s) ds \right\rangle = \left\langle w^*, \, \int_0^t k_i(t, \, s) w(s) ds \right\rangle,$$

for $i = 1, \, 2$, $w^* \in E^*$ and $t \in I_\lambda$. Furthermore, the function f is weakly-weakly sequentially continuous, so for each $t \in I_\lambda$, we obtain

$$\begin{aligned}
& \left\langle w^*, f\left(t, w_n(t), (HKP) \int_0^t k_1(t,s)\, w_n(s), (HKP) \int_0^t k_2(t,s)\, w_n(s) \right) ds \right\rangle \\
& \to \left\langle w^*, f\left(t, w(t), (HKP) \int_0^t k_1(t,s)\, w(s), (HKP) \int_0^t k_2(t,s)\, w(s) ds \right) \right\rangle,
\end{aligned}$$

as $n \to \infty$, in E. Hence, using Theorem 2.12, we obtain that for $t \in I_\lambda$

$$\langle w^*, G w_n(t) \rangle \to \langle w^*, \, G w(t) \rangle \,,$$

as $n \to \infty$ for each $w^* \in E^*$, which implies that $Gw_n(t)$ converges to $Gw(t)$ in $(C(I_\lambda, E), w)$.

Now, assume that $U \subset \widetilde{H}$ fulfills the assumption $U = \overline{\text{conv}}\,(\{y\} \cup G(U))$. It will be proved in the sequel that U is relatively weakly compact. As $U \subset \widetilde{H}$, $G(U) \subset X$, then $U \subset \overline{U} = \overline{\text{conv}}\,(\{y\} \cup G(U))$ is equicontinuous. Therefore, by Lemma 2.3 we can see that the function $t \mapsto u(t) = \beta(U(t))$ satisfies the condition of continuity on the interval I_λ. For fixed $t \in I_\lambda$ we divide the intervals $[0, t]$ into k parts where $t_j = \dfrac{jt}{k}$, $i = 1, 2, \ldots, k$, and for fixed $z \in [0, t]$, we divide the interval $[0, z]$ into k parts where $z_j = \dfrac{jz}{k}$, $j = 1, 2, \ldots, k$. Furthermore, we consider

$$U\left([z_j,\ z_{j+1}]\right) = \{w(s)\,|w \in U,\, z_i \le s \le z_{j+1}\}\ j = 1, 2, \ldots, k.$$

As u is a continuous function on a closed interval, therefore by the intermediate value property there exist $q_j \in J_j = [z_j,\, z_{j+1}]$ so that

$$\beta\left(U\left([z_j,\ z_{j+1}]\right)\right) = \sup\left\{\beta(U(s)),\, z_i \le s \le z_{j+1}\right\} := u(q_j).$$

Using Theorem 2.12 as well as properties of the Henstock–Kurzweil–Pettis integral we can obtain that for $w \in U$,

$$\begin{aligned} G(w)(t) &= w_0 \\ &\quad + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} f\Bigg(z,\, w(z),\, \sum_{j=0}^{k-1} (HKP) \int_{z_j}^{z_{j+1}} k_1\,(z, s)\, w(s) ds, \\ &\quad \sum_{j=0}^{k-1} (HKP) \int_{z_j}^{z_{j+1}} k_2\,(z, s)\, w(s) ds\Bigg) dz, \\ &\in w_0 + \sum_{i=0}^{k-1} (\Delta t_i) \overline{\text{conv}} f\Bigg(I_i,\, U(I_i),\, \sum_{j=0}^{k-1} (\Delta z_j) \overline{\text{conv}} (k_1\,(J_j,\, J_j)\, U(J_j)), \\ &\quad \sum_{j=0}^{k-1} (\Delta z_j) \overline{\text{conv}} (k_2\,(J_j,\, J_j)\, U(J_j))\Bigg), \end{aligned}$$

where $I_i = [t_i,\, t_{i+1}]$ and $J_j = [z_j,\, z_{j+1}]$ for $i, j = 0, 1, \ldots k - 1$. Applying β, the measure of weak noncompactness, and using its properties, we obtain the following inequalities,

$$\begin{aligned} \beta\left(GU\,(t)\right) &\le \sum_{i=0}^{k-1} (\Delta t_i) \beta\Bigg(f\Bigg(I_i, U(I_i), \sum_{j=0}^{k-1} (\Delta z_j) \overline{\text{conv}} (k_1\,(J_j, J_j)\, U(J_j)), \\ &\quad \sum_{j=0}^{k-1} (\Delta z_j) \overline{\text{conv}} (k_2\,(J_j, J_j)\, U(J_j))\Bigg)\Bigg), \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{k-1} \Delta t_i q_1 \beta(U(I_i)) \\
&\quad + \sum_{i=0}^{k-1} \Delta t_i q_2 \beta \left(\sum_{j=0}^{k-1} (\Delta z_j) \overline{\text{conv}}(k_1\,(J_j, J_j)\, U(J_j)) \right) \\
&\quad + \sum_{i=0}^{k-1} \Delta t_i q_3 \beta \left(\sum_{j=0}^{k-1} (\Delta z_j) \overline{\text{conv}}(k_2\,(J_j, J_j)\, U(J_j)) \right), \\
&\leq \sum_{i=0}^{k-1} (\Delta t_i) q_1 \beta(U(I_\lambda)) + \sum_{i=0}^{k-1} (\Delta t_i) q_2 \sum_{j=0}^{k-1} \Delta z_j \beta(k_1(J_j, J_j) U(J_j)) \\
&\quad + \sum_{i=0}^{k-1} (\Delta t_i) q_3 \sum_{j=0}^{k-1} \Delta z_j \beta(k_2(J_i, J_i) U(J_i)), \\
&\leq \beta(U(I_\lambda)) q_1 \lambda + q_4 \sum_{i=0}^{k-1} \Delta t_i q_2 \sum_{j=0}^{k-1} \Delta z_j \beta(U(J_j)) \\
&\quad + q_5 \sum_{i=0}^{k-1} \Delta t_i q_3 \sum_{j=0}^{k-1} \Delta z_j \beta(U(J_j)), \\
&\leq \beta(U(I_\lambda)) q_1 \lambda + \beta(U(I_\lambda)) q_2 q_4 \lambda^2 + \beta(U(I_\lambda)) q_3 q_5 \lambda^2 \\
&= \beta(U(I_\lambda)) \left(q_1 \lambda + q_2 q_4 \lambda^2 + q_3 q_5 \lambda^2 \right).
\end{aligned}
$$

Using the given stated assumptions, we reach to the conclusion that

$$\beta\,(GU(t)) \leq \beta(U(I_\lambda)).$$

Hence, we conclude that

$$u(t) = \beta U(t) = 0$$

for all $t \in I_\lambda$. All hypotheses of the Arzelà–Ascoli's theorem are satisfied, so the collection U is relatively weakly compact. Thus, by the fixed point theorem, Theorem 2.7, the operator $G(\cdot)$ has a fixed point, which results in the existence of a pseudo solution for the system (3.1). □

Theorem 3.3. *Assume that for each* $t \in I_b$, *the functions* $k_1(t,\,\cdot)w(\cdot)$, $k_2(t,\,\cdot)w(\cdot)$ *and*

$$f\left(\cdot,\, w(\cdot),\, (HKP)\int_0^{(\cdot)} k_1\,(\cdot, s)\, w(\cdot) ds,\, (HKP)\int_0^{(\cdot)} k_2\,(\cdot, s)\, w(\cdot) ds\right)$$

are Henstock Kurzweil Pettis integrable for every uniformly ACG^* *function* $w : I_b \to E$, *while* $k_1(t,\, s)w(\cdot)$, $k_2(t,\, s)w(\cdot)$ *and* $f(t, \cdot, \cdot, \cdot)$ *are weakly–weakly sequentially continuous functions. Assume there exist positive constants* r_2 *and* r_3 *and functions* C_{1-b} *and*

C_{2-b} that are continuous on I_b into $\mathbb{R}_+$ satisfying the following conditions,

$$\beta(f(I, D_1, D_2, D_3)) \leq r_2\beta(D_2) + r_3\beta(D_3) \text{ for each } D_1, D_2, D_3 \subset H \text{ and } I \subset I_b,$$
$$\beta(k_1(I,\, I)w(Y)) \leq \sup_{s\in I} C_{1-b}(s)\beta(Y) \text{ for each subset } Y \text{ of } H,\ I \subset I_b,$$
$$\beta(k_2(I,\, I)w(Y)) \leq \sup_{s\in I} C_{2-b}(s)\beta(Y) \text{ for each subset } Y \text{ of } H,\ I \subset I_b,$$

where

$$\begin{aligned} f(I, D_1, D_2, D_3) &= \{f(t, y_1, y_2, y_3) | (t, y_1, y_2, y_3) \in I \times D_1 \times D_2 \times D_3\}, \\ k(I, I) &= \{k(t, s) | (t, s) \in I \times I\}. \end{aligned}$$

Furthermore, let X, X_1 and X_2 be equicontinuous as well as uniformly ACG^* on I_b. Then there exists a pseudo-solution of the problem (3.1) *on I_λ, for some $0 < \lambda \leq b$.*

Proof. Following along the same lines as in Theorem 3.2, we can obtain the proof of the first part. Now it needs to be shown that the set U, where U is defined in Theorem 3.2, is relatively weakly compact. One can see that for z_j and $t \in I_\lambda$ as in Theorem 3.2, we have

$$\begin{aligned} \beta\left(U\left(t\right)\right) \ &\leq\ \sum_{i=0}^{k-1}(\Delta t_i)\beta\left(f\left(I_i,\, U(I_i),\, \sum_{j=0}^{k-1}(\Delta z_j)\overline{\text{conv}}k_1\left(J_j,\, J_j\right)U(J_j),\right.\right. \\ &\qquad \left.\left. \sum_{j=0}^{k-1}(\Delta z_j)\overline{\text{conv}}k_2\left(J_j,\, J_j\right)U(J_j)\right)\right) \\ &\leq\ \sum_{i=0}^{k-1}\Delta t_i r_2\beta\left(\sum_{j=0}^{k-1}(\Delta z_j)\overline{\text{conv}}k_1\left(J_j,\, J_j\right)U(J_j)\right) \\ &\qquad + \sum_{j=0}^{k-1}\Delta t_i r_3\beta\left(\sum_{j=0}^{k-1}(\Delta z_j)\overline{\text{conv}}k_2\left(J_j,\, J_j\right)U(J_j)\right) \\ &\leq\ \sum_{i=0}^{k-1}\Delta t_i r_2\sum_{j=0}^{k-1}\Delta z_j\beta\left(k_1(J_j,\, J_j)U(J_j)\right) \\ &\qquad + \sum_{i=0}^{k-1}\Delta t_i r_3\sum_{j=0}^{k-1}\Delta z_j\beta\left(k_2(J_j,\, J_j)U(J_j)\right) \\ &\leq\ \sum_{i=0}^{k-1}\Delta t_i r_2\sum_{j=0}^{k-1}\Delta z_j\sup_{s\in J_j} C_{1-b}(s)(\beta U(J_j)) \\ &\qquad + \sum_{i=0}^{k-1}\Delta t_i r_3\sum_{j=0}^{k-1}\Delta z_j\sup_{s\in J_j} C_{2-b}(s)\beta(U(J_j)) \end{aligned}$$

$$
\begin{aligned}
&\leq \lambda r_2 \sum_{j=0}^{k-1} \Delta z_j C_{1-b}(p_j) u(s_j) + \lambda r_3 \sum_{j=0}^{k-1} \Delta z_j C_{2-b}(p_j) u(s_j), \\
&= \lambda r_2 \left[\sum_{j=0}^{k-1} \Delta z_j C_{1-b}(p_i) u(p_j) + \sum_{j=0}^{k-1} \Delta z_j C_{1-b}(p_j) \left(u(s_j) - u(p_j) \right) \right] \\
&\quad + \lambda r_3 \left[\sum_{j=0}^{k-1} \Delta z_j C_{2-b}(p_j) u(p_j) + \sum_{j=0}^{k-1} \Delta z_j C_{2-b}(p_j) \left(u(s_j) - u(p_j) \right) \right],
\end{aligned}
$$

for some $p_j \in J_j$. Let us fix $\epsilon > 0$. Since the function $u(\cdot)$ is continuous, there exists a natural number k, large enough such that $u(s_j) - u(p_j) < \epsilon$, $j = 0, \ldots, k-1$.. Then,

$$
\begin{aligned}
\beta\left(U\left(t\right)\right) &\leq \lambda r_2 \left[\sum_{j=0}^{k-1} \Delta z_j C_{1-b}(p_j) u(p_j) + \sum_{j=0}^{k-1} \frac{z}{k} C_{1-b}(p_j) \epsilon \right] \\
&+ \lambda r_3 \left[\sum_{j=0}^{k-1} \Delta z_j C_{2-b}(p_j) u(p_j) + \sum_{j=0}^{k-1} \frac{z}{k} C_{1-b}(p_j) \epsilon \right] \\
&\leq \lambda r_2 \left[\sum_{j=0}^{k-1} \Delta z_j C_{1-b}(p_j) u(p_j) + \max_{0 \leq l \leq k-1} \frac{z}{k} C_{1-b}(p_l) \epsilon \right] \\
&+ \lambda r_3 \left[\sum_{j=0}^{k-1} \Delta z_j C_{2-b}(p_j) u(p_j) + \max_{0 \leq l \leq k-1} \frac{z}{k} C_{2-b}(p_l) \epsilon \right].
\end{aligned}
$$

Also, we can see that $z \max\limits_{0 \leq l \leq k-1} C_{1-b}(p_l)$ as well as $z \max\limits_{0 \leq l \leq k-1} C_{2-b}(p_l)$ are bounded, and if $\epsilon \to 0$, then both $\epsilon z \max\limits_{0 \leq l \leq k-1} C_{1-b}(p_l)$ and $\epsilon z \max\limits_{0 \leq l \leq k-1} C_{2-b}(p_l)$ approach zero. Thus,

$$
\beta(U(t)) \leq \lambda. \int_0^t \left(r_2 C_{1-b}(\tau) + r_3 C_{2-b}(\tau) \right) \beta U(\tau) d\tau,
$$

for $t \in [0\,, \lambda]$. Application of Gronwall's inequality give us the result,

$$
\beta(U(t)) = 0,
$$

for $t \in [0,\ \lambda]$. The hypotheses of Arzelà–Ascoli's theorem are fulfilled, so U is relatively weakly compact and hence by the fixed point theorem, Theorem 2.7, the operator $G(\cdot)$ has a fixed point. The fixed point of the operator $G(\cdot)$ is the required pseudo solution of our proposed problem. □

4 Topological Properties of the Solution Set

This section is devoted to study topological properties of the set of solutions of problem (3.1). Considering the problem (3.1), it will be shown that the related set of all solutions is weakly compact and connected in the space $(C(I_\lambda,\ E), \omega)$.

Remark 4.1. According to Definition 3.1 a weakly continuous function is solution of problem (3.1) if it satisfy all three conditions given in the definition. So before we move towards the topological properties of the solution set we came to know from the nature of solution of (3.1), that we are in the space, $(C(I_\lambda,\, E), \omega)$.

Theorem 4.2. *Assume that the hypotheses of Theorem* 3.2 *hold. Then the set* Ω *of all solutions, defined on the interval* I_λ*, of* (3.1) *is connected and weakly compact in the space* $(C(I_\lambda,\, E), \omega)$.

Proof. Assume that Ω is a set of all solutions of (3.1) defined on I_λ. Since $\Omega = G(\Omega)$, following in the same lines as in the proof of Theorem 3.2 with $U = \Omega$, we can obtain that the set Ω is relatively weakly compact in $(C(I_\lambda,\, E), \omega)$. As $G(\cdot)$ is a weakly continuous function on $\overline{\Omega^\omega(I_\lambda)}$ (the weak closure of set $\Omega(I_\lambda)$), Ω is closed with respect to the weak topology and consequently is weakly compact.

Now it will be shown that the solution set, Ω, is a connected set in $(C(I_\lambda,\, E), \omega)$. Therefore, for any $\rho > 0$, we represent by Ω_ρ, the set of those functions $v : I_\lambda \to E$ obeying the following properties:

1. $v(0) = w_0,\ v \in \widetilde{H}$,
2. $$\sup_{I_\lambda} \left\| v(t) - w_0 - \int_0^t f\left(t,\, v(t),\, (HKP)\int_0^t k_1(t,s)\, v(s)ds,\right.\right.$$ $$\left.\left.(HKP)\int_0^t k_2(t,s)\, v(s)ds\right)\right\| < \rho.$$

It is trivial to see that the set Ω_ρ is nonempty because $\Omega \subset \Omega_\rho$. If we assume there exists a real number ρ^0 such that $\rho^0 < \rho$, then by the equicontinuity of the family of functions X we can find ξ such that

$$\left\| \int_J f\left(t, w(t), (HKP)\int_0^t k_1(t,s)\, w(s)ds, (HKP)\int_0^t k_2(t,s)\, w(s)ds\right)\right\| \le \rho^0 < \rho,$$

for every weakly continuous function w defined on I_λ into the Banach space E_w, $J \subset I_\lambda$ and $|J| < \xi$. For any $\epsilon \in (0,\, \lambda)$, suppose $\chi(\cdot,\, \epsilon) : I_\lambda \to E$ is a function defined by

$$\chi(t,\, \epsilon) := \begin{cases} w_0, & \text{if } 0 \le t \le \epsilon, \\ w_0 + \displaystyle\int_0^{t-\epsilon} f\left(\zeta,\, \chi(\zeta,\epsilon),\, (HKP)\int_0^\zeta k_1(\zeta,s)\, \chi(s,\epsilon)ds,\right. & \\ \qquad\qquad \left.(HKP)\displaystyle\int_0^\zeta k_2(\zeta,s)\, \chi(s,\epsilon)ds\right) d\zeta, & \text{if } \epsilon \le t \le \lambda. \end{cases}$$

One can easily observe that $\chi(\cdot,\, \epsilon)$ satisfies the Condition 1. Moreover, for $0 < \epsilon \le \min(\xi, \lambda) := m$, we obtain a norm which is defined by

$$\left\| \chi(t,\, \epsilon) - w_0 - \int_0^t f\left(\zeta,\, \chi(\zeta,\epsilon),\, (HKP)\int_0^\zeta k_1(\zeta,s)\, \chi(s,\epsilon)ds,\right.\right.$$

$$(HKP)\int_0^{\zeta} k_2\left(\zeta,s\right)\chi(s,\epsilon)ds\Big)\, d\zeta\Big\|$$

$$=\begin{cases} \Big\|\displaystyle\int_0^t f\Big(\zeta,\,\chi(\zeta,\epsilon),\,(HKP)\int_0^{\zeta} k_1\left(\zeta,s\right)\chi(s,\epsilon)ds, \\ \qquad\qquad (HKP)\displaystyle\int_0^{\zeta} k_2\left(\zeta,s\right)\chi(s,\epsilon)ds\Big)\, d\zeta\Big\| & \text{if } 0\le t\le\epsilon, \\ \Big\|\displaystyle\int_{t-\epsilon}^t f\Big(\zeta,\,\chi(\zeta,\epsilon),\,(HKP)\int_0^{\zeta} k_1\left(\zeta,s\right)\chi(s,\epsilon)ds, \\ \qquad\qquad (HKP)\displaystyle\int_0^{\zeta} k_2\left(\zeta,s\right)\chi(s,\epsilon)ds\Big)\, d\zeta\Big\| & \text{if } \epsilon\le t\le\lambda. \end{cases} \tag{4.1}$$

It follows that the norm (4.1) is less than ρ^0, and furthermore, $\rho^0 \le \rho$, for all $t \in J$. Hence, we have that $\chi(\cdot,\,\epsilon)$ fulfills Condition 2.

In order to show that the set Ω_ρ is connected let us define a function

$$\chi_\epsilon(t) := \begin{cases} w_0 & \text{if } \ 0\le t\le\epsilon, \\ G\left(\chi_\epsilon\right)(t-\epsilon) & \text{if } \ \epsilon< t\le\lambda, \end{cases}$$

where $\chi_\epsilon = \chi\left(\cdot,\,\epsilon\right)$. Now, we will prove that the function $\epsilon \to \chi_\epsilon(\cdot)$ is sequentially continuous for $0<\epsilon\le\lambda$ such that $\chi_\epsilon(\cdot) \in \left(C(I_\lambda,\,E),\omega\right)$. Without loss of generality assume $0<\epsilon\le\delta\le\lambda$. Consider a functional $w^* \in E^*$, such that $\|w^*\| \le 1$. Now, for $t\in[0,\epsilon]$, by the definition of the function $\chi_\epsilon(t)$, it follows that,

$$\langle w^*, (\chi_\epsilon(t)-\chi_\delta(t))\rangle = 0. \tag{4.2}$$

Now for $t\in(\epsilon,\,\delta]$, we obtain

Thus, one can infer that

$$\langle w^*, \chi_\epsilon(t)-\chi_\delta(t)\rangle \le \Big\|\int_{t-\delta}^{t-\epsilon} f\Big(\zeta,\,\chi_\epsilon(\zeta),\,(HKP)\int_0^{\zeta} k_1\left(\zeta,s\right)\chi_\epsilon(s)ds, \\ (HKP)\int_0^{\zeta} k_2\left(\zeta,s\right)\chi_\epsilon(s)ds\Big)\, d\zeta\Big\| = \Xi_\delta. \tag{4.3}$$

As the space X is equicontinuous, an observation shows that $\Xi_\delta \to 0$ as δ approaches to ϵ. Also, for $\delta<t<2\delta$, we have

$$\begin{aligned} \langle w^*, \chi_\epsilon(t)-\chi_\delta(t)\rangle &\le \|w^*\|\,\|G(\chi_\epsilon)(t-\epsilon)-G(\chi_\epsilon)(t-\delta)\| \\ &+ \|w^*\|\,\|G(\chi_\epsilon)(t-\delta)-G(\chi_\delta)(t-\delta)\|\,. \end{aligned} \tag{4.4}$$

Now we consider a sequence $(\delta_n)_{n\in\mathbb{N}}$ that converges to ϵ as $n \to \infty$. Using (4.2) and (4.3) one can obtain that $\chi_{\delta_n}(t)$ converges weakly uniformly to $\chi_\epsilon(t)$ for $t \in [0,\delta]$. Therefore, $G(\chi_{\delta_n}) \to wG(\chi_\epsilon)(t))$ on the interval $[0,\delta]$. Now by (4.4), $\lim_{n\to\infty} \chi_{\delta_n}(t) = \chi_\epsilon(t)$, where the convergence is in the weak sense for $t \in [0, 2\delta]$. While using induction and following in the same lines as above we can get that the function from $(0,\lambda)$ to $(C(I_\lambda, E))$, defined by $\epsilon \to \chi_\epsilon(\cdot)$, is sequentially continuous. Hence, the set $V = \{\chi_\epsilon(.)|\epsilon \in (0\ \lambda)\}$ is connected in the space $(C(I_\lambda, E), \omega)$. Let $w \in \Omega_\rho$ and consider a parameter $\epsilon \in (0,\lambda)$ as well as

$$\begin{aligned}\sup_{t\in I_\lambda} \Bigg\| w(t) - w_0 - \int_0^t f\Bigg(z,\, w(z),\, (HKP)\int_0^z k_1(z,s)\,w(s)ds,\\ (HKP)\int_0^z k_2(z,s)\,w(s)ds\Bigg)\,dz\Bigg\|\\ +\Bigg\|\int_{I_\epsilon} f\Bigg(z,\, w(z),\, (HKP)\int_0^z k_1(z,s)\,w(s)ds,\\ (HKP)\int_0^z k_2(z,s)\,w(s)ds\Bigg)\,dz\Bigg\| < \rho.\end{aligned}$$

Let a function $\Theta(\cdot,q) : I_\lambda \to E$, where $q \in (0,\lambda)$, be defined piecewise by

$$\Theta(t,\,q) = \begin{cases} w(t); & \text{for } 0 \le t \le q\\ w(q) + \dfrac{w_0 - w(q)}{\epsilon}(t-q); & \text{for } q \le t \le \min(\lambda, q+\epsilon)\\ w_0 + \displaystyle\int_q^{t-\epsilon} f\Big(z,\, w(z),\, (HKP)\int_0^z k_1(z,s)\,w(s)ds, & \\ \qquad\qquad (HKP)\displaystyle\int_0^z k_2(z,s)\,w(s)ds\Big)dz; & \\ & \text{for } \min(\lambda,\ q+\epsilon) \le t \le \lambda, \end{cases}$$

and $\Theta(t,0) = \chi(t,\epsilon)$. Replacing $\chi(\cdot,\epsilon)$ by $\Theta(\cdot,q)$, and repeating the preceding argument, it is easy to get that $\Theta(\cdot,q) \in \Omega_\rho$, for such $q \in I_\lambda$. Moreover, the function $q \to \Theta(\cdot,q)$ with domain $[0,\ \lambda]$ into the space $(C(I_\lambda, E), \omega)$, is a sequentially continuous function. This results in the fact that, the set $T_w = \{\Theta(\cdot,q)|q \in [0,\ \lambda]\}$ is a connected subset of $(C(I_\lambda, E), \omega)$. As it is known that $\Theta(t,0) = \chi(t,\epsilon)$ is a connected set and is a subset of $V \cap T_w$, hence the set $Z = \cup_{w\in\Omega_\rho} T_w \cup V$ is a connected subset in the space $(C(I_\lambda, E), \omega)$. Furthermore, $\Omega_\rho \subseteq Z$, due to the fact that $w = \Theta(\cdot,q) \in T_w$, for which $w \in \Omega_\rho$. While $Z \subseteq \Omega_\rho$, as $T_w \subseteq \Omega_\rho$ and $v \subset \Omega_\rho$. Therefore, $\Omega_\rho = Z$ is a connected subset of $(C(I_\lambda, E), \omega)$.

We now suppose that the solution set Ω is disconnected. Since it has been proved that Ω is a weakly compact set, then we can find weakly compact sets $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$, such that $B_1 \cup B_2 = \Omega$. Therefore, there exist two disjoint weakly open subsets K_1 and K_2, so that $B_1 \cap B_2 = \emptyset$ and $B_2 \subset K_2$. Assume that, for each natural number

n, we can find $x_n \in V_n - U$, here $V_n = \overline{\Omega^\omega}_{\frac{1}{n}}$ and $U = K_1 \cup K_2$. This suggests that each V_n is a nonempty, weakly compact and connected subset of $(C(I_\lambda, E), \omega)$ which is also monotonically decreasing.

Assume that $J = \overline{\{x_n | n \in \mathbb{N}\}^\omega}$. We notice, $\lim_{n\to\infty} (x_n - G(x_n)) = 0$ in the space $(C(I_\lambda, E), \omega)$ and $J(t) \subset \{x_n(t) - G(x_n)(t) | x_n \in J\} + G(J)(t)$. Following in the same lines as we did in Theorem 3.2, it can be shown that we can find $x_0 \in J$ such that $x_0 = G(x_0)$, i.e., $x_0 \in \Omega$. Thus, the set U is open with respect to the weak topology and $x_n \in V_n - U$, hence $x_0 \notin U$, which contradicts the fact that $x_0 \in \Omega \subset U$. Hence, we can find $l \in \mathbb{N}$ such that $V_l \subset U = U_1 \cup K_2$ with$K_1 \cap K_2 = \emptyset$. Since, $\Omega \subset V_l$ we have that $K_1 \cap V_l \neq \emptyset \neq K_2 \cap V_l$. Therefore, V_l is not a connected set, and that is a contradiction to the fact that each V_n is connected. Therefore, the set of all solutions of problem (3.1) is a connected subset in $(C(I_\lambda, E), \omega)$. □

5 Conclusions

Functions that are of highly oscillatory behavior has wide range of applications in physics, engineering, finance etc. [16]. These kinds of functions are integrable in the sense of the Henstock–Kurzweil–Pettis(HKP) definition. This newly introduced work uses the HKP integral to formulate a novel integrodifferential equation. The results obtained in this paper are about existence of solutions, and some topological aspects of the underlying problem. Under different hypotheses various existence results have been explored and it is shown that the solution set is a compact and connected subset of $(C(I_\lambda,\ E), \omega)$. The results have been obtained via applying various tools of the fixed point theory and the measure of weak noncompactness.

The topological generalization of finiteness is considered as compactness. Basically topology deals with open sets and this assertion provides the information that behavior of some mathematical notion with open sets is same as its behavior with the entire ground set. Consequently, compactness of a space gives information that there exist at most finitely many behaviors. Similarly another important property of topological spaces is connectedness. This property of a space or subspace is among few properties of geometric figures that denote change under a homeomorphism. Without loss of generality if we assume Euclidean plane, then a point is limit point if there does not exist minimum distance from the point to elements of the set. So a set is not connected if it can be segregated into two subsets such that an element of one set is never a limit point of other subset. Therefore, our obtained result about the compactness is to show that there can be finitely many behaviors with the solution set of integrodifferential equations. Similarly, we have shown that the solution set Ω is connected, so it will predict that solution sets of other kinds of equations that are homeomorphic to Ω will have the same properties as Ω.

Concisely, the integrodifferential equations explored in the present manuscript are of fairly general nature and include a variety of particular cases. For $k_2(\cdot, \cdot) = 0$ we obtain the results carried out in [1]. Last but not least, the present work will provide a new way of thinking to formulate novel integrodifferential equations. In addition, this work will motivate researchers to use other kinds of vector valued integrals to formulate a variety of (ordinary / partial / fractional order) differential, integral and integrodifferential equations.

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