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The Role of Symmetry and Concavity in the Existence of Solutions of a Difference Equation with Dirichlet Boundary Conditions

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Abstract

In this paper, the layered compression-expansion fixed point theorem is applied to show the existence of solutions of a second order difference equation with Dirichlet boundary conditions where the nonlinearity is the sum of a monotonic increasing and a monotonic decreasing function. A cone consisting of nonnegative symmetric functions that satisfy a concavity condition is integral to the analysis.

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1 Introduction

Let $N \in \mathbb{N}$, $N > 1$. In this paper, we give an application of an Avery, Anderson, and Henderson fixed point theorem to obtain at least one positive symmetric solution of the difference equation

$$\Delta^2 u(k) + f(u(k)) = 0, \quad k \in \{0, 1, ..., N\}, \tag{1.1}$$

with boundary conditions

$$u(0) = u(N+2) = 0, \tag{1.2}$$

where $f : [0, \infty) \to [0, \infty)$ and Δ^2 is the second forward difference operator which acts on u by $\Delta^2 u(k) = u(k+2) - 2u(k+1) + u(k)$. We assume $f = f_\uparrow + f_\downarrow$ is the sum of a monotonic increasing and a monotonic decreasing function, respectively,

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where $f_\uparrow, f_\downarrow : [0,\infty) \to [0,\infty)$. Under certain conditions imposed on$f_\uparrow$ and $f_\downarrow$, we show (1.1), (1.2) has a positive symmetric solution in the sense that $u(k) \geq 0$ and $u(N+2-k) = u(k)$ for all $k \in \{0,1,...,N+2\}$.

Fixed point theorems due to Avery, Anderson, and Henderson and others, (see, for example, [4, 5, 7–9, 11–13]) when applied to boundary value problems with right focal boundary conditions, require functions from a cone to be nonnegative, nondecreasing and satisfy a concavity-like property [3, 14–17]. Boundary value problems with conjugate boundary conditions can also be studied using these fixed point theorems, but now the functions from the cone are required to be nondecreasing on half the interval and symmetric [1, 2, 10, 18]. Then the maximum value of the function occurs at the midpoint of the interval instead of the right endpoint, and by requiring symmetry, a similar approach can be taken.

In this paper, we see that by writing the nonlinearity f as a sum of a monotonic increasing and a monotonic decreasing function, the layered compression-expansion fixed point theorem [6] can be used to show the existence of positive symmetric solutions of a difference equation with Dirichlet boundary conditions. The proof of the main theorem relies on the fact that functions from the cone are symmetric and satisfy a concavity property.

2 The Fixed Point Theorem

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $\mathcal{P} \subset E$ is called a cone provided:

(i) $u \in \mathcal{P}$, $\lambda \geq 0$ implies $\lambda u \in \mathcal{P}$;

(ii) $u \in \mathcal{P}$, $-u \in \mathcal{P}$ implies $u = 0$.

Definition 2.2. A map α is said to be a nonnegative continuous concave functional on a cone $\mathcal{P}$ of a real Banach space E if

$$\alpha : \mathcal{P} \to [0,\infty)$$

is continuous and

$$\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v),$$

for all $u, v \in \mathcal{P}$ and $t \in [0,1]$.

Similarly, the map β is a nonnegative continuous convex functional on a cone $\mathcal{P}$ of a real Banach space E if

$$\beta : \mathcal{P} \to [0,\infty)$$

is continuous and

$$\beta(tu + (1-t)v) \leq t\beta(u) + (1-t)\beta(v),$$

for all $u, v \in \mathcal{P}$ and $t \in [0,1]$.

Let $\mathcal{P}$ be a cone, let u and v be real numbers, ϕ be a continuous concave functional and ξ a continuous convex functional. Define

$$\mathcal{P}(\phi, u, \xi, v) = \{x \in \mathcal{P} : \phi(x) < u \text{ and } \xi(x) > v\}.$$

We employ the following fixed point theorem to show the existence of positive symmetric solutions of (1.1), (1.2).

Lemma 2.3 (See [12]). *Suppose $\mathcal{P}$ is a cone in a real Banach space E, α and ψ are nonnegative continuous concave functionals on $\mathcal{P}$, β and θ are nonnegative continuous convex functionals on $\mathcal{P}$, and R, S, T are completely continuous operators on $\mathcal{P}$ with $T = R + S$. If there exist nonnegative real numbers a, b, c, d and $(r_0, s_0) \in \mathcal{P}(\beta, b, \alpha, a) \times \mathcal{P}(\theta, c, \psi, d)$ such that*

(A0) $\mathcal{P}(\beta, b, \alpha, a) \times \mathcal{P}(\theta, c, \psi, d)$ *is bounded;*

(A1) if $r \in \partial\mathcal{P}(\beta, b, \alpha, a)$ *with* $\alpha(r) = a$ *and* $s \in \overline{\mathcal{P}(\theta, c, \psi, d)}$, *then* $\alpha(R(r + s)) > a$;

(A2) if $r \in \partial\mathcal{P}(\beta, b, \alpha, a)$ *with* $\beta(r) = b$ *and* $s \in \overline{\mathcal{P}(\theta, c, \psi, d)}$, *then* $\beta(R(r + s)) < b$;

(A3) if $s \in \partial\mathcal{P}(\theta, c, \psi, d)$ *with* $\theta(s) = c$ *and* $r \in \overline{\mathcal{P}(\beta, b, \alpha, a)}$, *then* $\theta(S(r + s)) < c$; *and*

(A4) if $s \in \partial\mathcal{P}(\theta, c, \psi, d)$ *with* $\psi(s) = d$ *and* $r \in \overline{\mathcal{P}(\beta, b, \alpha, a)}$, *then* $\psi(S(r + s)) > d$;

then there exists an $(r^*, s^*) \in \mathcal{P}(\beta, b, \alpha, a) \times \mathcal{P}(\theta, c, \psi, d)$ such that $x^* = r^* + s^*$ is a fixed point for T.

3 Preliminaries

Let E be Banach space

$$E = \{u : \{0, \ldots, N + 2\} \to \mathbb{R}\}$$

with the usual supremum norm

$$\|u\| = \max_{k \in \{0,1,\ldots,N+2\}} |u(k)|.$$

The corresponding Green's function for $-\Delta^2 u = 0$ satisfying the boundary conditions (1.2) is given by

$$H(k, l) = \frac{1}{N + 2} \begin{cases} k(N + 2 - l), & k \in \{0, \ldots, l\}, \\ l(N + 2 - k), & k \in \{l + 1, \ldots N + 2\}. \end{cases}$$

Thus if u is a solution of the summation equation

$$u(k) = \sum_{l=1}^{N+1} H(k,l) f(u(l)),$$

then u is a solution of the boundary value problem (1.1), (1.2). We also point out that for fixed k,

$$\sum_{l=1}^{N+1} H(k,l) = \frac{k}{2}(N+2-k).$$

For notational purposes, define

$$\bar{N} = \left\lfloor \frac{N+2}{2} \right\rfloor$$

to be the greatest integer less than or equal to $\dfrac{N+2}{2}$. Define the cone $\mathcal{P} \subset E$ by

$$\mathcal{P} = \{u \in E : u(0) = 0,\ u(N+2-k) = u(k),\ u \text{ is nonnegative and nondecreasing on } \left\{0, 1, \ldots, \bar{N}\right\},\ \text{and } wu(y) \geq yu(w) \text{ for } w \geq y \text{ with } y, w \in \left\{0, 1, \ldots, \bar{N}\right\}\right\}.$$

Notice when N is even, if $u \in \mathcal{P}$, then the maximum value of u occurs when $k = \bar{N} = \dfrac{N+2}{2}$, and if $N+2$ is odd, the maximum value of u occurs when $k = \bar{N} = \dfrac{N+2-1}{2}$, and because of symmetry, at $k = \bar{N}+1 = \dfrac{N+2+1}{2}$.

Let $\tau, \nu \in \{1, \ldots, \bar{N}-1\}$. For $u \in \mathcal{P}$, define the nonnegative continuous convex functionals β and θ on $\mathcal{P}$ by

$$\beta(u) = \max_{l \in \{0,\ldots,\bar{N}\}} u(l) = u(\bar{N}),$$

and

$$\theta(u) = \max_{l \in \{0,\ldots,\tau\}} u(l) = u(\tau),$$

and the nonnegative continuous concave functionals α and ψ on $\mathcal{P}$ by

$$\alpha(u) = \min_{l \in \{\nu,\ldots,\bar{N}\}} u(l) = u(\nu),$$

and

$$\psi(u) = u(\bar{N}).$$

4 Positive Symmetric Solutions

Theorem 4.1. *Let* $\tau, \nu \in \{1, \dots, \bar{N}-1\}$ *and let* a, b, c, d *be nonnegative real numbers with* $b > a$. *If* $f_\downarrow, f_\uparrow : [0,\infty) \to [0,\infty)$ *are continuous with*

(1) $f_\uparrow\left(a + \frac{\bar{N}}{\nu} d\right) > \frac{2a}{\nu(N+1-2\nu)}$;

(2) $f_\uparrow\left(b + \frac{\bar{N}}{\tau} c\right) < \frac{2b}{(\bar{N}(N+2-\bar{N})}$;

(3) $f_\downarrow(0) < \frac{2c}{\tau(N+2-\tau)}$; *and*

(4) $f_\downarrow(b+d) > \frac{2d}{\bar{N}(N+2-\bar{N})}$.

then there exists a solution $u^* \in \mathcal{P}$ *of* (1.1)*,* (1.2)*.*

Proof. Define the operators $T, R, S : E \to E$ by

$$Tu(k) = \sum_{l=1}^{N+1} H(k,l) f(u(l)),$$

$$Ru(k) = \sum_{l=1}^{N+1} H(k,l) f_\uparrow(u(l)),$$

and

$$Su(k) = \sum_{l=1}^{N+1} H(k,l) f_\downarrow(u(l)).$$

Now $T = R + S$, and if u is a fixed point of T, then u is a solution of the boundary value problem (1.1), (1.2).

Notice H has the properties that

$$H\left(N+2-k, N+2-l\right) = H\left(k,l\right), \tag{4.1}$$

and

$$wH\left(y,l\right) \geq yH\left(w,l\right) \tag{4.2}$$

for all $w, y \in \{0, \dots, N+2\}$ with $w \geq y$. Therefore, $T, R, S : \mathcal{P} \to \mathcal{P}$. A standard application of the Arzelà–Ascoli theorem shows that T, R, and S are completely continuous.

We now show that (A0) holds. Let $(r,s) \in \mathcal{P}(\beta, b, \alpha, a) \times \mathcal{P}(\theta, c, \psi, d)$. Now,

$$r(k) \leq r(\bar{N}) = \beta(r) < b,$$

and

$$s(t) \leq s(\bar{N}) \leq \frac{\bar{N}}{\tau} s(\tau) < \frac{\bar{N}}{\tau} c.$$

Since r and s achieve their maximum values at $k = \bar{N}$, $\mathcal{P}(\beta, b, \alpha, a) \times \mathcal{P}(\theta, c, \psi, d)$ is bounded.

Here, we show (A1) holds. To do this, start by letting $r \in \partial\mathcal{P}(\beta, b, \alpha, a)$ with $\alpha(r) = a$ and $s \in \overline{\mathcal{P}(\theta, c, \psi, d)}$. Then, for $l \in \{\nu + 1, \ldots, N + 1 - \nu\}$,

$$r(l) + s(l) \geq r(\nu) + s(\nu) \geq r(\nu) + \frac{\bar{N}}{\nu} s(\bar{N}) \geq a + \frac{\bar{N}}{\nu} d.$$

This implies that if $l \in \{\nu + 1, \ldots, N + 1 - \nu\}$, $f_{\uparrow}(r(l) + s(l)) \geq f_{\uparrow}\left(a + \frac{\bar{N}}{\nu} d\right)$. By assumption (1),

$$\begin{aligned}
\alpha(R(r+s)) &= \sum_{l=1}^{N+1} H(\nu, l)\, f_{\uparrow}(r(l) + s(l)) \\
&\geq \sum_{l=\nu+1}^{N+1-\nu} H(\nu, l)\, f_{\uparrow}(r(l) + s(l)) \\
&\geq f_{\uparrow}\left(a + \frac{\bar{N}}{\nu} d\right) \sum_{l=\nu+1}^{N+1-\nu} \frac{\nu(N+2-l)}{N+2} \\
&= f_{\uparrow}\left(a + \frac{\bar{N}}{\nu} d\right) \frac{\nu}{2}(N + 1 - 2\nu) \\
&> a.
\end{aligned}$$

This implies that (A1) holds.

We show (A2) holds. Let $r \in \partial\mathcal{P}(\beta, b, \alpha, a)$ with $\beta(r) = b$ and $s \in \overline{\mathcal{P}(\theta, c, \psi, d)}$. Since r and s attain their maximum values at $l = \bar{N}$, for $l \in \{1, \ldots, \bar{N}\}$,

$$r(l) + s(l) \leq r(\bar{N}) + s(\bar{N}) \leq r(\bar{N}) + \frac{\bar{N}}{\tau} s(\tau) \leq b + \frac{\bar{N}}{\tau} c.$$

Therefore, for $l \in \{1, \ldots, N+1\}$, $f_{\uparrow}(r(l) + s(l)) \leq f_{\uparrow}(b + \frac{\bar{N}}{\tau} c)$. Assumption (2) gives that

$$\begin{aligned}
\beta(R(r+s)) &= \sum_{l=1}^{N+1} H(\bar{N}, l) f_{\uparrow}(r(l) + s(l)) \\
&\leq f_{\uparrow}\left(b + \frac{\bar{N}}{\tau} c\right) \sum_{l=1}^{N+1} H(\bar{N}, l)
\end{aligned}$$

$$\leq f_{\uparrow}\left(b + \frac{\bar{N}}{\tau}c\right)\frac{\bar{N}}{2}(N + 2 - \bar{N})$$
$$< b.$$

The next step in the proof is to show (A3) holds. To do that, let $s \in \partial\mathcal{P}(\theta, c, \psi, d)$ with $\theta(s) = c$ and $r \in \overline{\mathcal{P}(\beta, b, \alpha, a)}$. Now $f_{\downarrow}(0) \geq f_{\downarrow}(r(l) + s(l))$ for all $l \in \{1, \dots, \bar{N}\}$. By (3),

$$\begin{aligned}
\theta(S(r+s)) &= \sum_{i=1}^{N+1} H(\tau, l) f_{\downarrow}(r(l) + s(l)) \\
&\leq f_{\downarrow}(0) \sum_{i=1}^{N+1} H(\tau, l) \\
&= f_{\downarrow}(0)\frac{\tau}{2}(N + 2 - \tau) \\
&< c.
\end{aligned}$$

Finally, we show (A4) holds. Let $s \in \partial\mathcal{P}(\theta, c, \psi, d)$ with $\psi(s) = d$ and $r \in \overline{\mathcal{P}(\beta, b, \alpha, a)}$. For $l \in \{1, \dots, N+1\}$,

$$r(l) + s(l) \leq r(\bar{N}) + s(\bar{N}) \leq b + d.$$

Then for $l \in \{1, \dots, N+1\}$, $f_{\downarrow}(r(l) + s(l)) \geq f_{\downarrow}(b + d)$. By assumption (4), we have

$$\begin{aligned}
\psi(S(r+s)) &= \sum_{l=1}^{N+1} H(\bar{N}, l) f_{\downarrow}(r(l) + s(l)) \\
&\geq f_{\downarrow}(b + d) \sum_{l=1}^{N+1} H(\bar{N}, l) \\
&= f_{\downarrow}(b + d)\frac{\bar{N}}{2}(N + 2 - \bar{N}) \\
&> d.
\end{aligned}$$

Therefore, by Theorem 2.3, T must have a fixed point $u^* \in \mathcal{P}$, and u^* is a solution of (1.1), (1.2). □

Example 4.2. Let $N = 22$, $\tau = \nu = 6$, $a = d = 0$ and $b = c = 1$. Assumptions (1)–(4) of Theorem 4.1 reduce to

(1) $f_{\uparrow}(0) > 0$;

(2) $f_{\uparrow}(3) < \dfrac{1}{72}$;

(3) $f_{\downarrow}(0) < \dfrac{1}{54}$; and

(4) $f_{\downarrow}(2) > 0$.

If f is any function where $f = f_{\uparrow} + f_{\downarrow}$ with these four assumptions holding, then (1.1), (1.2) has a positive symmetric solution.

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