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Differentiation with Respect to Parameters of Solutions of Nonlocal Boundary Value Problems for Higher-Order Differential Equations

Jeffrey W. Lyons
The Citadel
Department of Mathematical Sciences
Charleston, SC 29409, USA
jlyons3@citadel.edu

Abstract

We consider the nth-order parameter dependent differential equation satisfying Dirichlet conditions and nonlocal boundary conditions. After imposing continuity and uniqueness conditions, solutions of the boundary value problem are differentiated with respect to the parameter. This new equation is shown to solve a nonhomogeneous boundary value problem similar to the associated variational equation.

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1 Introduction

Consider the nth-order parameter dependent boundary value problem

$$y^{(n)} = \left(x, y, y', y'', \ldots, y^{(n-1)}, \lambda\right),\ a < x < b, \tag{1.1}$$

satisfying the Dirichlet conditions

$$y^{(i-1)}\left(x_1\right) = y_i,\ 1 \leq i \leq n-1,$$

and nonlocal boundary condition

$$y\left(x_2\right) - \sum_{j=1}^{m} p_j y\left(\chi_j\right) = y_n,$$

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where $a < x_1 < \chi_1 < \cdots < \chi_m < x_2 < b$, $p_1, \ldots, p_m, y_1, y_2, \ldots, y_n \in \mathbb{R}$.

A few hypotheses are imposed upon (1.1):

(H1) $f(x, u_1, \ldots, u_n, \lambda) : (a, b) \times \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous,

(H2) $\dfrac{\partial f}{\partial u_i}(x, u_1, \ldots, u_n, \lambda) : (a, b) \times \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous, $i = 1, 2, \ldots, n$,

(H3) $\dfrac{\partial f}{\partial \lambda}(x, u_1, \ldots, u_n, \lambda) : (a, b) \times \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous, and

(H4) solutions of initial value problems for (1.1) extend to (a, b).

Remark 1.1. Note that (H4) is not a necessary condition but lets us avoid continually making statements about maximal intervals of existence inside (a, b).

The main motivation for this work is a recent paper by Henderson and Jiang [13] in which the authors considered an nth-order parameter dependent difference equation with Dirichlet and nonlocal boundary conditions. Henderson and Jiang imposed continuity conditions upon the nonlinearity and uniqueness assumptions upon solutions of the boundary value problem and the associated variational equation. These conditions allowed the authors to seek a derivative of the solution of the boundary value problem with respect to the parameter. This derivative solved an associated nonhomogeneous equation of the given difference equation.

Research on the relationship between the solution of a differential, difference, or dynamic equation and its associated variational equation is rooted in a result found in Hartman [10] attributed to Peano that discusses initial value problems. Since then, authors have used continuous dependence on boundary value results to establish analogous results for boundary value problems. In the realm of differential equations, one finds results for right–focal problems [11], functional problems [7,8], and nonlocal, multipoint, and integral problems [9, 16, 18, 19, 22]. For difference equations, we point to analogous results to those of differential equations [1,3–6,12,15,20]. Lyons [21] published a second-order dynamic equation result for the $h\mathbb{Z}$ time scale which was later generalized to a second-order dynamic equation on an arbitrary time scale [2].

In Section 2, the reader will find preliminary definitions, assumptions, and results to include Peano's theorem with a parameter. Section 3 is where our main result is found; an analogue of both Peano's theorem and Henderson and Jiang's theorem.

2 Preparatory Work

First, we present the variational equation and a related nonhomogeneous equation followed by Peano's result with a parameter.

Definition 2.1. Given a solution $y(x)$ of (1.1), we define the variational equation along $y(x)$ by

$$z^{(n)} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} \left(x, y, y', \ldots, y^{(n-1)}, \lambda\right) z^{(i-1)}. \tag{2.1}$$

Definition 2.2. Given a solution $y(x)$ of (1.1), an associated nonhomogeneous equation is given by

$$z^{(n)} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} \left(x, y, y', \ldots, y^{(n-1)}, \lambda\right) z^{(i-1)} + \frac{\partial f}{\partial \lambda} \left(x, y, y', \ldots, y^{(n-1)}, \lambda\right). \tag{2.2}$$

Theorem 2.3 (parameter Peano)**.** *Assume that, with respect to* (1.1)*, conditions (H1)–(H4) are satisfied. Let* $x_0 \in (a, b)$ *and let* $y(x) := y\left(x; x_0, c_1, c_2, \ldots, c_n, \lambda\right)$ *denote the solution of* (1.1) *satisfying the initial conditions* $y^{(i-1)}\left(x_0\right) = c_i,\ i = 1, \ldots, n$*. Then*

(A) for each $j = 1, \ldots, n$*,* $\alpha_j(x) := \dfrac{\partial y}{\partial c_j}(x)$ *exists on* (a, b) *and is the solution of the variational equation* (2.1) *along* $y(x)$ *satisfying the initial conditions*

$$\alpha_j^{(i-1)}\left(x_0\right) = \delta_{ij},\ i = 1, \ldots, n.$$

(B) $\beta(x) := \dfrac{\partial y}{\partial x_0}(x)$ *exists on* (a, b) *and is the solution of the variational equation* (2.1) *along* $y(x)$ *satisfying the initial conditions*

$$\beta^{(i-1)}\left(x_0\right) = -y^{(i)}\left(x_0\right),\ i = 1, \ldots, n.$$

(C) $\Lambda(x) := \dfrac{\partial y}{\partial \lambda}(x)$ *exists on* (a, b) *and is the solution of the nonhomogeneous equation* (2.2) *along* $y(x)$ *satisfying the initial conditions*

$$\Lambda^{(i-1)}\left(x_0\right) = 0,\ i = 1, \ldots, n.$$

Remark 2.4. The primary focus of this work is differentiation with respect to the parameter λ; a BVP analogue to part (C) of Peano's theorem. Therefore, we will not consider differentiation with respect to the boundary data. However, the approach is quite similar to that presented here, and we refer the interested reader to the referenced works in the introduction for those details.

Next, we present two uniqueness conditions that allow us to connect initial value problems to boundary value problems. The first assumption applies to (1.1) and second to (2.1).

(H5) If $y(x)$ and $z(x)$ are solutions of (1.1) such that $y^{(i-1)}(x_1) = z^{(i-1)}(x_1)$, $i = 1, \dots, n$ and $y(x_2) - \sum_{j=1}^{m} p_j y(\chi_j) = z(x_2) - \sum_{j=1}^{m} p_j z(\chi_j)$, then $y(x) \equiv z(x)$ on (a, b).

(H6) Let $y(x)$ be a solution of (1.1) and $u(x)$ be a solution of (2.1) along $y(x)$. Then if $u^{(i-1)}(x_1) = 0,\ i = 1, \dots, n$ and $u(x_2) - \sum_{j=1}^{m} p_j u(\chi_j) = 0$, then $u(x) \equiv 0$ on (a, b).

We also will make use of continuous dependence on boundary values and parameters. We refer the avid reader to [14, 17] for proof ideas.

Theorem 2.5 (Continuous dependence on boundary conditions and parameters)**.** *Assume (H1)–(H5) are satisfied with respect to* (1.1)*. Let $y(x)$ be a solution of* (1.1) *on (a, b). Then there exists a $\delta > 0$ such that, for $|x_1 - t_1| < \delta$, $|x_2 - t_2| < \delta$, $|p_j - \rho_j| < \delta$, $j = 1, \dots, m$, $|\chi_j - X_j| < \delta$, $j = 1, \dots, m$, $\left|y^{(i-1)}(x_1) - y_i\right| < \delta$, $i = 1, \dots, n-1$, $\left|y(x_2) - \sum_{j=1}^{m} p_j y(\chi_j) - y_n\right| < \delta$, and $|\lambda - L| < \delta$, there exists a unique solution $y_\delta(x)$ of* (1.1) *with respect to parameter L such that*

$$y_\delta^{(i-1)}(t_j) = y_i,\ i = 1, \dots, n-1,$$

and

$$y_\delta(t_2) - \sum_{j=1}^{m} \rho_j y(X_j) = y_n.$$

In addition, for $i = 1, \dots, n$, $\left\{y_\delta^{(i-1)}(x)\right\}$ converges uniformly to $y^{(i-1)}(x)$ as $\delta \to 0$ on $[\alpha, \beta] \subset (a, b)$.

3 Main Result

Now, we present our analogue of Theorem 2.3 with respect to the parameter λ.

Theorem 3.1. *Assume conditions (H1)–(H6) are satisfied.*

Let $y(x) = y(x; x_1, x_2, y_1, \dots, y_n, p_1, \dots, p_m, \chi_1, \dots, \chi_m, c, d, \lambda)$ be the solution of (1.1) *on (a, b) satisfying*

$$y^{(i-1)}(x_1) = y_i,\ i = 1, \dots, n-1,$$

and

$$y(x_2) - \sum_{j=1}^{m} p_j y(\chi_j) = y_n.$$

Then $L(x) := \dfrac{\partial y}{\partial \lambda}(x)$ *exists on* (a, b) *and is the solution of the nonhomogeneous equation* (2.2) *along* $y(x)$ *satisfying the boundary conditions*

$$L^{(i-1)}(x_1) = 0,\ i = 1, \ldots, n-1,$$

and

$$L(x_2) - \sum_{j=1}^{m} p_j L(\chi_j) = 0.$$

Proof. To ease the burdensome notation and realizing that everything is fixed except x and λ, we denote $y\left(x; x_1, x_2, \chi_1, \ldots, \chi_m, y_1, \ldots, y_n, p_1, \ldots, p_m, \lambda\right)$ by $y(x; \lambda)$. Let $\delta > 0$ be as in Theorem 2.5 with $0 \leq |h| \leq \delta$, and define the difference quotient for λ by

$$L_h(x) = \frac{1}{h}[y(x; \lambda + h) - y(x; \lambda)].$$

First, we substitute the boundary conditions into $L_h(x)$. For $h \neq 0$,

$$\begin{aligned} L_h^{(i-1)}(x_1) &= \frac{1}{h}\left[y^{(i-1)}(x_1; \lambda + h) - y^{(i-1)}(x_1; \lambda)\right] \\ &= \frac{1}{h}[y_i - y_i] = 0,\ i = 1, \ldots, n-1, \end{aligned}$$

and

$$\begin{aligned} L_h(x_2) - \sum_{j=1}^{m} p_j L_h(\chi_j) &= \frac{1}{h}\left[y(x_2; \lambda + h) - y(x_2; \lambda)\right] - \sum_{j=1}^{m} \frac{1}{h}\left[y(x; \lambda + h) - y(x; \lambda)\right] \\ &= \frac{1}{h}[y_n - y_n] = 0. \end{aligned}$$

Next, we show that $L_h(x)$ is a solution of the nonhomogeneous equation (2.2). To that end, define $\mu = y^{(n-1)}(x_1; \lambda)$ and $\nu = \nu(h) = y^{(n-1)}(x_1; \lambda + h) - \mu$.

Note by Theorem 2.5, $\nu = \nu(h) \to 0$ as $h \to 0$. View $y(x)$ as the solution of an initial value problem at x_1 and use the notation of initial values problems to get $y(x) = y(x; x_1, y_1, \ldots, y_n, \mu, \lambda)$. Thus, we have

$$L_h(x) = \frac{1}{h}\left[y(x; x_1, y_1, \ldots, y_{n-1}, \mu + \nu, \lambda + h) - y(x; x_1, y_1, \ldots, y_{n-1}, \mu, \lambda)\right].$$

Next, by utilizing telescoping sums to vary only one component at a time, we have

$$\begin{aligned} L_h(x) = \frac{1}{h}[&y(x; x_1, y_1, \ldots, y_{n-1}, \mu + \nu, \lambda + h) - y(x; x_1, y_1, \ldots, y_{n-1}, \mu, \lambda + h) \\ &+ y(x; x_1, y_1, \ldots, y_{n-1}, \mu, \lambda + h) - y(x; x_1, y_1, \ldots, y_{n-1}, \mu, \lambda)]. \end{aligned}$$

By Theorem 2.3 and the mean value theorem, we obtain

$$\begin{aligned}L_h(x) =&\frac{1}{h}[\alpha_n(x;y(x;x_1,y_1,\ldots,y_{n-1},\mu+\bar{\nu},\lambda+h))(\mu+\nu-\mu)\\&+\Lambda(x;y(x;x_1,y_1,\ldots,y_{n-1},\mu,\lambda+\bar{h}))(\lambda+h-\lambda)]\\=&\frac{\nu}{h}\alpha_n(x;y(x;x_1,y_1,\ldots,y_{n-1},\mu+\bar{\nu},\lambda+h))\\&+\Lambda(x;y(x;x_1,y_1,\ldots,y_{n-1},\mu,\lambda+\bar{h})),\end{aligned}$$

where $\alpha_n(x;y(x;x_1,y_1,\ldots,y_{n-1},\mu+\bar{\nu},\lambda+h))$ solves the variational equation (2.1) and $\Lambda(x;y(x;x_1,y_1,\ldots,y_{n-1},\mu,\lambda+\bar{h}))$ solves the nonhomogeneous equation (2.2). Furthermore, $\mu+\bar{\nu}$ is between μ and $\mu+\nu$ and $\lambda+\bar{h}$ is between λ and $\lambda+h$.

Thus, to show $\lim_{h\to 0} L_h(x)$ exists, it suffices to show, $\lim_{h\to 0}\frac{\nu}{h}$ exists. Recall, from the construction of $L_h(x)$, we have

$$L_h^{(i-1)}(x_1)=0,\ i=1,\ldots,n-1,$$

and

$$L_h(x_2)-\sum_{j=1}^m p_j L_h(\chi_j)=0.$$

From the latter condition, we have

$$\begin{aligned}&\frac{\nu}{h}\alpha_n(x_2;y(x;x_1,y_1,\ldots,y_{n-1},\mu+\bar{\nu},\lambda+h))+\Lambda(x_2;y(x;x_1,y_1,\ldots,y_{n-1},\mu,\lambda+\bar{h}))\\&\quad-\sum_{j=1}^m p_j\Big[\frac{\nu}{h}\alpha_n(\chi_j;y(x;x_1,y_1,\ldots,y_{n-1},\mu+\bar{\nu},\lambda+h))\\&\quad+\Lambda(\chi_j;y(x;x_1,y_1,\ldots,y_{n-1},\mu,\lambda+\bar{h}))\Big]=0.\end{aligned}$$

Solve for $\frac{\nu}{h}$ to get

$$\frac{\nu}{h}=\frac{-\Lambda(x_2;y(\cdot))+\sum_{j=1}^m p_j\Lambda(\chi_j;y(\cdot))}{\alpha_n(x_2;y(\cdot))-\sum_{j=1}^m p_j\alpha_n(\chi_j;y(\cdot))}.$$

Since $\alpha_n^{(n-1)}(x_1;y(x;x_1,y_1,\ldots,y_{n-1},\mu,\lambda))=1$, we have

$$\alpha_n(x;y(x;x_1,y_1,\ldots,y_{n-1},\mu,\lambda))\not\equiv 0.$$

Coupled with hypothesis (H6),

$$\alpha_n(x_2;y(x;x_1,y_1,\ldots,y_{n-1},\mu,\lambda))-\sum_{j=1}^m p_j\alpha_n(\chi_j;y(x;x_1,y_1,\ldots,y_{n-1},\mu,\lambda))\neq 0.$$

By Theorem 2.5 and for sufficiently small h, we have

$$\alpha_n(x_2; y(x; x_1, y_1, \ldots, y_{n-1}, \mu + \bar{\nu}, \lambda + h)) \\ - \sum_{j=1}^{m} p_j \alpha_n(\chi_j; y(x; x_1, y_1, \ldots, y_{n-1}, \mu + \bar{\nu}, \lambda + h)) \neq 0$$

implying $E := \lim_{h \to 0} \dfrac{\nu}{h}$ exists. Hence, we have

$$L(x) = \frac{\partial u}{\partial \lambda} = \lim_{h \to 0} L_h(x) = E\alpha_n(x; y(x)) + \Lambda(x; y(x))$$

exists on (a, b) and solves the nonhomogeneous equation (2.2).

Finally, we have

$$L^{(i-1)}(x_1) = \lim_{h \to 0} L_h^{(i-1)}(x_1) = \lim_{h \to 0} 0 = 0, \ i = 1, \ldots, n-1,$$

and

$$L(x_2) - \sum_{j=1}^{m} p_j L(\chi_j) = \lim_{h \to 0} \left[L_h(x_2) - \sum_{j=1}^{m} p_j L_h(\chi_j) \right] = \lim_{h \to 0} 0 = 0.$$

Therefore, $L(x)$ is the solution to the nonhomogeneous equation (2.2) satisfying

$$L^{(i-1)}(x_1) = 0, \ i = 1, \ldots, n-1 \ \text{ and } \ L(x_2) - \sum_{j=1}^{m} p_j L(\chi_j) = 0.$$

This concludes the proof. □

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